Discrete Mathematics

Counting

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6.1: The Basics of Counting

Basic principles

Product rule

Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and for each of these days of doing the first task there are n_2 ways to do the second task, then there are $n_1 \cdot n_2$ ways to do the procedure.

Basic principles

Example

How many functions are there from a set with m elements to a set with n elements?

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How many one-to-one functions are there from a set with m elements to one with n elements?

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Example

Use the product rule to show that the number of different subsets of a finite set S is $2^{|S|}$.

Basic principles

Sum rule

If a task can be done either in one of n_1 ways or in one of n_2 ways, and none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.

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Example

Each user on a computer system has a password, which is six to eight characters long, and each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

Basic principles

Subtraction rule

If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two collections of options.

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Example

How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

Basic principles

Division rule

There are n/d ways to do a task if it can be done using a procedure that can be carried out in n ways, and for every way w, exactly d of the n ways correspond to the way w.

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Example

How many different ways are there to seat four people around a circular table, where seatings are considered the same when each person has the same left neighbor and the same right neighbor?

Tree diagrams

Example

How many bit strings of length four do not have two consecutive 1s?

6.2: The Pigeonhole Principle

The Pigeonhole Principle

Theorem

If k is a positive integer and $k+1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

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Example

Show that for every integer n there is a multiple of n that has only 0s and 1s in its decimal expansion.

The Pigeonhole Principle

Theorem

If N objects are placed in k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

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Example

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Example (Ramsey Theory)

Every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length $n + 1$ which is either strictly increasing or strictly decreasing.

6.3: Permutations and Combinations

Permutations

Definition

A *permutations* of a set of distinct objects is an ordered arrangement of these objects. We also are interested in ordered arrangements of some of the elements of a set. An arrangement of r elements of a set is called an *r-permutation*. The number of r-permutations of a set with n elements is denoted by $P(n, r)$.

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Theorem

If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+2)(n-r+1)$$

r-permutations of a set with n distinct elements.

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Permutations

Example

How many permutations of the letters A, B, C, D, E, F, G, and H contain the string "ABC"?

Combinations

Definition

An *r-combination* of elements of a set is an unordered selection of r elements from the set. Thus, an r-combination is simply a subset of cardinality r of the parent set. The number of r-combinations of a set with n distinct elements is denoted by $C(n, r)$.

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Theorem

The number of r-combinations of a set with n elements, where n is a nonnegative integer and r is an integer with $0 \leq r \leq n$, equals

$$C(n, r) = \frac{n!}{r!(n-r)!}.$$

Combinations

Example

How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a standard deck of 52 cards?

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Theorem

Let n and r be nonnegative integers with $r \leq n$. Then $C(n, r) = C(n, n - r)$.

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Theorem

Let n and r be nonnegative integers with $r \leq n$. Then $C(n, r) = C(n, n-r)$.

Example

How many bit strings of length n contain exactly r 1s?