Discrete Mathematics

Advanced Counting Techniques

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8.4: Generating Functions

Generating functions

Definition

The *generating function* for the sequence $a_0, a_1, \dots, a_k, \dots$ of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + \cdots + a_k x^k + \cdots = \sum_{k=0}^{\infty} a_k x^k.$$

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Remark

We can define generating functions for finite sequences of real numbers by extending a finite sequence $a_0, a_1, \ldots, a_n$ into an infinite sequence by setting $a_{n+1} = 0$, $a_{n+2} = 0$, and so on. The generating function $G(x)$ of this infinite sequence $\{a_n\}$ is a polynomial of degree n:

$$G(x) = a_0 + a_1 x + \cdots + a_n x^n.$$

Generating functions

Example

Let m be a positive integer. Let $a_k = C(m, k)$ for $k = 0, 1, 2, \ldots, m$. What is the generating function for the sequence $a_0, a_1, \ldots, a_m$?

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The generating function for this sequence is

$$G(x) = \binom{m}{0} + \binom{m}{1}x + \binom{m}{2}x^2 + \cdots + \binom{m}{m}x^m$$

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The generating function for this sequence is

$$G(x) = \binom{m}{0} + \binom{m}{1}x + \binom{m}{2}x^2 + \cdots + \binom{m}{m}x^m = (1 + x)^m.$$

Useful facts about power series

Example

The function $f(x) = 1/(1 - ax)$ is the generating function of the sequence $1, a, a^2, a^3, \ldots,$ because

$$\frac{1}{1 - ax} = 1 + ax + a^2x^2 + \cdots$$

when $|ax| < 1$, or equivalently, for $|x| < 1/|a|$ for $a \neq 0$.

Useful facts about power series

Theorem

Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $g(x) = \sum_{k=0}^{\infty} b_k x^k$. Then:

$$f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k,$$

$$f(x) \cdot g(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j} \right) x^k.$$

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We know that $1/(1-x) = 1 + x + x^2 + \cdots$. Using the product expansion theorem, we then have

$$1/(1-x)^2 = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} 1 \right) x^k = \sum_{k=0}^{\infty} (k+1)x^k.$$

Useful facts about power series

Definition

Let u be a real number and k a nonnegative integer. Then the *extended binomial coefficient* $\binom{u}{k}$ is defined as:

$$\binom{u}{k} = \begin{cases} u(u-1)\cdots(u-k+1)/k! & \text{if } k > 0, \\ 1 & \text{if } k = 0. \end{cases}$$

Useful facts about power series

Example

When $u = -n$ is a negative integer, the extended binomial coefficient can be expressed in terms of an ordinary one:

$$\binom{-n}{r} = \frac{(-n)(-n-1)\cdots(-n-r+1)}{r!}$$

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Useful facts about power series

Theorem

Let x be a real number with $|x| < 1$ and let u be a real number. Then:

$$(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k.$$

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Example

Find the generating function for $(1+x)^{-n}$ and $(1-x)^{-n}$, where n is a positive integer.

Useful facts about power series

Solution

Using the extended binomial theorem, we have

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Replacing x by $-x$, we also find that

$$(1-x)^{-n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k.$$

Useful facts about power series

Example

Find the number of solutions of $e_1 + e_2 + e_3 = 17$, where e_1, e_2, and e_3 are nonnegative integers with $1 + i \leq e_i \leq 4 + i$.

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The number of solutions with the indicated constraints is the coefficient of x^{17} in the expansion of

$$(x^2 + x^3 + x^4 + x^5)(x^3 + x^4 + x^5 + x^6)(x^4 + x^5 + x^6 + x^7).$$

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Explicit expansion shows this coefficient to be 3. (Note that performing the expansion is about as much work as an explicit enumeration of the solutions.)

Useful facts about power series

Example

Use generating functions to determine the number of ways to insert tokes with \$1, \$2, and \$5 into a vending machine to pay for an item that costs r dollars in both the cases when the order in which the tokens are inserted does not matter and when the order does matter. (For example: there are two ways to pay for an item that costs \$3 when the order does not matter and three when it does.)

Useful facts about power series

Solution: Unordered

Because we can use any number of \$1, \$2, and \$5 tokens, the answer is the coefficient of x^r in the generating function

$$(1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)(1 + x^5 + x^{10} + \cdots).$$

For example, the number of ways to pay for an item costing \$7 is given by the coefficient of x^7, which is 6.

Useful facts about power series

Solution: Ordered

The number of ways to insert exactly n tokens to produce a total of r dollars is the coefficient of r in $(x + x^2 + x^5)^r$. Because any number of tokens can be inserted, we are really interested in the coefficient of x^r in the sum

$$1 + (x + x^2 + x^5) + (x + x^2 + x^5)^2 + \cdots = \frac{1}{1 - (x + x^2 + x^5)}.$$

The number of ways to pay for an item costing \$7 is 26.

Using generating functions to solve recurrence relations

Example

Solve the recurrence relation $a_k = 3a_{k-1}$ and initial condition $a_0 = 2$.

Using generating functions to solve recurrence relations

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Solve the recurrence relation $a_k = 3a_{k-1}$ and initial condition $a_0 = 2$.

Let $G(x)$ be the associated generating function. Note that $xG(x)$ is the generating function for a shift of this sequence, and hence the recurrence relation becomes

$$G(x) - 3xG(x) = \sum_{k=0}^{\infty} a_k x^k - 3 \sum_{k=1}^{\infty} a_{k-1} x^k$$

$$= a_0 + \sum_{k=1}^{\infty} (a_k - 3a_{k-1}) x^k = 2.$$

Thus, $G(x) - 3xG(x) = 2$.

Using generating functions to solve recurrence relations

Example

Suppose that a valid codeword is an n-digit number in decimal notation containing an even number of 0s, and let a_n denote the number of valid codewords of length n. We've previously shown this satisfies the recurrence $a_n = 8a_{n-1} + 10^{n-1}$ with initial condition $a_1 = 9$. Use generating functions to find an explicit formula for a_n.

Using generating functions to solve recurrence relations

Solution

To make life simpler, we set $a_0 = 1$, which is consistent with our recurrence. Multiplying the recurrence by x^n, we get $a_n x^n = 8a_{n-1}x^n + 10^{n-1}x^n$. Writing $G(x)$ for the generating function for a_n and summing this equation over n, we find

$$\begin{aligned} G(x) - 1 &= \sum_{n=1}^{\infty} (8a_{n-1}x^n + 10^{n-1}x^n) \\ &= 8x \sum_{n=1}^{\infty} a_{n-1}x^{n-1} + x \sum_{n=1}^{\infty} 10^{n-1}x^{n-1} \\ &= 8xG(x) + x/(1 - 10x). \end{aligned}$$

Using generating functions to solve recurrence relations

Solution, continued

$$G(x) - 1 = 8xG(x) + 1/(1 - 10x).$$

Solving for $G(x)$ gives

$$G(x) = \frac{1 - 9x}{(1 - 8x)(1 - 10x)} = \frac{1}{2}\left(\frac{1}{1 - 8x} + \frac{1}{1 - 10x}\right).$$

Using generating functions to solve recurrence relations

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Expanding this last expression using series, we find

$$G(x) = \frac{1}{2}\left(\sum_{n=0}^{\infty} 8^n x^n + \sum_{n=0}^{\infty} 10^n x^n\right) = \sum_{n=0}^{\infty} \frac{1}{2}(8^n + 10^n)x^n.$$

Consequently, $a_n = \frac{1}{2}(8^n + 10^n)$.

Proving identities via generating functions

Example

Use generating functions to show that $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.

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Use generating functions to show that $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.

First note that $\binom{2n}{n}$ is the coefficient of x^n in $(1+x)^{2n}$. However, we also have

$$(1+x)^{2n} = ((1+x)^n)^2 = \left(\sum_{j=0}^{n} \binom{n}{j} x^j\right)^2 .$$

The coefficient of x^n in this expression is

$$\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \cdots + \binom{n}{n}\binom{n}{0} = \sum_{k=0}^{n} \binom{n}{k}^2 .$$

8.5: Inclusion-Exclusion

The principle of inclusion-exclusion

Principle

The number of elements in the union of the two sets A and B is the sum of the numbers of elements in the sets minus the number of elements in their intersection. That is,

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

The principle of inclusion-exclusion

Theorem

Let $A_1, \ldots, A_n$ be finite sets, and let $A = A_1 \cup \cdots \cup A_n$. Then

$$|A| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \cdots + (-1)^{n+1} |A_1 \cap \cdots \cap A_n|.$$

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$$|A| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \cdots + (-1)^{n+1} |A_1 \cap \cdots \cap A_n|.$$

Proof

Suppose that a is a member of exactly r of the sets. Then a is counted $\binom{r}{1}$ times by the first sum, $\binom{r}{2}$ times by the second sum — and counted $\binom{r}{m}$ by the m^{th} sum. Thus, a is counted

$$\binom{r}{1} - \binom{r}{2} + \binom{r}{3} - \cdots + (-1)^r \binom{r}{r}$$

times in all. Considering the expansion of $(1-1)^r$, we see that the above expression is equal to 1. Thus, each member of A is counted exactly once by the right-hand sequence of sums, so it equals $|A|$.

The principle of inclusion-exclusion

Example

Give a formula for the number of elements in the union of four arbitrary sets: A_1, A_2, A_3, A_4.

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$$
\begin{aligned}
|A_1 \cup A_2 \cup A_3 \cup A_4| = {} & |A_1| + |A_2| + |A_3| + |A_4| \\
& - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4| \\
& - |A_2 \cap A_3| - |A_2 \cap A_4| - |A_3 \cap A_4| \\
& + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| \\
& + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4| \\
& - |A_1 \cap A_2 \cap A_3 \cap A_4|.
\end{aligned}
$$