Discrete Mathematics

Advanced Counting Techniques

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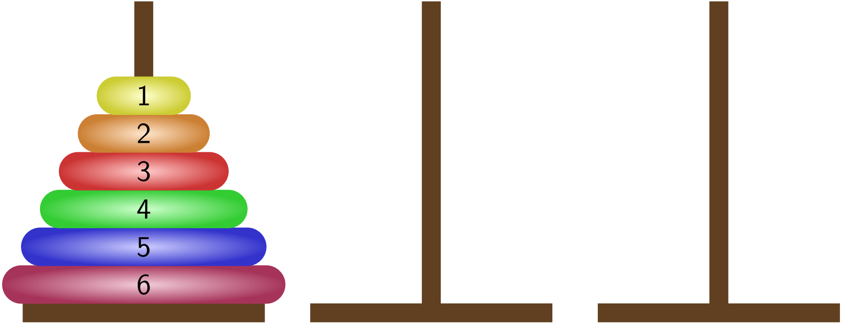
8.1: Applications of Recurrence Relations

The Tower of Hanoi

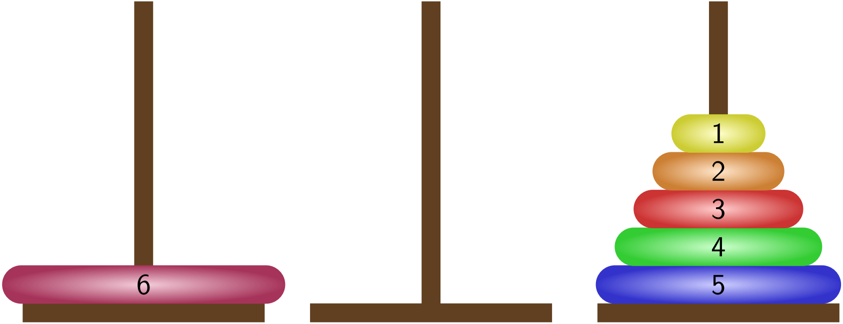
Example

A popular puzzle of the late ninetheen century invented by the French mathematician Édouard Lucas, called the Tower of Hanoi, consists of three pegs mounted on a board together with disks of different sizes. Initially these disks are placed ont he first peg in order of size, with the largest on the bottom. The rules of the puzzle allow disks to be moved on eat a time from one peg to another as long as a disk is never placed on top of a smaller disk. The goal of the puzzle is to have all the disks on the second peg in order of size, with the largest on the bottom.

Tower of Hanoi



Tower of Hanoi



The Tower of Hanoi

Example

Let H_n denote the number of moves needed to solve the Tower of Hanoi problems with n disks. Set up a recurrence relation for the sequence $\{H_n\}$.

Codeword enumeration

Example

A computer system considers a string of decimal digits a valid codeword if it contains an even number of 0 digits. For instance, 1230407869 is valid, whereas 120987045608 is not valid. Let a_n be the number of valid n-digit codewords. Find a recurrence relation for a_n.

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Note that $a_1 = 9$ because there are 10 one-digit strings, and only one, namely, the string 0, which is not valid. Then, a valid string of n digits can be obstained by appending a valid string of $n - 1$ digits with a digit other than 0. Also, a valid string of n digits can be obstained by appending a 0 to a string of length $n - 1$ that is not valid. Summing these two cases, we find:

$$
\begin{aligned}
a_n &= 9a_{n-1} + (10^{n-1} - a_{n-1}) \\
&= 8a_{n-1} + 10^{n-1}.
\end{aligned}
$$

Associations

Example

Find a recurrence relation for C_n, the number of ways to parenthesize the product of $n+1$ numbers $x_0, \ldots, x_n$ to specify the order of multiplication. For example, $C_3 = 5$:

$$((x_0 x_1)x_2)x_3, \quad (x_0(x_1 x_2))x_3, \quad (x_0 x_1)(x_2 x_3),$$
$$x_0((x_1 x_2)x_3), \quad x_0(x_1(x_2 x_3)).$$

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$$x_0 ((x_1 x_2) x_3), \qquad x_0 (x_1 (x_2 x_3)).$$

This recurrence relation can be solved using the method of generating functions, which we'll come to in a bit. It can be shown that $C_n = \frac{1}{n+1}\binom{2n}{n}$. The sequence $\{C_n\}$ is the sequence of *Catalan numbers*, named after Eugène Charles Catalan.

8.2: Solving Linear Recurrence Relations

Linear homogeneous recurrences

Definition

A *linear homogeneous recurrence relation of degree k with constant coefficients* is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k},$$

where $c_1, \ldots, c_k$ are real numbers and $c_k \neq 0$.

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where $c_1, \ldots, c_k$ are real numbers and $c_k \neq 0$.

Example

The recurrence relation $P_n = (1.11)P_{n-1}$ is a linear homogeneous recurrence relation of degree one. The recurrence relation $f_n = f_{n-1} + f_{n-2}$ is a linear homogeneous recurrence relation of degree two. The recurrence relation $a_n = a_{n-5}$ is a linear homogeneous recurrence relation of degree five.

Solving L.H.R.R.w.C.C.

The basic approach for solving linear homogeneous recurrence relations is to look for solutions of the form $a_n = r^n$, where r is a constant. Note that $a_n = r^n$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k}$ if and only if

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$$r^n = c_1 r^{n-1} + \cdots + c_k r^{n-k}.$$

When both sides of this equation are divided by r^{n-k} and the right-hand side is subtracted from the left, we obtain the equation

$$r^k - c_1 r^{k-1} - \cdots - c_{k-1} r - c_k = 0.$$

Solving L.H.R.R.w.C.C.

$$r^k - c_1 r^{k-1} - \cdots - c_{k-1} r - c_k = 0.$$

Consequently, the sequence $\{a_n\}$ with $a_n = r^n$ is a solution if and only if r is a solution of this last equation. We call this the *characteristic equation* of the recurrence relation. The solutions of this equation are called the *characteristic roots* of the recurrence relation. As we will see, these characteristic roots can be used to give an explicit formula for all the solutions of the currence relation.

Solving L.H.R.R.w.C.C.

Theorem

Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2. Then, the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots$ where α_1 and α_2 are constants.

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We must do two things to prove the theorem. First, it must be shown that if r_1 and r_2 are the roots of the characteristic equation, and α_1 and α_2 are constants, then the sequence $\{a_n\}$ with $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ is a solution of the recurrence relation. Second, it must be shown that if the sequence $\{a_n\}$ is a solution, then $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for some constants α_1 and α_2.

Solving L.H.R.R.w.C.C.

Proof, continued

First we will show that equations of this form are solutions to the recurrence. Because r_1 and r_2 satisfy $r^2 - c_1 r - c_2 = 0$, it follows that $r_1^2 = c_1 r_2 + c_2$ and $r_2^2 = c_1 r_2 + c_2$. From these equations, we see

$$c_1 a_{n-1} + c_2 a_{n-2} = c_1(\alpha_1 r_1^{n-1} + \alpha_2 r_2^{n-1}) + c_2(\alpha_1 r_1^{n-2} + \alpha_2 r_2^{n-2})$$

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$$\begin{aligned} c_1 a_{n-1} + c_2 a_{n-2} &= c_1(\alpha_1 r_1^{n-1} + \alpha_2 r_2^{n-1}) + c_2(\alpha_1 r_1^{n-2} + \alpha_2 r_2^{n-2}) \\ &= \alpha_1 r_1^{n-2}(c_1 r_1 + c_2) + \alpha_2 r_2^{n-2}(c_1 r_2 + c_2) \end{aligned}$$

Solving L.H.R.R.w.C.C.

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Proof, continued

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$$\begin{aligned} c_1 a_{n-1} + c_2 a_{n-2} &= c_1(\alpha_1 r_1^{n-1} + \alpha_2 r_2^{n-1}) + c_2(\alpha_1 r_1^{n-2} + \alpha_2 r_2^{n-2}) \\ &= \alpha_1 r_1^{n-2}(c_1 r_1 + c_2) + \alpha_2 r_2^{n-2}(c_1 r_2 + c_2) \\ &= \alpha_1 r_1^{n-2} r_1^2 + \alpha_2 r_2^{n-2} r_2^2 \\ &= \alpha_1 r_1^n + \alpha_2 r_2^n \\ &= a_n. \end{aligned}$$

Solving L.H.R.R.w.C.C.

Proof, continued

Now, suppose that $\{a_n\}$ is a solution of the recurrence and that the initial conditions $a_0 = C_0$ and $a_1 = C_1$ hold. We can solve for candidate values for α_1 and α_2 by substituting into the equation $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$:

$$a_0 = C_0 = \alpha_1 + \alpha_2,$$
$$a_1 = C_1 = \alpha_1 r_1 + \alpha_2 r_2.$$

Solving this linear system for α_1 and α_2 gives

$$\alpha_1 = \frac{C_1 - C_0 r_2}{r_1 - r_2},$$
$$\alpha_2 = \frac{C_0 r_1 - C_1}{r_1 - r_2}.$$

Solving L.H.R.R.w.C.C.

Proof, continued

$$\alpha_1 = \frac{C_1 - C_0 r_2}{r_1 - r_2},$$
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(Note that it is now critical that $r_1 \neq r_2$.)

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Proof, continued

$$\alpha_1 = \frac{C_1 - C_0 r_2}{r_1 - r_2},$$
$$\alpha_2 = \frac{C_0 r_1 - C_1}{r_1 - r_2}.$$

(Note that it is now critical that $r_1 \neq r_2$.) Hence, we have produced the formula $\alpha_1 r_1^n + \alpha_2 r_2^n$ which solves the recurrence and satisfies the two initial condiitons. By uniqueness of solutions to LHRRwCC with full initial conditions, it follows that this formula expresses the sequence $\{a_n\}$.

L.H.R.R.w.C.C.s

Example

Find an explicit formula for the Fibonacci numbers.

L.H.R.R.w.C.C.s

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Find an explicit formula for the Fibonacci numbers.

Recall that the Fibonacci numbers are characterized by $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$. The roots of the characteristic equation are $r_1 = (1+\sqrt{5})/2$ and $r_2 = (1-\sqrt{5})/2$. Hence,

$$f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

L.H.R.R.w.C.C.s

$$f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2} \right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2} \right)^n .$$

To solve for the constants, we substitute the known values $f_0 = 0$ and $f_1 = 1$ to get...

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$$\alpha_1 = 1/\sqrt{5}, \qquad \alpha_2 = -1/\sqrt{5}.$$

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Consequently:

$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n + \frac{-1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n .$$

L.H.R.R.w.C.C.s

Theorem

Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0. A sequence a_n is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$, where α_1 and α_2 are constants.

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Example

What is the solution of the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with initial conditions $a_0 = 1$ and $a_1 = 6$?

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Solution: The sole root of the characteristic equation is 3, hence a_n takes the form $a_n = \alpha_1 3^n + \alpha_2 n 3^n$.

L.H.R.R.w.C.C.s

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Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0. A sequence a_n is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$, where α_1 and α_2 are constants.

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Solution: The sole root of the characteristic equation is 3, hence a_n takes the form $a_n = \alpha_1 3^n + \alpha_2 n 3^n$. Using the initial conditions, one finds $\alpha_1 = \alpha_2 = 1$, and hence $a_n = 3^n + n3^n$.

L.H.R.R.w.C.C.s

Theorem

Let $c_1, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has k distinct roots $r_1, \ldots, r_k$. Then, a sequence $\{a_n\}$ is a solution of the recurrence

$$a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k}$$

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n,$$

where $\alpha_1, \ldots, \alpha_k$ are constants.

L.H.R.R.w.C.C.s

Theorem

Let $c_1, \ldots, c_k \in \mathbb{R}$, and suppose that the equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has t distinct roots $r_1, \ldots, r_t$ with multiplicities $m_1, \ldots, m_t \geq 1$ respectively. Then, $\{a_n\}$ is a solution of the recurrence

$$a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k}$$

if and only if it has the form

$$a_n = \sum_{i=1}^{t} \sum_{j=0}^{m_i - 1} \alpha_{i,j} n^j r_i^n,$$

where the $\alpha_{i,j}$ are constants.

L.N.R.R.w.C.C.s

Definition

The recurrence relation $a_n = 3a_{n-1} + 2n$ is an example of a *linear nonhomogeneous recurrence relation with constant coefficients*, that is, a recurrence relation of the form

$$a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k} + F(n),$$

where $c_1, \ldots, c_k$ are real numbers and $F(n)$ is a function not identically zero depending only on n.

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where $c_1, \ldots, c_k$ are real numbers and $F(n)$ is a function not identically zero depending only on n. The same recurrence with $F(n)$ omitted is called the *associated homogeneous recurrence relation*. It plays an important role in the solution of the nonhomogeneous recurrence relation.

L.N.R.R.w.C.C.s

Theorem

If $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous linear recurrence relation with constant coefficients

$$a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k} F(n),$$

then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$, where $\{a_n^{(h)}\}$ is a solution of the associated homogeneous recurrence relation.

L.N.R.R.w.C.C.s

Proof

Suppose that $\{b_n\}$ is a solution of the nonhomogeneous recurrence relation. Substituting both it and $\{a_n^{(p)}\}$ into the recurrence relation formula and subtracting, one finds

$$b_n - a_n^{(p)} = c_1(b_{n-1}a_{n-1}^{(p)}) + \cdots + c_k(b_{n-k} - a_{n-k}^{(p)}) + (F(n) - F(n)).$$

It follows that $\{b_n - a_n^{(p)}\}$ is a solution of the associated homogeneous linear recurrence, and hence $b_n = a_n^{(p)} + a_n^{(h)}$.

L.N.R.R.w.C.C.s

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Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1 = 3$?

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The associated homogeneous recurrence is $a_n = 3a_{n-1}$, which is solved by $a_n^{(h)} = \alpha 3^n$ for any constant α.

L.N.R.R.w.C.C.s

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Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1 = 3$?

The associated homogeneous recurrence is $a_n = 3a_{n-1}$, which is solved by $a_n^{(h)} = \alpha 3^n$ for any constant α.
Because $F(n) = 2n$ is a polynomial in n of degree 1, a reasonable trial solution is $p_n = cn + d$ for constants c and d. Suppose that there is a such a solution p_n; then,

$$\begin{aligned} a_n &= 3a_{n-1} + 2n \\ cn + d &= 3(c(n-1) + d) + 2n \\ 0n + 0 &= (2c+2)n + (2d - 3c). \end{aligned}$$

L.N.R.R.w.C.C.s

Solution, continued

It follows that $c = -1$ and $d = -3/2$, and hence all solutions take the form

$$a_n = a_n^{(p)} + a_n^{(h)} = -n - \frac{3}{2} + \alpha 3^n.$$

To satisfy the initial condition, we set $\alpha = 11/6$.

L.N.R.R.w.C.C.s

Example

Find all solutions of the recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2} + 7^n.$$

L.N.R.R.w.C.C.s

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The solutions of the associated homogeneous recurrence are

$$a_n^{(h)} = \alpha_1 3^n + \alpha_2 2^n.$$

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The solutions of the associated homogeneous recurrence are

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Because $F(n) = 7^n$, a reasonable trial solution is $a_n^{(p)} = C \cdot 7^n$. Making the substitution and solving for C gives $C = 49/20$, and hence $a_n^{(p)} = (49/20) \cdot 7^n$ is a particular solution.

L.N.R.R.w.C.C.s

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Because $F(n) = 7^n$, a reasonable trial solution is $a_n^{(p)} = C \cdot 7^n$. Making the substitution and solving for C gives $C = 49/20$, and hence $a_n^{(p)} = (49/20) \cdot 7^n$ is a particular solution. Hence, a complete parametrization of solutions is:

$$a_n = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n + (49/20) \cdot 7^n.$$

L.N.R.R.w.C.C.s

Theorem

Suppose that $\{a_n\}$ satisfies the linear nonhomogeneous recurrence relation

$$a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k} + F(n),$$

where $c_1, \ldots, c_k$ are real numbers and

$$F(n) = (b_t n^t + \cdots + b_1 n + b_0)s^n,$$

where $b_0, \ldots, b_t$ and s are real numbers. When s is not a root of the characteristic equation of the associated linear homogeneous recurrence, there is a particular solution of the form $(p_t n^t + \cdots + p_1 n + p_0)s^n$. When s is a root of this characteristic equation and its multiplicity is m, there is a particular solution of the form

$$n^m(p_t n^t + \cdots + p_1 n + p_0)s^n.$$

L.N.R.R.w.C.C.s

Example

What form does a particular solution of the linear nonhomogeneous recurrence

$$a_n = 6a_{n-1} - 9a_{n-2} + F(n)$$

have when $F(n) = 3^n$, $F(n) = n3^n$, $F(n) = n^2 2^n$, and $F(n) = (n^2 + 1)3^n$?

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$F(n)$	$a_n^{(p)}$
3^n	$p_0 n^2 3^n$
$n3^n$	$n^2(p_1 n + p_0)3^n$

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have when $F(n) = 3^n$, $F(n) = n3^n$, $F(n) = n^2 2^n$, and $F(n) = (n^2+1)3^n$?

$F(n)$	$a_n^{(p)}$
3^n	$p_0 n^2 3^n$
$n3^n$	$n^2(p_1 n + p_0)3^n$
$n^2 2^n$	$(p_2 n^2 + p_1 n + p_0)2^n$

L.N.R.R.w.C.C.s

Example

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$(n^2+1)3^n$	$n^2(p_2 n^2 + p_1 n + p_0)3^n$