Solutions to Homework #1

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1 For submission to Thayer Anderson

Problem 1.1. 1. The average of real numbers $a_1, \dots, a_n$ is given by

$$M = \frac{a_1 + \cdots + a_n}{n}.$$

Show that at least one of the numbers a_j satisfies $a_j \geq M$.

2. Arrange the numbers $1, \dots, 9$ in a circle. Show that there must exist three adjacent numbers whose sum is at least 16, no matter what circular arrangement you pick.

Solution. 1. Suppose for the sake of contradiction that all a_j satisfy $a_j < M$ where m is the average of the numbers. We see that if $x < y$ and $p < q$ then $x + p < y + q$. Therefore,

$$\begin{aligned} a_1 + a_2 + \cdots + a_n &< \underbrace{M + \ldots + M}_{n \text{ times}} \\ \Rightarrow a_1 + \cdots a_n &< nM \\ \Rightarrow M = \frac{a_1 + \cdots + a_n}{n} &< M \end{aligned}$$

This is a contradiction, and therefore at least one a_j satisfies $a_j \geq M$.

2. Suppose we have a circular arangement of the numbers 1 through 9. Denote this arrangement with the string $a_1 a_2 \cdots a_9$ where the a_i are the distinct integers between 1 and 9. Note that a_1 and a_9 are understood to be adjacent.

For $1 \leq i \leq 9$, define the sets, G_i, as follows:

$$\begin{aligned}
G_1 &= \{a_1, a_2, a_3\}, \\
G_2 &= \{a_2, a_3, a_4\}, \\
&\vdots \\
G_9 &= \{a_9, a_1, a_2\}
\end{aligned}$$

We see that the G_i are exactly the sets of three adjacent numbers. Let s_i be the sum of the elements in G_i. Suppose for the sake of contradiction that $s_i \leq 15$ for all i. Let T denote the total sum over all of the groups. Then we have

$$T \leq 9 \cdot 15 = 145 \tag{1}$$

with equality exactly when $s_i = 15$ for all i. Moreover, each a_k is a member of exactly three G_i. Summing over all these sets, we achieve a total T as follows:

$$T = 3 \cdot 1 + 3 \cdot 2 + \ldots + 3 \cdot 9 = 3 \left(\sum_{i=1}^{9} i \right) = 145.$$

Thus, by (1), $s_i = 15$ for all i. Then consider groups $G_1 = \{a_1, a_2, a_3\}$ and $G_2 = \{a_2, a_3, a_4\}$. We have:

$$\begin{aligned}
15 = a_1 + a_2 + a_3 &= a_2 + a_3 + a_4 \\
&\Rightarrow a_1 = a_4
\end{aligned}$$

This is a contradiction as the numbers in the circle are distinct. Therefore $s_i > 15$ for at least one fixed i and we are done. (TA)

Problem 1.2. Give an example of a function $f : \mathbb{N} \to \mathbb{N}$ which is...

1. ...injective but not surjective.
2. ...surjective but not injective.
3. ...surjective and injective, but different from the "identity function" $f(x) = x$.
4. ...neither surjective nor injective.

Each time, justify your example.

Solution. There are numerous solutions, but here I provide an example for each using my favorite definition of the natural numbers, $\mathbb{N} = \{0, 1, \ldots\}$.

1. Let $f(n) = 2n$. We see the outputs of f are exactly the even numbers. 3 is not even, and therefore f is not surjective. Suppose $f(n) = f(m)$. Then $2n = 2m \Rightarrow n = m$. Therefore, f is injective.

2. Define f as follows (with $k \in \mathbb{N}$):

$$f(n) = \begin{cases} 0 & \text{if } n = 2k \\ k & \text{if } n = 2k+1 \end{cases}$$

We see that this function is well defined as no number is both odd and even. We see that $f(0) = f(2) = 0$. Thus, f is not injective. Suppose $k \in \mathbb{N}$, then $k = f(2k+1)$ and thus f is surjective.

3. Let f be defined as follows:

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n = 1 \\ n & \text{otherwise} \end{cases}$$

Suppose $f(n) = f(m)$. If $f(n) = 1$ then $n = m = 0$. If $f(n) = 0$ then $n = m = 1$. Otherwise, $f(n) = f(m)$ immediately implies that $n = m$. Therefore f is injective. Suppose $n \in \mathbb{N}$ arbitrary. If $n = 0$ then $n = f(1)$. If $n = 1$, then $n = f(0)$. Otherwise, $n = f(n)$. Therefore, f is surjective.

4. Let $f(n) = 0$ for all n. There is no n such that $f(n) = 1$. Therefore f is not surjective. Additionally, $f(0) = f(1) = 0$ so f is not injective. (TA)

Problem 1.3. A guest at a party is a *celebrity* if this person is known by every other guest, but knows none of them. There is at most one celebrity at a party — if there were two, they would know each other. On the other hand, it is possible that no guest is a celebrity. Devise a method for finding the celebrity at a party of n people which involves only asking questions of the form "Person A, do you know Person B?" and which takes no more than $3(n-1)$ questions.

Solution. Suppose we ask Person A if they know Person B. If Person A does know Person B, then Person A cannot be the celebrity. Otherwise, Person B cannot be the celebrity. It follows that as long as there are two or more people who *might* be celebrities, we can eliminate one from contention with a single question. Applying this inductively, we can narrow down to one potential celebrity after $n-1$ questions. Call that person, Person C. Then we ask Person C if they know each of the the $n-1$ other people. Then we ask each of the $n-1$ other people if they know Person C.

If the answer to all those questions is "Yes," then Person C is a celebrity. If not, then there is no celebrity. This process takes a total of

$$(n-1)+(n-1)+(n-1)=3(n-1)$$

questions. (TA)

2 For submission to Davis Lazowski

Problem 2.1. Prove that for any n, the sum of the first n odd integers is n^2.

Solution. We give a proof by induction.

Base case. For $n=1$, $1=1^2$, therefore done.

Inductive step. Let the sum of the first n odd integers equal n^2. Then $(n+1)^2-n^2=2n+1$. But the $n+1$–th odd integer is precisely $2n+1$. Therefore the sum of the first $n+1$ odd integers equals $(n+1)^2$. (DL)

Problem 2.2. Show that if x is an irrational number, then there is an integer n such that the distance between x and n is less than $1/2$. (Feel free to use that real numbers have decimal expansions.)

Solution. Every real number has a decimal expansion. For $r \in \mathbb{R}$, denote the fractional part of r as $F(r)$. Then $1 > F(r) \geq 0$, and $r - F(r) \in \mathbb{N}$.

Let q an irrational number. Let $F(q) < \frac{1}{2}$. Then we are done, because the distance between $q - F(q)$ and q is less than $\frac{1}{2}$. Observe that $F(q) \neq \frac{1}{2}$ for all irrational numbers because if $F(q) = \frac{1}{2}$, then $2q$ is an integer.

Therefore, the only other possibility is that $F(q) > \frac{1}{2}$. Then $\frac{1}{2} > 1 - F(q)$, and $q + 1 - F(q) \in \mathbb{N}$, so that the distance between $q+1-F(q)$ and q is less than $\frac{1}{2}$. (DL)

Problem 2.3. Consider a set of $n+1$ positive integers, each less than or equal to $2n$. By inducting on $n \geq 1$, show that there must always exist a pair of integers in the set, one dividing the other.

Solution. Suppose such a list existed. The list must have a number less than or equal to n, because there are only n numbers in the interval $[n+1, 2n]$. Let u_1 be the greatest such number.

Claim. There exists $c_1 : n < 2^{c_1} u_1 \leq 2n$.

Proof. Clearly $2u_1 \leq 2n$. If $2u_1 > n$, then we are done. If $2u_1 \leq n$, then still $2 \cdot 2u_1 \leq 2n$. Repeating this process, we must arrive at some $2^j u_1 > 2n$, so there must be some $c_1 : n < 2^{c_1} u_1 \leq 2n$.

Clearly $2^{c_1} u_1$ and u_1 cannot be on the same list, so $2^{c_1} u_1$ cannot be on the list.

So there are at most $n-1$ numbers on the list greater than n. Therefore there are at least 2 numbers less than or equal to n. Call the second greatest u_2.

Repeat the same process, which will work exactly as before.

The only concern is if $2^{c_1}u_1 = 2^{c_2}u_2$. But then either $2^{c_1-c_2} \in \mathbb{N}$, so $2^{c_1-c_2}u_1 = u_2$, or $2^{c_2-c_1}u_2 = u_1$. In either case, one of u_1 or u_2 divides the other, so u_1 and u_2 cannot be on the same list.

Inducting, we get that there must be at least $n+1$ numbers less than or equal to n, but this is a contradiction. (DL)

Solution. Here's an alternative inductive solution. The claim is true when $n = 1$: in this case, the set must be the two elements $\{1, 2\}$, and 1 divides 2. Suppose in general that the claim is true for some fixed $n = j$ and consider the case $n = j + 1$. There are three cases to consider:

- Suppose that the subset S contains only one of $2j + 1$ and $2j + 2$. In this case, $S \cap [1, 2j]$ contains exactly $j + 2 - 1 = j + 1$ elements, and hence the inductive assumption applies directly to give a division pair in $S \cap [1, 2j]$.

- Suppose that the subset S contains neither $2j+1$ nor $2j+2$. In that case, $S \cap [1, 2j]$ contains $n + 2$ elements. If we discard *any* element — it doesn't matter which — we get a set with $n + 1$ elements, to which the inductive assumption applies.

- Finally, suppose that the subset S contains both $2j + 1$ and $2j + 2$. In this case, $S \cap [1, 2j]$ contains only n elements, so we cannot directly apply the inductive assumption. We split into two further cases:
  - If $S \cap [1, 2j]$ contains $j+1$, then we are done: S is also assumed to contain $2j+2 = 2(j+1)$ in this case.
  - If $S \cap [1, 2j]$ does not contain $j + 1$, then we add $j + 1$ to the restricted set to form $T = (S \cap [1, 2j]) \cup \{j + 1\}$, a set of $j + 1$ elements in the range $[1, 2j]$. The inductive hypothesis applies to give us a pair of elements $a, b \in T$ with a dividing b. We, again, fork into cases:
    * If neither a nor b is $j + 1$, then they both belong to the original set $S \cap [1, 2j]$, and hence they give a division pair in S.
    * If b is $j + 1$, we need to modify the pair, since $j + 1$ does not belong to S. In this case, $b = j + 1$ divides $2(j + 1)$, which is in S, and hence a divides $2b = 2(j + 1)$ as well. So, we can take $(a, 2(j + 1))$ as the division pair in S.
    * The other option is for a to be $j + 1$. This can never happen: there must be some element $b \leq 2j$ which a divides, yet the smallest element which a divides (which is not a itself) is $2a = 2(j + 1) > 2j$. Hence, a can never actually be $j + 1$. (ECP)

3 For submission to Handong Park

Problem 3.1. Formulate a conjecture about the final decimal digit (i.e., the "ones" digit) of the 4^{th} power of an integer. Prove your conjecture by cases.

Solution. We're asked to formulate a conjecture about the final decimal digit of the fourth power of an integer, and to prove this by cases.
So we can do some preliminary exploring:

$$0^4 = 0, 1^4 = 1, 2^4 = 16, 5^4 = 625, 24^4 = 331776, ...$$

After trying to raise many integers to the fourth power, we observe that the last digit seems to be either a $0, 1, 5$, or 6. So, we claim the following:

Claim: The final decimal digit of the fourth power of an integer is either $0, 1, 5$, or 6.

How can we go about proving our claim? Well, consider the decimal expansion of a generic integer Z with n places - for some integer coefficients a_i that range between 0 and 9, where $i \in \mathbb{N}$, we have

$$Z = a_n \cdot 10^n + a_{n-1} \cdot 10^{n-1} + ... + a_1 \cdot 10 + a_0$$

Suppose we were to raise Z to the fourth power. We'd then have

$$\begin{aligned} Z^4 &= (a_n \cdot 10^n + a_{n-1} \cdot 10^{n-1} + ... + a_1 \cdot 10 + a_0)^4 \\ &= (a_n \cdot 10^n + a_{n-1} \cdot 10^{n-1} + ... + a_1 \cdot 10 + a_0) \cdot (a_n \cdot 10^n + a_{n-1} \cdot 10^{n-1} + ... + a_1 \cdot 10 + a_0) \\ &\cdot (a_n \cdot 10^n + a_{n-1} \cdot 10^{n-1} + ... + a_1 \cdot 10 + a_0) \cdot (a_n \cdot 10^n + a_{n-1} \cdot 10^{n-1} + ... + a_1 \cdot 10 + a_0) \end{aligned}$$

Now, when we multiply these terms, we find that almost all of these terms cannot contribute to the ones digit of Z^4. In fact, any cross-term that is multiplied by a power of 10 cannot contribute to the ones digit of Z^4 at all - it will have a 0 in its ones digit because it is multiplied by a power of 10.
Thus, the only term that can contribute to the ones digit of Z^4 is a_0^4, the only term not involving any powers of 10. Thus, the ones digit of Z^4 will be given by the ones digit of a_0^4.
Now, examine all possible cases for a_0^4, by considering all possible a_0's, the possible ones digits of possible integers. We have

$$0^4 = 0, 1^4 = 1, 2^4 = 16, 3^4 = 81, 4^4 = 256, 5^4 = 625$$

$$6^4 = 1296, 7^4 = 2401, 8^4 = 4096, 9^4 = 6561$$

In all of the possible cases, the ones digit is given by $0, 1, 5$, or 6, proving our conjecture. (HP)

Problem 3.2. Suppose that $n = a/b$ is a rational number, where a and b are integers with no common factors (meaning, for instance, that they cannot both be even). Show that n^2 cannot be 2 — i.e., $\sqrt{2}$ cannot be rational.

Solution. Let's prove that $\sqrt{2}$ is irrational by contradiction. First, assume that $\sqrt{2}$ is indeed rational, so that

$$\sqrt{2} = \frac{a}{b}$$

where a and b have no common factors, so that this fraction is in lowest terms. If we square both sides, we have

$$2 = \frac{a^2}{b^2}$$

Moving the b^2 over, we have

$$2b^2 = a^2$$

But $2b^2$ is even, which implies that a^2 must be even. However, only the square of an even number can be even. Thus, a itself must be even, meaning $a = 2c$ for some integer c, giving us

$$2b^2 = (2c)^2 = 4c^2$$

Now divide both sides by 2, then we have

$$b^2 = 2c^2$$

But since $2c^2$ is even, b^2 must also be even! This means that b must also be even, by the same logic as above.

However, if both a and b are even, then that contradicts our assumption, since a and b were assumed to have no common factors (so that the fraction is in lowest terms). By contradiction, our assumption must have been incorrect, implying that we actually cannot find a and b in lowest terms such that $\sqrt{2} = \frac{a}{b}$.

Thus, since we cannot find such a and b without running into this contradiction, we prove that $\sqrt{2}$ cannot be a rational number. (HP)

Problem 3.3. Explain what is wrong with this "proof":

> We would like to show that all horses have the same color. Toward that end, let $P(n)$ denote the claim "Any collection of n horses have the same color." The first claim, $P(1)$, is true: any collection consisting of a single horse has only one color. Then, suppose that $P(j)$ is true for some j, and consider a collection of $j+1$ horses. Numbering the horses, the first j of them must have the same color by $P(j)$, and the final j of them must have the same color by $P(j)$. Since the middle $j-1$ horses from these two sets

overlap, all the horses in the collection of $j+1$ of them must have the same color by transitivity. Hence, $P(j+1)$ follows from $P(j)$. By induction, $P(n)$ is true for all values n.

Solution. We've taken the base case to be $n = 1$ horses, where a single horse has only one color. Now, let's attempt to induct from $P(1)$ to $P(2)$ by the logic proposed above. Assume that any collection of a single horse has only one color. We wish to show that any collection of 2 horses has only one color as well. By the logic above, for any given $j+1$ horses (in this case, $j+1 = 2$), we should have that the middle $j-1$ horses overlap, giving us the same color for all the horses.
Yet for $j+1 = 2$, this fails. There are no "middle" $j-1$ horses, because $j-1 = 0$ - the two horses have no overlapping horses in between them of the same color, because there are no other horses between them.
Thus, this proof fails - we've chosen a base case $n = 1$ where the inductive step does not hold for ALL $j \geq 1$; the inductive step fails when we attempt to go from 1 horse to 2 horses. (HP)

Problem 3.4. Suppose that five 1s and four 0s are arranged around a circle. Form a new circle by placing a 0 between any two unequal adjacent numbers and a 1 between any two equal values, then erasing the original values. Show that, no matter how many times you repeat this and no matter what the initial configuration is, you will never get a circle of all 0s.

Solution. We are given five 1s and four 0s arranged in a circle. We form a new circle, all at once, by placing a 0 between any two unequal bits and a 1 between any two equal bits, then erasing all of the original values.

Claim: No matter what arrangement we start with, and no matter how many steps we take, there is no way to obtain a circle of all 0s at the end.

Suppose by some divine providence that we happened to have all 0s at the end. Then, let's attempt to backtrack - what did the circle look like just before we arrived at all 0s?
Since the only way to insert a 0 between two numbers is to have to unequal numbers, we must have had that any two adjacent numbers had unequal values, so that a 0 could be placed in between every two numbers. In other words, we needed to have a circle that fully alternated, with no two adjacent numbers having the same value.
However, we have 9 numbers in the circle - so no matter what we do, we cannot completely alternate. Either we'll have five 1s and four 0s or four 1s and five 0s, but because we have an unequal number of 1s and 0s, a repetition will occur, due to our uneven number of slots in the circle.
Thus, there is no way to obtain all 0s at the end, and we're done. (HP)

4 For submission to Rohil Prasad

Problem 4.1. 1. Prove or disprove that the product of two *rational* numbers is *rational.*

2. Prove or disprove that the product of two *irrational* numbers is *irrational*.

Solution. (a) A real number is **rational** if and only if it can be expressed as a ratio p/q where p, q are integers.

Given two rational numbers a, b, we can thus write them as $a = p_1/q_1$ and $b = p_2/q_2$ for p_i, q_i integers. Their product is just $ab = p_1 p_2/q_1 q_2$. The product of two integers is again an integer, so $p_1 p_2$ and $q_1 q_2$ are integers themselves. It follows that ab is itself rational by our definition.
(b) This is false. We can produce a counterexample, namely that the product of the two irrational numbers $\sqrt{2}$ and $\sqrt{2}$ is the rational number 2. (RP)

Problem 4.2. Show that there is no rational number r satisfying

$$r^3 + r + 1 = 0.$$

(Hint: set $r = a/b$, clear the denominators, and consider the parities of a and b.)

Solution. Assume for the sake of contradiction that a rational solution r exists for this equation.

Set $r = a/b$ where a, b are integers. Furthermore, we can also assume that a, b have no common factors (why?).

Then our equation above becomes $(a/b)^3 + (a/b) + 1 = 0$. Simplifying, we get

$$\begin{aligned} \frac{a^3}{b^3} + \frac{a}{b} + 1 &= 0 \\ \frac{a^3 + a^2 b + b^3}{b^3} &= 0 \\ a^3 + a^2 b + b^3 &= 0 \end{aligned}$$

Since $a^3 + a^2 b + b^3 = 0$, it follows that the left-hand side must be even, which can be expressed in modular arithmetic terms as $a^3 + a^2 b + b^3 \equiv 0 \mod 2$

Now we can do casework to evaluate $a^3 + a^2 b + b^3 \bmod 2$ given the values of $a, b \bmod 2$. Modular arithmetic here is nice because it makes our "bookkeeping" of our cases a little bit easier.

Case 1: $a \equiv 0 \mod 2$, $b \equiv 0 \mod 2$

We assumed a, b have no common factors. In this case, a, b would have a common factor of 2, so it cannot occur.

Case 2: $a \equiv 1 \mod 2$, $b \equiv 0 \mod 2$

We have $a^3 \equiv 1^3 \equiv 1 \mod 2$, $a^2 b \equiv 1^2 \cdot 0 \equiv 0 \mod 2$, and $b^3 \equiv 0^3 \equiv 0 \mod 2$, so their sum is $1 + 0 + 0 \equiv 1 \mod 2$.

Case 3: $a \equiv 0 \mod 2$, $b \equiv 1 \mod 2$

We have $a^3 \equiv 0^3 \equiv 0 \mod 2$, $a^2 b \equiv 0^2 \cdot 1 \equiv 0 \mod 2$, and $b^3 \equiv 1^3 \equiv 1 \mod 2$, so their sum is $0 + 0 + 1 \equiv 1 \mod 2$.

Case 4: $a \equiv 1 \mod 2$, $b \equiv 1 \mod 2$

We have $a^3 \equiv 1^3 \equiv 1 \mod 2$, $a^2 b \equiv 1^2 \cdot 1 \equiv 1 \mod 2$, and $b^3 \equiv 1^3 \equiv 1 \mod 2$, so their sum is $1 + 1 + 1 \equiv 1 \mod 2$.

In all of our cases, we have therefore found that $a^3 + a^2 b + b^3$ is actually *odd*, so it cannot be equal to 0. Therefore, our original assumpption is false and we find that there is no rational solution r to this equation. (RP)

Problem 4.3. Write the numbers $1, \ldots, 2n$ on a blackboard, where n is an odd integer. Pick any two numbers j and k from this list, erase them, and add $|j - k|$ to the list. Repeat this step until there is a single integer remaining. Show that this last integer must be odd.

Solution. This problem was quite tricky! The key insight here is that for some reason we specified that n must be *odd* in order to get an *odd* number on the board. In fact, if you tried this out with n *even*, you would get an *even* number left on the board.

To show that the last number on the board is odd, we will actually show something a little stronger. We will show that the *sum* of the numbers *at any time* during the process is odd (why is this a stronger statement?).

First, we should check that this holds true for our starting set of numbers. We can calculate the sum of 1 through $2n$ to be $\frac{2n(2n+1)}{2} = n(2n+1)$. Note that $2n + 1$ is always odd for any integer, and n is odd by the conditions of the problem, so their product is odd as well.

Now, I claim that it suffices to show that our erasing "operation" preserves the parity of the sum of the numbers on the chalkboard. If this is true, then we can see in an inductive fashion that the sum of the numbers is odd. Namely, assume there are $k < 2n$ numbers on the board. Then, we know that these k numbers must have been obtained by performing $2n - k$ erasing steps on the board. Since these steps each preserve the parity of the sum, and the initial sum is odd, it follows that the sum of the k numbers is odd.

The erasing step takes two numbers j, k with $j \geq k$ and erases them and replaces them with $j - k$. If S is the sum of the numbers initially, then after the erasing step we are left with the sum $S - j - k + j - k = S - 2k$. Therefore, we are subtracting $2k$ from the sum in our erasing step. The number $2k$ is *even* no matter what, so *the parity of the sum remains unchanged* as desired and we are done! (RP)