Homework #10

Math 25a

Due: November 16th, 2016

Guidelines:

- You must type up your solutions to this assignment in LaTeX. There's a template available on the course website.
- This homework is divided into four parts. You will turn each part in to a separate CA's mailbox on the second floor of the science center. So, be sure to do the parts on *separate* pieces of paper.
- If your submission to any particular CA takes multiple pages, then *staple them together*. If you don't own one (though you should), a stapler is available in the Cabot Library in the Science Center.
- Be sure to put your *name* at the top of each part, so that we know who to score!
- If you collaborate with other students, please announce that somewhere (ideally: next to the problems you collaborated on) so that we don't get suspicious of hyper-similar answers.

Failure to meet these guidelines may result in loss of points. (Staple your pages!)

1 For submission to Thayer Anderson

Problem 1.1. Consider the matrix

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 4 & 1 \end{pmatrix}.$$

Compute a square root of M.

Problem 1.2. Prove or give a counterexample: for V a complex vector space, $f\colon V \to V$ a linear operator, and $n = \dim V < \infty$, the operator $f^{\circ n}$ is diagonalizable.

Problem 1.3. Suppose $f\colon V \to V$ is a linear operator on a finite-dimensional complex vector space, and let $v \in V$ be a vector. Prove that there exists a unique monic polynomial p of smallest degree such that $p(f)(v) = 0$. Then, prove that this $p(z)$ divides the minimal polynomial of f.

2 For submission to Davis Lazowski

Problem 2.1. Let $f\colon V \to V$ be a linear operator on a finite-dimensional vector space. Prove that $\ker f^{\circ m} = \ker f^{\circ(m+1)}$ if and only if $\operatorname{im} f^{\circ m} = \operatorname{im} f^{\circ(m+1)}$. Conclude that $\operatorname{im} f^{\circ N}$ is stable for $N \gg 0$.

Problem 2.2. Give an example of an operator $f\colon V \to V$ on a finite-dimensional *real* vector space such that 0 is the only eigenvalue of f but f is not nilpotent.

Problem 2.3. Let $f\colon V \to V$ be an operator on a finite-dimensional complex vector space. Prove that there exist operators $D, N\colon V \to V$ such that D is diagonalizable, N is nilpotent, $f = D + N$, and they satisfy the commutation relation $DN = ND$.

Problem 2.4. Suppose that $f\colon V \to V$ has characteristic polynomial

$$p(z) = 4 + 5z - 6z^2 - 7z^3 + 2z^4 + z^5.$$

Calculate the minimal polynomial of f. Prove that f^{-1} must exist and calculate its minimal polynomial as well.

3 For submission to Handong Park

Problem 3.1. Suppose A and B are block diagonal matrices of the form

$$A = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_m \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_m \end{pmatrix},$$

where A_j and B_j are of size $n_j \times n_j$ for $j = 1, \dots, m$. Show that AB is a block diagonal matrix of the form

$$AB = \begin{pmatrix} A_1B_1 & & 0 \\ & \ddots & \\ 0 & & A_mB_m \end{pmatrix}.$$

Problem 3.2. Suppose V is a finite-dimensional complex vector space and $f\colon V \to V$ is a linear operator.

1. Prove that V has a basis consisting of eigenvectors of f if and only if every generalized eigenvector of f is a classical eigenvector of f.
2. Prove that V has a basis consisting of eigenvectors of f if and only if the minimal polynomial of f has no repeated zeroes.

Problem 3.3. Suppose V is a finite-dimensional complex vector space and $f\colon V \to V$ is a linear operator. Prove that there does not exist a direct sum decomposition of V into two proper invariant subspaces if and only if the minimal polynomial of f is of the form $(z - \lambda)^{\dim V}$ for some $\lambda \in \mathbb{C}$.

Problem 3.4. For coefficients $a_0, \dots, a_{n-1} \in \mathbb{C}$, consider the matrix

$$\begin{pmatrix} 0 & & & & -a_0 \\ 1 & 0 & & & -a_1 \\ & 1 & \ddots & & -a_2 \\ & & \ddots & & \vdots \\ & & & 0 & -a_{n-2} \\ & & & 1 & -a_{n-1} \end{pmatrix}.$$

Calculate the characteristic and minimal polynomials of this matrix.

4 For submission to Rohil Prasad

Problem 4.1. Let $f, g\colon V \to V$ be two linear operators related by an isomorphism $h\colon V \to V$ via the equation $f = hgh^{-1}$. Relate the generalized eigenspaces for f and g through h. Conclude that f and g have the same eigenvalues with the same algebraic multiplicities (i.e., these are *invariants* of f and g).

Problem 4.2. Suppose $f\colon \mathbb{C}^4 \to \mathbb{C}^4$ is such that the eigenvalues of f are 3, 5, and 8. Prove that

$$(f-3)^2(f-5)^2(f-8)^2 = 0.$$

Problem 4.3. Suppose $f\colon V \to V$ is invertible. Prove that there exists a polynomial $p(z)$ (dependent upon f) such that $f^{-1} = p(f)$.