Homework #1

Math 25a

Due: September 14, 2016

Guidelines:

- This homework is divided into four parts. You will turn each part in to a separate CA's mailbox, so be sure to do the parts on *separate* pieces of paper (stapling together multiple pages if necessary).
- Be sure to put your *name* at the top of each part, so that we know who to score!
- If you collaborate with other students, please announce that somewhere (ideally: next to the problems you collaborated on) so that we don't get suspicious of hyper-similar answers.
- For the first few assignments, we would like you to keep track of how long it takes you to complete the (whole) assignment and to notate that somewhere on your submission, so that we can calibrate the difficulty of the homeworks.

1 For submission to Thayer Anderson

1. (a) The average of real numbers $a_1, \ldots, a_n$ is given by

$$M = \frac{a_1 + \cdots + a_n}{n}.$$

   Show that at least one of the numbers a_j satisfies $a_j \geq M$.

   (b) Arrange the numbers $1, \ldots, 9$ in a circle. Show that there must exist three adjacent numbers whose sum is at least 16, no matter what circular arrangement you pick.

2. Given an example of a function $f : \mathbb{N} \to \mathbb{N}$ which is...

   (a) ... injective but not surjective.

   (b) ... surjective but not injective.

   (c) ... surjective and injective, but different from the "identity function" $f(x) = x$.

   (d) ... neither surjective nor injective.

   Each time, justify your example.

3. A guest at a party is a *celebrity* if this person is known by every other guest, but knows none of them. There is at most one celebrity at a party — if there were two, they would know each other. On the other hand, it is possible that no guest is a celebrity. Devise a method for finding the celebrity at a party of n people which involves only asking questions of the form "Person A, do you know Person B?" and which takes no more than $3(n-1)$ questions.

2 For submission to Davis Lazowski

1. Prove that for any n, the sum of the first n odd integers is n^2.
2. Show that if x is an irrational number, then there is an integer n such that the distance between x and n is less than $1/2$. (Feel free to use that real numbers have decimal expansions.)
3. Consider a set of $n+1$ positive integers, each less than or equal to $2n$. By inducting on $n \geq 1$, show that there must always exist a pair of integers in the set, one dividing the other.

3 For submission to Handong Park

1. Formulate a conjecture about the final decimal digit (i.e., the "ones" digit) of the 4^{th} power of an integer. Prove your conjecture by cases.
2. Suppose that $n = a/b$ is a rational number, where a and b are integers with no common factors (meaning, for instance, that they cannot both be even). Show that n^2 cannot be 2 — i.e., $\sqrt{2}$ cannot be rational.
3. Explain what is wrong with this "proof":

   > We would like to show that all horses have the same color. Toward that end, let $P(n)$ denote the claim "Any collection of n horses have the same color." The first claim, $P(1)$, is true: any collection consisting of a single horse has only one color. Then, suppose that $P(j)$ is true for some j, and consider a collection of $j+1$ horses. Numbering the horses, the first j of them must have the same color by $P(j)$, and the final j of them must have the same color by $P(j)$. Since the middle $j-1$ horses from these two sets overlap, all the horses in the collection of $j+1$ of them must have the same color by transitivity. Hence, $P(j+1)$ follows from $P(j)$. By induction, $P(n)$ is true for all values n.

4. Suppose that five 1s and four 0s are arranged around a circle. Form a new circle by placing a 0 between any two unequal adjacent numbers and a 1 between any two equal values, then erasing the original values. Show that, no matter how many times you repeat this and no matter what the initial configuration is, you will never get a circle of all 0s.

4 For submission to Rohil Prasad

1. (a) Prove or disprove that the product of two *rational* numbers is *rational*.

   (b) Prove or disprove that the product of two *irrational* numbers is *irrational*.
2. Show that there is no rational number r satisfying
   $$r^3 + r + 1 = 0.$$
   (Hint: set $r = a/b$, clear the denominators, and consider the parities of a and b.)
3. Write the numbers $1, \ldots, 2n$ on a blackboard, where n is an odd integer. Pick any two numbers j and k from this list, erase them, and add $|j-k|$ to the list. Repeat this step until there is a single integer remaining. Show that this last integer must be odd.