Discrete Mathematics

Discrete Probability

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7.1: An Introduction to Discrete Probability

Finite probability

Definition

An *experiment* is a procedure that yields one of a given set os possible outcomes. The *sample space* of the experiment is the set of possible outcomes. An *event* is a subset of the sample space. Laplace's definition of the probability $p(E)$ of an event E in a sample space S with finitely many equally possible outcomes is

$$p(E) = \frac{|E|}{|S|}.$$

Finite probability

Example

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By the product rule, the number of hands containing a full house is the product of the number of ways to pick two kinds in order, the number of ways to pick three out of four for the first kind, and the number of ways to pick two out of the four for the second kind.
We see that the number of hands containing a full house is

$$P(13, 2) \cdot C(4, 3) \cdot C(4, 2) = 13 \cdot 12 \cdot 4 \cdot 6 = 3744.$$

Because there are $C(52, 5) = 2,598,960$ poker hands, the probability of a full house is

$$\frac{3744}{2598960} \approx 0.0014.$$

Finite probability

Example

What is the probability that the numbers 11, 4, 17, 39, and 23 are drawn in that order from a bin containing 50 balls labeled with the numbers 1, 2, . . . , 50 if the ball selected *is not* returned to the bin before the next ball is selected?

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By the product rule, there are $50 \cdot 49 \cdot 48 \cdot 47 \cdot 46 = 254251200$ ways to select the balls because each time a ball is drawn there is one fewer ball to choose from. Consequently, the probability that the specified balls are drawn in that order is $1/254251200$.

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By the product rule, there are $50^5 = 312500000$ ways to select the balls because each time a ball is drawn there is one fewer ball to choose from. Consequently, the probability that the specified balls are drawn in that order is $1/312500000$.

Probabilities of complements and unions of events

Theorem

Let E be an event in a sample space S. The probability of the event $\overline{E} = S - E$, the complementary event of E, is given by $p(\overline{E}) = 1 - p(E)$.

Probabilities of complements and unions of events

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Example

A sequence of 10 bits is randomly generated. What is the probability that at least one of these bits is 0?

Probabilities of complements and unions of events

Theorem

Let E_1 and E_2 be events in the sample space S. Then

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2).$$

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Example

What is the probability that a positive integer selected at random from the set of positive integers not exceeding 100 is divisible by either 2 or 5?

7.2: Probability Theory

Assigning probabilities

Definition

Let S be the sample space of an experiment with a finite or countable number of outcomes. We assign a probability $p(s)$ to each outcome s. We require that two conditions be met:

1. $0 \leq p(s) \leq 1$ for each $s \in S$.
2. $\sum_{s \in S} p(s) = 1$.

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The function p from the set of all outcomes of the sample space S is called a *probability distribution*.

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Example

Suppose that a die is biased (or loaded) so that 3 appears twice as often as each other number but that all other outcomes are equally likely. What is the probability that an odd number appears when we roll this die?

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Theorem (Redux)

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Probabilities of complements and unions of events

Theorem (Redux)

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2. $p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$.

Theorem

If $E_1, E_2, \ldots$ is a sequence of pairwise disjoint events in a sample space S, then

$$p\left(\bigcup_i E_i\right) = \sum_i p(E_i).$$

Conditional probability

Definition

Let E and F be events with $p(F) > 0$. The *conditional probability* of E given F, denoted by $p(E \mid F)$, is defined as

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Example

What is the conditional probability that a family with two children has two boys, given they have at least one boy? Assume that each of the possibilities BB, BG, GB, and GG is equally likely, where B represents a boy and G represents a girl. (Note that BG represents a family with an older boy and a younger girl, while GB represents a girl with an older girl and a younger boy.)

Independence

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The events E and F are *independent* if and only if $p(E \cap F) = p(E)p(F)$.

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Example

Are the events E, that a family with three children has children of both sexes, and F, that this family has at most one boy, independent? Assume that the eight ways a family can have three children are equally likely.

Independence

Definition

The events E_1, E_2, ..., E_n are *pairwise independent* if and only if

$$p(E_i \cap E_j) = p(E_i)p(E_j)$$

for all pairs of integers i and j with $1 \leq i < j \leq n$. These events are *mutually independent* if

$$p(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_m}) = p(E_{i_1}) \cdots p(E_{i_m})$$

whenever i_j, $j = 1, 2, \ldots, m$, are integers with $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ for $m \geq 2$.

Bernoulli trials and the binomial distribution

Definition

Suppose that an experiment can have only two possible outcomes. For instance, when a bit is generated at random, the possible outcomes are 0 and 1. When a coin is flipped, the possible outcomes are heads and tails. Each performance of an experiment with two possible outcomes is called a *Bernoulli trial*. In general, a possible outcome of a Bernoulli trial is called a *success* or a *failure*. If p is the probability of a success and q is the probability of a failure, it follows that $p + q = 1$.

Bernoulli trials and the binomial distribution

Theorem

The probability of exactly k successes in n independent Bernoulli trials, with probability of success p and probability of failure $q = 1 - p$, is

$$b(k; n, p) = C(n, k)p^k q^{n-k}.$$

Fixing n and p, the function $b(k) = b(k; n, p)$ is called a *binomial distribution*.

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Example

Suppose that the probability that a 0 bit is generated is 0.9, that the probability that a 1 bit is generated is 0.1, and that bits are generated independently. What is the probability that exactly eight 0 bits are generated when 10 bits are generated?

Random variables

Definition

A *random variable* is a function from the sample space of an experiment to the set of real numbers. That is, a random variable assigns a real number to each possible outcome.

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A *random variable* is a function from the sample space of an experiment to the set of real numbers. That is, a random variable assigns a real number to each possible outcome.
The *distribution* of a random variable X on a sample space S is the set of pairs $(r, p(X = r))$ for all $r \in X(S)$, where $p(X = r)$ is the probability that X takes the value r.

Random variables

Example

Each of the eight possible outcomes when a fair coin is flipped three times has probability $1/8$. So, the distribution of the random variable $X(t)$ in the previous example is determined by the probabilities

$$P(X=3)=1/8, \qquad P(X=2)=3/8,$$
$$P(X=1)=3/8, \qquad P(X=0)=1/8.$$

Consequently, the distribution of $X(t)$ in the previous example is the set of pairs

$$\{(3,1/8),(2,3/8),(1,3/8),(0,1/8)\}.$$

The birthday problem

Example

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Assumptions: We assume that the birthdays of the people in the room are independent. Furthermore, we assume that each birthday is equally likely and that there are 366 days in the year. (In reality, more people are born on some days of the year than others, such as days nine months after some holidays including New Year's Eve, and only leap years have 366 days.)