DETECTING NONTRIVIAL HOMOTOPY ELEMENTS VIA (CO)HOMOLOGY

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INTRODUCTION

A major part of algebraic topology is centered around computing the homotopy groups of a space. One can easily visualize these groups in small degree - maping points, circles and spheres into spaces - and it's also easy to vizualize (when the degree of the source sphere is smaller than the target space) whether or not such embeddings are homotopic. These groups are also very important when it comes to checking if two spaces are homotopy equivalent. Indeed, from Whitehead we know that if there exists a map $f : X \to Y$ between spaces (by 'space' we mean connected CW-complex, unless specified) which induces an isomorphism between the homotopy groups then X and Y are homotopy equivalent. The list of useful facts arising from knowing the homotopy groups of a space is extremely long and the different questions one can answer when they know such groups is equally impressive. The immense power of these groups is, however, rivaled by how difficult it is to actually compute them. It's really hard to compute all the homotopy groups of even simple (geometrically speaking) spaces. One only has to consider $X = S^2$ and you're stuck.

Rather than asking about the homotopy groups of a space we could also ask about its (co)homology groups. These groups are much more difficult to vizualize but they are typically easier to compute. It would be nice to be able to infer properties about homotopy groups from (co)homology groups. This will be the theme of these notes.

As a first example of using cohomology to detect homotopy, let's consider the setting where X is an $(n-1)$-connected $(n > 1)$ space with $H\mathbb{Z}^n(X) = \mathbb{Z}$. Recall, for such spaces X, the Hurewicz homomorphism $h : \pi_n(X) \to \mathrm{Hom}(H\mathbb{Z}^n(X), H\mathbb{Z}^n(S^n))$, $(f : S^n \to X) \mapsto (f^* : H\mathbb{Z}^n(X) \to H\mathbb{Z}^n(S^n))$, is an isomorphism. Therefore, since $\mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \cong \mathbb{Z}$, we see that cohomology via the Hurewicz isomorphism is detecting non-trivial maps from S^n into X; i.e., it's detecting non-trivial homotopy for X. In particular, the degree of a map $f : S^n \to S^n$ is detected by $h : \pi_n(S^n) \to \mathbb{Z}$.

STABLE COHOMOLOGY OPERATIONS

We can continue along the same path, but let's introduce more algebraic structure on cohomology and use it tackle deeper questions. We know about cup products, which endow cohomology with the structure of a ring, but these alone will not suffice for what we want to do. Instead we'd really like to have cohomology operations that commute with suspension $\Sigma \circ \theta = \theta \circ \Sigma$; i.e., we'd like stable operations. This is really important as it implies these operations thrive in the stable category, along with cohomology. This is not the case for the cup product, suspending a space results in annihilating all cup products. This point should really be hammered home. Suppose we want to show that there exists non-trivial (in homotopy) maps from S^n to some space X. It might be the case that we'd rather work with a space Y instead, with $\Sigma^j Y = X$. The plan has now changed to showing non-trivial maps $\Sigma^{-j} S^n \to Y$ exist and then suspending up to show that they also exist between $S^n \to X$. Since cup products are trivial on suspended spaces we cannot use them, but, since stable operations commute with suspensions we can use them. Another way to view this is as follows. Cohomology is a stable invariant telling us about the stable homotopy groups of a space, and so we shouldn't have to leave the stable category by requesting the presence of cup squares. In addition to commuting with the suspension functor, the operations should obey a naturality condition:

given a map $f : X \to Y$ the following diagram commutes (here G is arbitrary)

$$\begin{array}{ccc} HG^n(Y) & \xrightarrow{\theta_Y} & HG^m(Y) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ HG^n(X) & \xrightarrow[\theta_X]{} & HG^m(X) \end{array}$$

This is telling us that stable cohmology operations live in the endomorphism ring of the cohomology functor we wish to use; that is, letting $\mathscr{O}_E$ denote the algebra of cohomology operations associated with a cohomology theory E, we have $\mathscr{O}_E \cong E^*(E)$. For the case when $G = \mathbb{F}_2$ we get something called the Steenrod algebra, denoted $\mathscr{A}^*$ (it's really a Hopf algebra). Through the use of the Serre spectral sequence one can show that the Steenrod algebra is the algebra generated by the Steenrod squares along with their relations (see below). Let's now discuss these operations.

The Steenrod squares are (mod 2) cohomology operations $\mathrm{Sq}^i : H\mathbb{F}_2^n(X) \to H\mathbb{F}_2^{n+i}(X)$ obeying:

(1) $\mathrm{Sq}^0 = 1$
(2) Sq^1 is the Bockstein (connecting homomorphism) associated with the coefficient sequence

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0;$$

(3) if $x \in H\mathbb{F}_2^n(X)$ then $\mathrm{Sq}^n x = x^2$;
(4) if $x \in H\mathbb{F}_2^n(X)$ then $\mathrm{Sq}^i x = 0$ for $i > n$;
(5) (Cartan formula) $\mathrm{Sq}^i(xy) = \sum_j (\mathrm{Sq}^j x)(\mathrm{Sq}^{i-j} y)$;
(6) (Adem relations) for $a < 2b$, $\mathrm{Sq}^a \mathrm{Sq}^b = \sum_c \binom{b-c-1}{a-2c}_2 \mathrm{Sq}^{a+b-c} \mathrm{Sq}^c$.

With this new algebraic structure, cohomology being a $\mathscr{A}^*$-module, we can start answering more advanced questions. A slightly more interesting question we could ask is what happens for higher homotopy groups; can we detect non-trivial maps from $S^m \to X$, where $m > \dim X$? In order to address this question let's fix $X = S^n$. The question now becomes, can we detect nontrivial elements living in the higher homotopy groups of spheres using cohomology?

To begin, recall that for a map $f : X \to Y$ one can construct a cofibre sequence

$$X \xrightarrow{f} Y \xrightarrow{\iota} Cf$$

with 'cofibre' $Cf = Y \cup_f CX$. Now, let's suppose we have a map $f : S^{n-1} \to S^m$ that is homotopy equivalent to the constant map f_0. In this case the cofibre is the complex $Cf = S^m \cup_f e^n$. This complex is homtopy equivalent to the complex $S^m \cup_{f_0} e^n = S^m \vee S^n$, and so, $H\mathbb{F}_2^*(Cf)$ has the two generators $x \in H\mathbb{F}_2^m(Cf)$ and $y \in HF_2^n(Cf)$ along with the $\mathscr{A}^*$-module isomorphism

$$H\mathbb{F}_2^*(Cf) \xrightarrow{\cong} H\mathbb{F}_2^*(S^m) \oplus H\mathbb{F}_2^*(S^n).$$

Now, for any $\tau \in \mathscr{A}^*$ of positive degree, $\tau x = 0$. In conclusion, we see that for any map $f : S^{n-1} \to S^m$ if there is an $\tau \in \mathscr{A}^*$ of positive degree with $\tau x \neq 0$, it follows that f is not null homotopic. So, in this case, we're using the Steenrod algebra to detect non-trivial homotopy classes of maps from S^{m-1} into S^n.

As a concrete example we can show that the suspended Hopf map $\Sigma\eta : \Sigma S^3 \to \Sigma S^2$ is not nullhomotopic. To do so, we'll first start with the cofibre of the Hopf map, compute its cohomology and then suspend up, showing the nontriviality of $\Sigma\eta$. It should be noted that this does not follow from Freudenthal, as $\pi_3(S^2) \to \pi_4(S^3)$ is not in the stable range.

To begin, the Hopf map $\eta : S^3 \to S^2$ begets a cofibre $C\eta = S^2 \cup_\eta e^4$, which is homotpic to CP^2. Thus, $H\mathbb{F}_2^*(C\eta)$ has the structure of the polynomial ring $\mathbb{F}_2[y]/(y^3)$, with $|y| = 2$. Suspending $C\eta$ gives us the cofibre $X = C\Sigma\eta = S^3 \cup_{\Sigma\eta} S^5$, of $\Sigma\eta : S^4 \to S^3$. From the cell decomposition of X it follows that $H\mathbb{F}_2^*(X)$ has the two generators $x \in H\mathbb{F}_2{}^3(X)$, $z \in H\mathbb{F}_2{}^5(X)$. Further, from the suspension isomorphism $\sigma : E^{*+1} \circ \Sigma \xrightarrow{\cong} E^*$ for cohomology, we immediately know $x = \sigma^{-1}(y)$ and $z = \sigma^{-1}(y^2) = \sigma^{-1}(\mathrm{Sq}^2 y)$ and so, we have

$$\mathrm{Sq}^2 x = \mathrm{Sq}^2 \sigma^{-1}(y) = \sigma^{-1}(\mathrm{Sq}^2 y) = \sigma^{-1}(y^2) = z.$$

Therefore $\Sigma\eta$ cannot be null homotopic.

In a similar fashion we have the Hopf invariant one problem. The setup is easy to comprehend, but the full solution requires immense amounts of high-brow machinery. The setup is as follows. Suppose we have a map $f : S^{2n-1} \to S^n$ (we'll assume that $n > 1$). The cofibre of this map will be a CW complex with one n-cell and one $2n$-cell. Hence, its integral cohomology is $\mathbb{Z}$ concentrated in degrees 0, n, with generator say $x \in HZ^n(Cf)$, and $2n$, with generator $y \in H\mathbb{Z}^{2n}(Cf)$. We immediately observe that x^2 is an integral multiple of y. This multiple is denoted $H(f)$ and called the Hopf invariant of f (H defines a map $\pi_{2n-1} \to \mathbb{Z}$). The Hopf invariant one problem asks for which values of n does there exist maps f with $H(f) = 1$. The answer turns out to be $n = (1), 2, 4, 8$. We'll not be able to prove this deep result, but we can go quite far with only using mod 2 cohomology.

We'll show that if f has Hopf invariant one (mod 2) then $n = 2^k$, $k \geq 0$. In order to do so we must first take a deeper look at the Steenrod squares and the Adem relations. From these relations one finds that certain squares decompose into linear combinations other squares with lower degree. For example, $\mathrm{Sq}^3 = \mathrm{Sq}^2\,\mathrm{Sq}^1$ and $\mathrm{Sq}^6 = \mathrm{Sq}^2\,\mathrm{Sq}^4 + \mathrm{Sq}^5\,\mathrm{Sq}^1$. Now, it's a fact that if $n \neq 2^k$ then Sq^n is decomposable. Hence, for $n \neq 2^k$, Sq^n decomposes into a sum of a Steenrod squares which individually have degree less than n, implying that to go from x to $\mathrm{Sq}^n x = x^2$ we have to take intermediate steps through cohomology of degrees less than $2n$ (see figure). But then there's no way to get anything nonzero as those groups are all zero. Hence, we conclude, that $\mathrm{Sq}^n x = 0$; i.e., $H(f) \neq 1$.

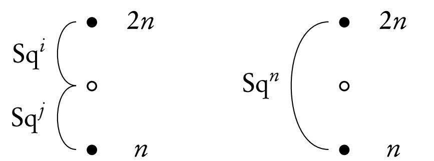
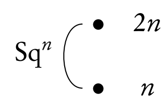


FIGURE 1. When Sq^n is decomposable we must factor through trivial cohomology groups, whereas when Sq^n is indecomposable we don't.

Thus, we've just seen that the indecomposables in $\mathscr{A}^*$, denoted $Q\mathscr{A}$, detect odd Hopf invariant maps.

Alternatively, the Hopf invariant one problem can be re-formulated as follows. For which values of n does there exist a map $f : S^{2n-1} \to S^n$ with cofibre X such that

$H\mathbb{F}_2^*(X) =$



Equivalently, we're asking for spaces X with nontrivial extensions[1]

$$0 \to \mathbb{F}_2 \to H\mathbb{F}_2^*(X) \to \mathbb{F}_2[n] \to 0;$$

that is, since $H\mathbb{F}_2^*(X)$ is an $\mathscr{A}^*$-module, we are looking for non-trivial elements in $\mathrm{Ext}^1_{\mathscr{A}^*}(\mathbb{F}_2[t], \mathbb{F}_2)$ which, to help motivate the next section, is equivalent to $\mathrm{Ext}^1_{\mathscr{A}^*}(H\mathbb{F}_2^*(\mathbb{S})[t], \mathbb{F}_2)$ (here $\mathbb{S}$ is the sphere spectrum).

A QUICK PREVIEW OF THE ADAMS SPECTRAL SEQUENCE

By now the reader should see that the aproach we take is to add more and more structure to (co)homology and then enjoy the fruits of our labor by using this deeper algebraic machinery to answer more interesting questions. With this motto in mind, we'll now introduce a truley magical piece of algebraic machinery, the Adams spectral sequence. This spectral sequence, commonly called ASS, allows one to (in certain cases) compute the stable homotopy groups of a space via its homology groups and a knowledge of $\mathscr{A}^*$. This is quite different than what we've done so far. Until now we've only been able to detect if specific maps lead to nontrivial elements in homotopy, the

[1]The trivial extension $H\mathbb{F}_2^*(X) \cong H\mathbb{F}_2^*(S^n) \oplus H\mathbb{F}_2^*(S^{2n})$ corresponds to null homotopic maps f.

ASS will actually compute the groups (rather, the p-completed groups). Of course it takes a lot of hard work to get the crank turning, but once it starts extremely deep results tend to pop out. Let's state the theorem.

Theorem (Adams). For a connective spectrum X (satisfying some properties) there is a spectral sequence, whose E_2-page is given by

$$E_2^{s,t} = \mathrm{Ext}^s_{\mathscr{A}^*}(H\mathbb{F}_2^*(X)[t], \mathbb{F}_2),$$

converging to the 2-completed stable homotopy groups of X, $\pi^S_{t-s}(X)^\wedge_2$. This is quite a remarkable fact. It's saying that knowing the homology of a space along with the Steenrod algebra gives you enough to compute the stable homotopy groups, no more algebraic machinery is needed.

We see that for $X = \mathbb{S}$ the ASS will compute the stable homotopy groups of spheres. Of course computing with the ASS is not easy and typically not quickly done. Hence, we'll not be able to really do any kind of examples in the talk. Instead we'd like to mention some interesting things that happen inside the ASS when $X = \mathbb{S}$.

Let's first discuss the filtration $s = 0$. In this case we're looking at the groups $\mathrm{Ext}^0_{\mathscr{A}^*}(\mathbb{F}_2[t], \mathbb{F}_2)$. Now, we know from homological algebra that $\mathrm{Ext}^0_R(M,N) \cong \mathrm{Hom}_R(M,N)$. Hence, non-trivial elements living in $\mathrm{Ext}^0_{\mathscr{A}^*}(\mathbb{F}_2[t], \mathbb{F}_2)$ naturally correspond to non-trivial elements living in $\mathrm{Hom}_{\mathscr{A}^*}(\mathbb{F}_2[t], \mathbb{F}_2)$, which, as we've seen, naturally correspond to elements which are non-zero under the Hurewicz isomorphism. Thus, the $s = 0$ filtration in the ASS tells you about the Hurewicz map and so, we should expect to see a non-trivial element in the $(s = 0, t = 0)$ spot.

The filtration $s = 1$ elements, as we've mentioned, correspond to Hopf invariant odd maps. Therefore, we expect non-trivial elements in the $(s = 1, t = 1)$, $(1,2)$, $(1,4)$, $(1,8)$ entries. The statement that these are the only elements which survive to the E_∞ page is the Hopf invariant one problem. More precisely, the Hopf invariant one theorem (due to Adams) is that $d_2(h_i) = h_0 h_{i-1}^2$, $i \geq 4$.

For completeness, we now post a portion of the E_2 page:

s \ $t-s$	0	1	2	3	4	5	6	7	8
	$\vdots$								
4	h_0^4	0	0	0	0	0	0	$h_0^3 h_3$	0
3	h_0^3	0	0	h_1^3	0	0	0	$h_0^2 h_3$	α
2	h_0^2	0	h_1^2	$h_0 h_2$	0	0	h_2^2	$h_0 h_3$	$h_1 h_3$
1	h_0	h_1	0	h_2	0	0	0	h_3	0
0	1	0	0	0	0	0	0	0	0

From the E_2 page we see the generator in $(0,0)$ corresponding to the Hurewicz isomorphism, the generators along the $s = 1$ line corresponding to the Hopf invariant one maps, along with their products. The higher filtrations $(s > 1)$ also correspond to deep theorems/ideas in algebraic topology but due to lack of time these explanations will have to wait.