THE SUM $\sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)}$

Hi, everyone,

During class today, we produced a series which computed π:

$$\pi = \sum_{k=0}^{\infty} (-1)^k \frac{4}{2k+1}.$$

We got this using the Taylor expansion for arctangent:

$$\tan^{-1}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1},$$

then we set $x = 1$ and noted that $\tan^{-1}(1) = \pi/4$.

At the very end of class, I wrote the following series on the board:

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots$$

and asked whether we could compute its sum. We can solve this problem by working the arctangent problem backwards: we will rewrite this so that it looks like a sum of coefficients of a power series, then we will produce a guess power series, and then we will try to find a function which has that power series as its Taylor expansion. If all the stars align, this will give us a way to compute the sum, by computing the function instead.

Step 1: Let's start by writing this sum in a way that looks more like a power series. Power series are commonly expressed in Σ–notation, so let's rewrite our sum to use Σ–notation as well:

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots = \sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)}.$$

Step 2: The above sum is a sum of numbers. Remember that we can gets sums of numbers by evaluating a power series at a particular point. So, for instance, if we have a power series

$$\sum_{j=0}^{\infty} a_j x^j,$$

then by setting $x = 2$ we get

$$\left. \sum_{j=0}^{\infty} a_j x^j \right|_{x=2} = \sum_{j=0}^{\infty} a_j 2^j,$$

or if we set $x = 1$ then we get

$$\left. \sum_{j=0}^{\infty} a_j x^j \right|_{x=1} = \sum_{j=0}^{\infty} a_j 1^j = \sum_{j=0}^{\infty} a_j.$$

If we pick a value other than 1 (like $x = 2$), then we see that the x leaves a lingering geometric factor in the sum, like the 2^j above. Our term $\frac{1}{(k+1)(k+2)}$ doesn't have any geometric terms, so we will imagine that it came from a power series where we picked $x = 1$. Specifically, if we use the power series

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)} x^{k?}$$

and we set $x = 1$, then we get

$$f(1) = \sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)} \cdot 1^{k?} = \sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)}.$$

So, all we need to do is figure out what $f(x)$ is, then plug in 1, and we'll know our answer. (This really is just like what we did for arctangent, when we set $x = 1$ in $\tan^{-1)}(x)$ and computed $\tan^{-1}(1) = \pi/4$.)

Before we move on, here'ss an important and subtle point: I've written "k?" in the exponents, because it doesn't really matter if we have k or $k+1$ or $2k$ or whatever. All that matters is that when we plug in $x = 1$, this factor of $x$es disappears. Its precise form will come out of how we solve the problem.

Step 3: Now we turn to finding a compact expression for the function $f(x)$, rather than just writing it as a series. Remembering that integration of a power series looks like

$$\int \sum_{j=0}^{\infty} a_j x^j \, dx = \sum_{j=0}^{\infty} a_j \frac{x^{j+1}}{j+1},$$

the series expression for $f(x)$ looks like it's been integrated a couple of times. That would explain the k pieces in the denominator. So, let's start by finding a function that expresses a series without these things in the denominator, which is now geometric:

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}.$$

Now, in order to morph this series into one with ks in the denominator, we integrate both sides:

$$\begin{aligned} \int \sum_{k=0}^{\infty} x^k \, dx &= \int \frac{1}{1-x} \, dx \\ \left(\sum_{k=0}^{\infty} \frac{1}{k+1} x^{k+1} \right) + C &= -\ln|1-x| \end{aligned}$$

By evaluating the series at its center $x = 0$, we discover what C is:

$$\begin{aligned} (0) + C &= -\ln|1-0| \\ C &= 0 \end{aligned}$$

and hence

$$\sum_{k=0}^{\infty} \frac{1}{k+1} x^{k+1} = -\ln|1-x|.$$

So, we've produced a series with just $(k+1)$ in the denominator, and we've found an easy function that it's equal to. To get the missing $(k+2)$, we integrate again:

$$\begin{aligned} \int \sum_{k=0}^{\infty} \frac{1}{k+1} x^{k+1} \, dx &= \int -\ln|1-x| \, dx \\ \left(\sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)} x^{k+2} \right) + C &= |1-x| \ln|1-x| - |1-x|. \end{aligned}$$

The second line comes from knowing the antiderivative of the logarithm, which you get by using integration by parts:

$$\int \ln u \, du = u \ln u - u + C.$$

Now, again, we set $x = 0$ to evaluate at the center of the series and solve for C:

$$\begin{aligned} (0) + C &= |1-0| \ln|1-0| - |1-0| \\ C &= -1, \end{aligned}$$

from which this follows:

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{1}{(k+1)(k+2)}x^{k+2} &= (1-x)\ln|1-x|-(1-x)+1\\ &= (1-x)\ln(1-x)+x = f(x).\end{aligned}$$

Step 4: We've done it! We found a function $f(x)$ (on the last line, right above) whose series expansion at 0 is exactly the power series we want.[1] So, if we set $x = 1$, we should get an expression for our original sum. Unfortunately, the function we came up with is not defined at 1 (since ln(0) is undefined), but the function and the series are both defined (and agree) on the region $-1 < x < 1$. So, we can take the limit of both sides as x tends to 1 from below:

$$\begin{aligned}\lim_{x\to 1^-}\sum_{k=0}^{\infty}\frac{1}{(k+1)(k+2)}x^{k+2} &= \lim_{x\to 1^-}((1-x)\ln(1-x)+x)\\ \sum_{k=0}^{\infty}\frac{1}{(k+1)(k+2)} &= \lim_{x\to 1^-}((1-x)\ln(1-x))+1.\end{aligned}$$

On the left, we pulled the limit inside the sum and evaluated. On the right, we evaluated the easy part of the limit (the lone "x" on the end), and what's left is the indeterminate form $0\cdot\infty$. We can resolve this using L'Hôpital's rule:

$$\lim_{x\to 1^-}(1-x)\ln(1-x) = \lim_{x\to 1^-}\frac{\ln(1-x)}{\frac{1}{1-x}} \overset{H}{=} \lim_{x\to 1^-}\frac{\frac{-1}{1-x}}{\frac{1}{(1-x)^2}} = \lim_{x\to 1^-}(x-1) = 0.$$

Plugging this back in to the above, we finally arrive at

$$\sum_{k=0}^{\infty}\frac{1}{(k+1)(k+2)} = 0+1 = 1.$$

Step 5: Celebrate! What a miraculous sum. You could have checked that it converged using the comparison and p–series tests, but you wouldn't have been able to know what it converged *to*. Other simple–looking sums that you write down turn out to take on quite complicated values. For instance, the harmonic series gives the undesirable value

$$\sum_{k=1}^{\infty}\frac{1}{k} = +\infty,$$

and the sums of reciprocal squares gives the mysterious value

$$\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6},$$

but this is quite hard to calculate and beyond the techniques of our class. Yet, this similar-looking sum was within our reach and gave the simple value 1. Neat!

Step 6: Now go looking for a geometric interpretation, like we had the subdivided square picture for $\sum_{j=1}^{\infty} r^j \ldots$. Maybe this is best left until after your exam on Tuesday, though.

[1] You'll notice that "k?" earlier was justified: we ended up with $k+2$ in the exponent.

Appendix A. Finding a geometric interpretation

We actually can visualize these as rectangles nesting into a unit square. The idea is to carve up a square by alternatingly decreasing either side length. Beginning with a unit square, carve half of it out to leave a rectangle of dimensions $\frac{1}{2} \times 1$, and note that we removed a rectangle of area $\frac{1}{1\cdot 2}$ to do so. Since the remaining rectangle has area $1/2$, by removing one-third of it we leave a rectangle of dimensions $\frac{1}{2} \times \frac{2}{3}$ and we removed a rectangle of area $\frac{1}{2\cdot 3}$. Since the remaining rectangle has area $\frac{1}{3}$, we remove one-fourth of it to leave a rectangle of dimensions $\frac{3}{8} \times \frac{2}{3}$ and in the process we removed a rectangle of area $\frac{1}{3\cdot 4}$. This continues ad infinitum.

To show formally that this works, we define a sequence of remaining heights h_n and a sequence of remaining widths w_n. These are subject to the recursion relations

$$\begin{aligned} h_0 &= 1, & w_0 &= \frac{1}{2}, \\ h_1 &= \frac{2}{3}, & w_1 &= \frac{3}{8}, \\ h_2 &= \frac{8}{15}, & w_2 &= \frac{5}{16}, \\ &\vdots & &\vdots \\ h_n &= h_{n-1} \cdot \left(1 - \frac{1}{2n+1}\right), & w_n &= w_{n-1} \cdot \left(1 - \frac{1}{2n+2}\right). \end{aligned}$$

We then need to show that this process of carving out rectangles is *exhaustive*, in the sense that eventually every point in the square gets captured. This is the same as the pair of assertions

$$h_n \xrightarrow{n\to\infty} 0, \qquad w_n \xrightarrow{n\to\infty} 0.$$

Out of laziness, I leave these to the reader.

Appendix B. Alternative approach

This problem is amusingly simple to solve using telescoping series, which we did not cover in class. If you apply partial fraction decomposition to the summation terms, you'll discover that essentially everything cancels, giving an easy expression for the n^{th} partial sum.