SPECTRA AND STABILITY

ERIC PETERSON
SEPTEMBER 10, 2013

1. Stable homotopy groups and a category

Suspension is an extraordinarily important operation that gets short-shrift with the definition

$$\Sigma X = \frac{X \times [0,1]}{\{x_0\} \times [0,1] \cup X \times \{0,1\}}.$$

A more telling account of why we care so much about it comes from two of its properties:

(1) Suspension increments the dimension of spheres and disks: $\Sigma S^n \simeq S^{n+1}$, $\Sigma D^n \simeq D^{n+1}$.
(2) Suspension preserves cofiber sequences: the gluing sequence

$$Y \xrightarrow{f} X \to X \cup_f CY$$

is sent by suspension to another gluing sequence

$$\Sigma Y \xrightarrow{\Sigma f} \Sigma X \to \Sigma X \cup_{\Sigma f} C\Sigma Y.$$

Together, these turn a CW-structure on a space X into a CW-structure on ΣX by shifting all the cells of X up one dimension. There's an analogous shift operation in graded algebraic contexts: the suspension of a graded group, for instance, is given by the formula

$$(\Sigma G)_{n+1} = G_n.$$

Much of modern homotopy theory revolves around the study of *stable invariants*, which are invariants that intertwine the topological and algebraic notions of suspension. Homology is one example of a stable invariant: the Eilenberg–Steenrod axioms can be used to produce the familiar identity

$$H_n(X) \cong H_{n+1}(\Sigma X).$$

Homotopy, however, is not stable — but a theorem of Freudenthal says that a piece of it is.

Freudenthal Suspension Theorem. *Let X and Y be spaces which are s-connected and t-dimensional respectively. The map of function spaces*

$$F(Y,X) \to F(\Sigma Y, \Sigma X)$$

is a $(2s - t)$-equivalence. In particular, the same map on homotopy groups

$$\begin{array}{ccc} \pi_n X & \longrightarrow & \pi_{n+1}\Sigma X \\ \downarrow{\cong} & & \downarrow{\cong} \\ \pi_n F(S^0, X) & \longrightarrow & \pi_n F(\Sigma S^0, \Sigma X) \end{array}$$

is an isomorphism in the range $n \leq 2s$.[1]

Freudenthal's theorem gives no control over what happens outside of this range — and, indeed, all sorts of unstable phenomena arise. Nonetheless, we can leverage it to build a stable invariant: notice that when X is s-connected, ΣX is then $(s+1)$-connected. Applying Freudenthal's theorem again, this implies that the map $\pi_n \Sigma X \to \pi_{n+1}\Sigma^2 X$ is an isomorphism for $n \leq 2s+2$, a slight improvement over the previous range because of the factor of 2. Iterating this process, we produce groups called the *stable homotopy groups of X*:

$$\operatorname{colim}\big([S^n, X] \to [\Sigma S^n, \Sigma X] \to [\Sigma^2 S^n, \Sigma^2 X] \to \cdots\big) =: \pi_n \Sigma^\infty X.$$

[1] In the setting of cell complexes, "s-connected" means that X can be constructed (up to homotopy type) with bottom cell in dimension $s+1$, and "d-dimensional" means Y can be constructed (up to homotopy type) using cells only of dimension at most d.

Freudenthal's theorem says this isn't much of a colimit: the sequence becomes constant at a finite stage.

That's neat, I guess, but so far it's just another example of a stable invariant. What we'd rather be doing is constructing a category in which stable invariants live so that we can study them en masse — a *stable homotopy category*. Freudenthal's theorem gives us a foothold: for any space X we associate an object "$\Sigma^\infty X$", called *the suspension spectrum of* X. The mapping space between such suspension spectra is given by the formula

$$[\Sigma^\infty Y, \Sigma^\infty X] := \operatorname{colim}\big([Y,X] \to [\Sigma Y, \Sigma X] \to [\Sigma^2 Y, \Sigma^2 X] \to \cdots\big).$$

Again, Freudenthal's theorem says this system becomes constant at a finite stage. This definition of the mapping space also justifies the word "stable" floating around: there's an equality $[\Sigma^\infty X, \Sigma^\infty Y] = [\Sigma^\infty \Sigma X, \Sigma^\infty \Sigma Y]$.[2]

2. Infinite loopspaces as objects in the stable homotopy category

Suspension spectra like $\Sigma^\infty X$ will play the role of the "input data" for our stable invariants: eventually we'll be able to see that there's a factorization

$$\begin{tikzcd} \mathsf{Spaces} \arrow[rr, "H_*"] \arrow[dr, "\Sigma^\infty_+"'] & & \mathsf{GradedAbelianGroups} \\ & \mathsf{SuspensionSpectra} \arrow[ur, dotted, "H_*"'] & \end{tikzcd}$$

What we'd really like to do, however, is incorporate the stable invariants themselves into a nice category. This will take a little doing. The first thing to notice is that these stable mapping groups $[\Sigma^\infty Y, \Sigma^\infty X]$ can be computed without ever leaving the category of spaces. Namely, we have an equivalence of sequences

$$\begin{tikzcd} {[Y,X]} \arrow[r] \arrow[d, "\cong"] & {[\Sigma Y, \Sigma X]} \arrow[r] \arrow[d, "\cong"] & {[\Sigma^2 Y, \Sigma^2 X]} \arrow[r] \arrow[d, "\cong"] & \cdots \\ {[Y,X]} \arrow[r] & {[Y, \Omega\Sigma X]} \arrow[r] & {[Y, \Omega^2\Sigma^2 X]} \arrow[r] & \cdots. \end{tikzcd}$$

By defining the space $\Omega^\infty(\Sigma^\infty X) = \operatorname{colim}_n \Omega^n \Sigma^n X$ (sometimes denoted QX), we have an isomorphism

$$[\Sigma^\infty Y, \Sigma^\infty X] \cong [Y, \Omega^\infty \Sigma^\infty X].$$

This space has an interesting property related to suspension invariance:

$$[Y, QX] = [\Sigma^\infty Y, \Sigma^\infty X] = [\Sigma^\infty \Sigma Y, \Sigma^\infty \Sigma X] = [\Sigma Y, Q\Sigma X] = [Y, \Omega Q \Sigma X].$$

It follows that $Q\Sigma X$ is a *delooping* of the space QX. Continuing in this fashion, there's an infinite sequence $\{Q\Sigma^n X\}_n$ of such deloopings, making QX into an *infinite loopspace*.

This is reminiscent of something about ordinary cohomology. Cohomology with coefficients in G is represented by *Eilenberg–Mac Lane spaces*, and the same argument can be used to prove an analogous relationship:

$$\begin{tikzcd} H^n(Y;G) \arrow[r, "\cong"] \arrow[d, "\cong"] & H^{n+1}(\Sigma Y;G) \arrow[r, "\cong"] \arrow[d, "\cong"] & H^{n+2}(\Sigma^2 Y;G) \arrow[r, "\cong"] \arrow[d, "\cong"] & \cdots \\ {[Y, K(G,n)]} \arrow[r, "\cong"] \arrow[d, "\cong"] & {[\Sigma Y, K(G,n+1)]} \arrow[r, "\cong"] \arrow[d, "\cong"] & {[\Sigma^2 Y, K(G,n+2)]} \arrow[r, "\cong"] \arrow[d, "\cong"] & \cdots \\ {[Y, K(G,n)]} \arrow[r, "\cong"] & {[Y, \Omega K(G,n+1)]} \arrow[r, "\cong"] & {[Y, \Omega^2 K(G,n+2)]} \arrow[r, "\cong"] & \cdots \end{tikzcd}$$

Since $K(G,n)$ has an infinite sequence of deloopings, it looks like it could fit into the story we've been telling thus far: perhaps there's an object "HG" with the property that $K(G,n) = \Omega^\infty \Sigma^n HG$. The key property of such an object is that its stable homotopy would take the form

$$\pi_n HG = \begin{cases} G & \text{when } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

[2]We can also justify our use of the words "stable homotopy groups": set the *stable n-sphere* or *nth sphere spectrum* to be $\mathbb{S}^n = \Sigma^\infty S^n$. The definition of mapping space then gives the familiar formula $\pi_n \Sigma^\infty X = [\mathbb{S}^n, \Sigma^\infty X]$.

It turns out that we can't quite get this with the one tool Σ^∞ that we have, but we can get close. Start by considering the suspension spectrum $\Sigma^\infty K(G,n)$. Again appealing to Freudenthal, we see that its homotopy is:

$$\pi_m \Sigma^\infty K(G,n) = \begin{cases} G & \text{when } m = n, \\ 0 & \text{when } m \leq 2n \text{ and } m \neq n, \\ \text{mystery groups} & \text{when } m > 2n. \end{cases}$$

The shift is something we can quickly fix: since we have stability invariance, we can define *desuspensions* or *negative suspensions* of suspension spectra, whose mapping spaces are given by the formula

$$[\Sigma^\infty Y, \Sigma^{-n}\Sigma^\infty X] = [\Sigma^\infty \Sigma^n Y, \Sigma^\infty X].$$

Next, there are natural maps

$$\Sigma^{-n}\Sigma^\infty K(G,n) \to \Sigma^{-(n+1)}\Sigma^\infty K(G,n+1),$$

so if we had sequential colimits we could set

$$HG := \operatorname{colim}\left(\Sigma^\infty K(G,0) \to \Sigma^{-1}\Sigma^\infty K(G,1) \to \cdots \to \Sigma^{-n} K(G,n) \to \cdots\right).$$

This is exactly what we do: set *the (homotopy) category of spectra* to be the ind-completion of the category of desuspensions of suspension spectra.[3]

3. The relationship to homological algebra

Let's just take a moment to just marvel at this thing we've built.

OK. Now for some intriguing facts. The first thing to notice is that all of the stable mapping sets defined above are actually abelian groups, just like (and for the exact same reasons that) $\pi_2 X$ is an abelian group. This makes the category of spectra into an *additive category*. The other additive categories you know about are probably categories of modules — the stable category is not like those. Instead, it is a triangulated category: given any map of spaces $Y \to X$, we can extend it to a *cofiber sequence*

$$Y \to X \to X \cup_f CY \to \Sigma Y,$$

and we define cofiber sequences of spectra to be those arising from applying Σ^∞ to cofiber sequences of spaces (plus some closure properties). This structure means that the category of spectra is more like the derived category of chain complexes in an abelian category.

It's worth noting that it's *not* such a derived category, though. There are two competing facts: first, in the derived category $D(\mathsf{A})$ of an abelian category A, for any object X we can define an object $X/2$ by the cofiber sequence

$$X \xrightarrow{\cdot 2} X \to X/2.$$

The homotopy of such an object is always 2-torsion. Secondly, in the stable category one can compute that $\pi_1(\mathbb{S}^0/2) = \mathbb{Z}/4$. Hence, spectra cannot be the derived category of an abelian category.

Nonetheless, it is useful to be in the mindset of derived categories when working in the stable category — just expect some facts to be more complicated or even false.

[3]One failing of these notes is that we have constructed only a suitable *homotopy* category of spectra. Consider the analogous classical situation: the homotopy category of spaces is not particularly well-behaved, and many constructions (especially those rooted in geometry) rely on having a "background" category to work in before passing to the homotopy category. This is also a necessary headache when working with spectra up to homotopy, and it quickly becomes desirable to have access to a background category of spectra, together with a notion of passing to its homotopy category. In fact, when I say "ind-completion", I am blurring exactly this line. There are many different constructions of such a background category, each with a different set of positive attributes. I hope someone will tell us some of that story in a future talk.

4. GENERALIZED HOMOLOGY THEORIES AND THE STABLE CATEGORY

In this section, we'll focus on the most delightful property of the stable category: it factors in a very particular way every stable invariant satisfying Mayer–Vietoris. This known as Brown representability, and it is a generalization of the property mentioned before that it factors the homology functor H_*.

Let's see why this is true. First, remember that the category of spaces has a *smash product*:

$$X \wedge Y = \frac{X \times Y}{X \vee Y}.$$

It has three important properties:

(1) It specializes to suspension: $\Sigma X \simeq S^1 \wedge X$.
(2) It commutes with (nice) colimits: $Y \wedge \operatorname{hocolim}_\alpha X_\alpha \simeq \operatorname{hocolim}_\alpha (Y \wedge X_\alpha)$. In particular, it passes through cofiber sequences.
(3) Accordingly, it participates in an adjunction with function spaces: $[Z \wedge Y, X] \cong [Z, F(Y, X)]$.

We can quickly bootstrap up to a smash product of spectra. To begin, we define

$$\Sigma^\infty X \wedge \Sigma^\infty Y := \Sigma^\infty (X \wedge Y).$$

Using the identity $S^n \wedge S^m \simeq S^{n+m}$, we can lift this to desuspended suspension spectra as well:

$$\Sigma^n \Sigma^\infty X \wedge \Sigma^m \Sigma^\infty Y \simeq \Sigma^{n+m} \Sigma^\infty (X \wedge Y).$$

Finally, given that a general spectrum can be presented as a colimit of such things, we make the final definition:

$$\begin{aligned} X \wedge Y &= \operatorname{colim}_\alpha \{\Sigma^{n_\alpha} \Sigma^\infty X_\alpha\} \wedge \operatorname{colim}_\beta \{\Sigma^{m_\beta} \Sigma^\infty Y_\beta\}. \\ &= \operatorname{colim}_{\alpha,\beta} \{\Sigma^{n_\alpha + m_\beta} \Sigma^\infty (X_\alpha \wedge Y_\beta)\}. \end{aligned}$$

This endows the homotopy category with a symmetric monoidal structure. Moreover, all the above properties of smash products also hold in the stable category: it specializes to suspension, it plays nicely with homotopy colimits, and it begets a notion of function spectrum by a fancy version of the adjoint functor theorem.

Incredibly, this is all it takes to construct our functor $H_*(-; G)$ inside spectra: it is given by the formula

$$H_*(X; G) = \pi_*(\Sigma^\infty_+ X \wedge HG).$$

All the Eilenberg–Steenrod axioms are satisfied: because it is defined by taking homotopy groups, it is homotopy invariant; the dimension axiom follows from the homotopy groups $\pi_* HG$; additivity follows from additivity of the stable category; and exactness follows from the commutability of $- \wedge HG$ with cofiber sequences.

It turns out that this incredible fact is generic:

Brown Representability Theorem. *For any homology theory E_* satisfying the generalized Eilenberg–Steenrod axioms (i.e., discarding the dimension axiom), there is a spectrum E such that if X is a finite CW-complex, there is a natural isomorphism*

$$E_* X \cong \pi_*(\Sigma^\infty_+ X \wedge E).$$

(Conversely, every spectrum E begets a homology functor by the formula $E_* X := \pi_*(E \wedge \Sigma^\infty X)$.)

This is a wonderful theorem: what Brown's theorem is saying is that the category of spectra is an enrichment of the category of homology theories in such a way that verbs like "quotient" and "glue" make good sense. The input data for homology theories also live in this category as suspension spectra. Finite spectra are thus particularly fertile: since they are finite, they arise as suspension spectra, and since they are finite, they arise from finitely many cofiber operations applied to the sphere spectrum, a kind of homology theory. So, finite spectra bridge these two worlds.

Let's now explore two wonderful consequences of our wonderful theorem. First, this allows for the insertion of some algebra into the world of spectra. Since localization maps like $\mathbb{Z} \to \mathbb{Z}_{(p)}$ are flat, this implies that the tensor functor carries long exact sequences

$$\cdots \to \pi_n Y \to \pi_n X \to \pi_n(X \cup_f CY) \to \cdots$$

to other long exact sequences:

$$\cdots \to \mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} \pi_n Y \to \mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} \pi_n X \to \mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} \pi_n(X \cup_f CY) \to \cdots.$$

It follows that $\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} \pi_* -$ is a generalized homology theory, and Brown's theorem implies that it has the form of smashing with a spectrum $\mathbb{S}^0_{(p)}$:

$$\pi_* X \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \cong \pi_*(X \wedge \mathbb{S}^0_{(p)}).$$

This simple fact also holds for $\mathbb{Z}[\frac{1}{p}]$ and $\mathbb{Q}$, and this shows the beginning of what turns out to be a deep and mysterious interplay between algebraic localizations and features of the stable category.

Secondly, let's construct a more exotic sort of homology theory. Ordinary homology is commonly defined via singular homology, which comes from defining a chain complex $C_*(X)$ with $C_n(X)$ the free abelian group on the set of continuous maps off the n-simplex:

$$C_n(X) = \mathbb{Z}\{f \mid f : \Delta^n \to X\}.$$

A slight tweak of this definition begets a more interesting chain complex: rather than using just the n-simplex, we use any n-manifold with boundary. The boundary of an n-manifold is a closed $(n-1)$-manifold, and this fact is used to construct the boundary map on chains. One quickly checks that the homology of this new chain complex also satisfies the Eilenberg–Steenrod axioms, and hence begets a spectrum known as Ω^O or MO, called *unoriented bordism*. Thom computed the stable homotopy groups of this spectrum in his thesis, where he found

$$\pi_* MO = \mathbb{F}_2[x_i \mid i \neq 2^j - 1].$$

It turns out that MO actually contains the same information as ordinary mod-2 homology:

$$MO \simeq \bigvee_m \Sigma^{n_m} H\mathbb{Z}/2.$$

A more interesting variant on this same construction gives a more interesting theory: simply restrict to oriented manifolds. This, too, begets a homology theory, and hence a spectrum MSO, the *oriented bordism spectrum*. This spectrum is not 2-torsion, and so does not fall prey to this splitting — instead, its structure is immensely complicated to describe, making it of much interest to everyone. There's also a form of this construction for complex manifolds, and the spectrum MU turns out to have incredible arithmetical information. Much of the modern portrait of stable homotopy theory revolves around this connection to algebraic and arithmetic geometry, but *why* the connection is there — why manifolds have such rich arithmetic information — is not known.