Discrete Mathematics

Basic Structures:
Sets, Functions, Sequences, and Sums

Prof. Steven Evans

2.3: Functions

Functions

Definition

Let A and B be nonempty sets. A *function* f from A to B is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ to denote that b is the element assigned to a, and we write $f : A \to B$ to signify that f is a function from A to B.

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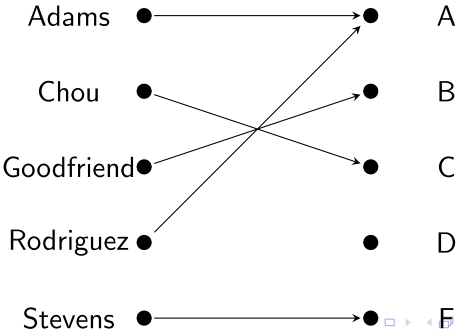


Functions

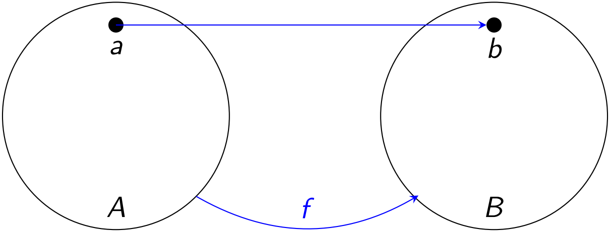
Remark

A function can be defined in terms of relations. Recall that a relation from A to B is a subset of $A \times B$. A relation that contains exactly one ordered pair (a, b) for every element $a \in A$ defines a function $f : A \to B$. The function is given by $f(a) = b$, where (a, b) is the unique pair in the relation with a as its first component.

Functions

Definition

If f is a function from A to B, we say that A is its *domain* and B its *codomain*. If $f(a) = b$, we say that b is the *image* of a and a is the *preimage* of b. The *range* or *image* of f is the set of all the images of all the elements of A. Also, we sometimes say f *maps* A to B.



Functions

Equality

When specifying a function, we specify its domain, its codomain, and how it maps elements of the domain to the codomain. Two functions are equal when

1. their domains are equal,
2. their codomains are equal,
3. **and** they map each element in their common domain to the same element in their common codomain.

Functions

Definition

Let f and g be functions from a set A to $\mathbb{R}$, the set of real numbers. Then there are functions $f+g$ and $f \cdot g$ defined by the formulas

$$(f+g)(a) = f(a) + g(a)$$
$$(f \cdot g)(a) = f(a) \cdot g(a).$$

Functions

Definition

Let f be a function from A to B and let S be a subset of A. The *image of* S *under* f is the subset of B of images of elements of S. We write

$$f(S) = \{t \in B \mid \exists s \in S(t = f(s))\}.$$

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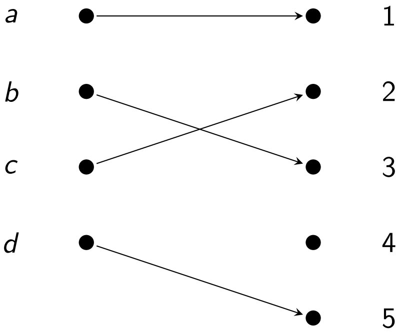
$$f(S) = \{t \in B \mid \exists s \in S(t = f(s))\}.$$

We also use the shorthand $\{f(s) \mid s \in S\}$ to refer to this set.

One-to-one and onto functions

Definition

A function f is said to be *one-to-one*, *injective*, or *an injection* when $f(a) = f(b)$ implies $a = b$ for all a and b in the domain of f.



One-to-one and onto functions

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Note that a function is one-to-one if and only if $f(a) \neq f(b)$ whenever $a \neq b$. This is the contrapositive of the definition given previously.

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Remark

We can also express this property using quantifiers as:

$$\forall a \forall b (f(a) = f(b) \rightarrow a = b),$$

where the universe of discourse is the domain of the function.

One-to-one and onto functions

Definition

Suppose that f is a function whose domain and codomain are subsets of the real numbers. Then f is called

- *increasing* if $f(x) \leq f(y)$
- *strictly increasing* if $f(x) < f(y)$
- *decreasing* if $f(x) \geq f(y)$
- *strictly decreasing* if $f(x) > f(y)$

whenever x and y are in the domain of f and $x < y$.

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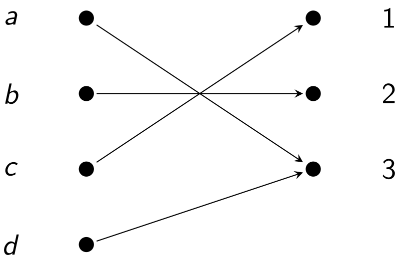
Remark

Note that if f is strictly increasing or strictly decreasing, then it is one-to-one.

One-to-one and onto functions

Definition

A function $f : A \to B$ is called *onto*, *surjective*, or *a surjection* when for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.

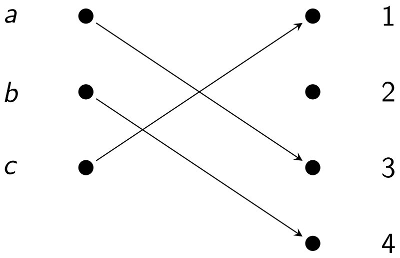


One-to-one and onto functions

Definition

We say that a function is *a one-to-one correspondence*, *bijective*, or *a bijection* if it is both one-to-one and onto.

One-to-one and onto functions



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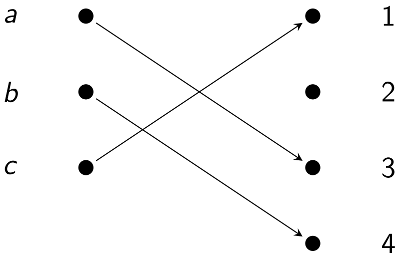
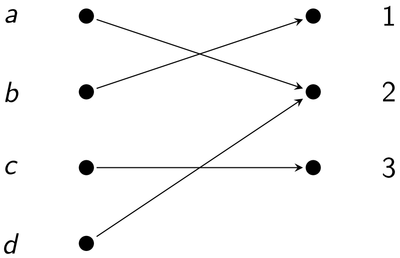


One-to-one, not onto.

One-to-one and onto functions



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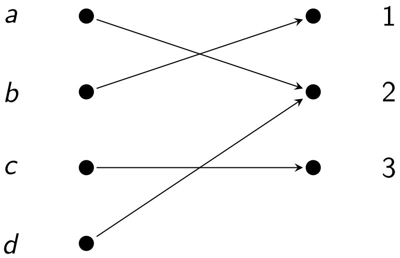
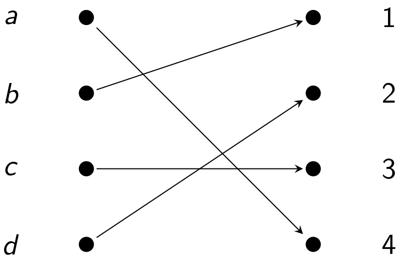


Onto, not one-to-one.

One-to-one and onto functions



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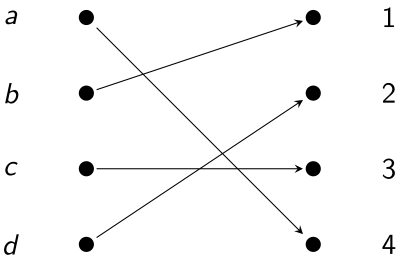

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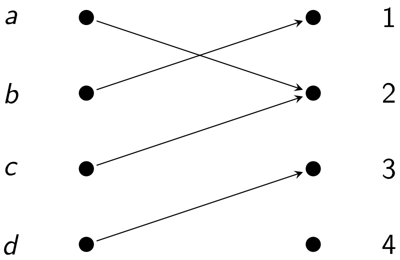

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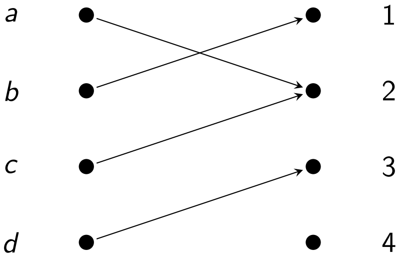
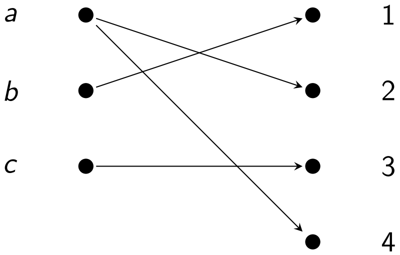


Neither one-to-one nor onto.

One-to-one and onto functions



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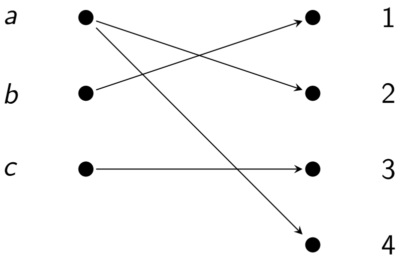
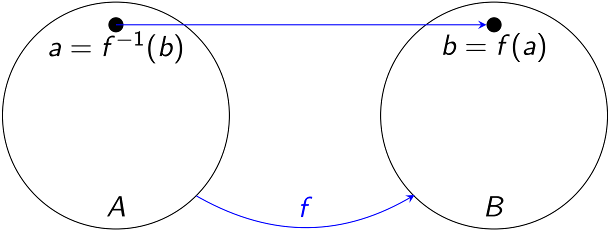


Not a function!

Inverse functions and function composition

Definition

Let f be a one-to-one correspondence from A to B. The *inverse function* of f is the function f^{-1} that assigns to $b \in B$ the unique element $a \in A$ with $f(a) = b$. Hence, when $f(a) = b$, then $f^{-1}(b) = a$.



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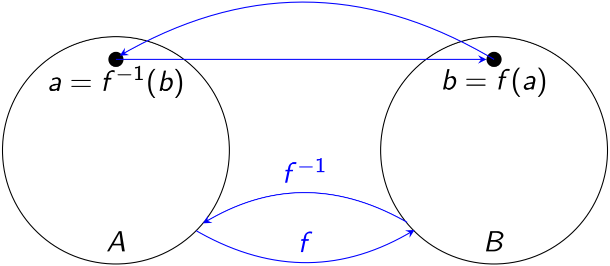
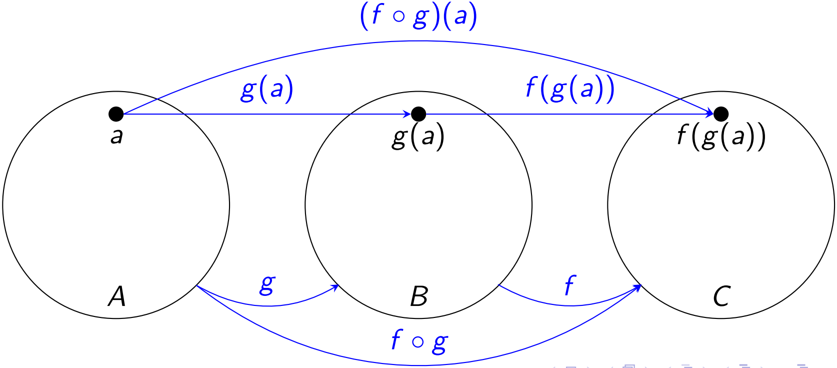


Inverse functions and function composition

Definition

Let $g : A \to B$ and $f : B \to C$ be two functions. Then their *composition* is a function $f \circ g : A \to C$, pronounced "f after g", is defined by the formula $(f \circ g)(a) = f(g(a))$.



Graphs of functions

Definition

Let f be a function $f : A \to B$; then the *graph* of f is the subset of $A \times B$ of ordered pairs

$$\{(a, b) \mid a \in A, b = f(a)\}.$$

Some important functions

Definition

The *floor function* assigns to a real number x the largest integer which is less than or equal to it, denoted $\lfloor x \rfloor$. The *ceiling function* assigns to a real number x the smallest integer which is greater than or equal to it, denoted $\lceil x \rceil$.

Graphs of the (a) Floor and (b) Ceiling Functions.

Useful properties of the floor and ceiling functions

Let x be a real number and n an integer.
Value inequalities:

- $\lfloor x \rfloor = n$ if and only if $n \leq x < n + 1$.
- $\lceil x \rceil = n$ if and only if $n - 1 < x \leq n$.
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Negative identities:

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Translation identities:

- $\lfloor x+n \rfloor = \lfloor x \rfloor + n$.
- $\lceil x+n \rceil = \lceil x \rceil + n$.

Some important functions

Definition

We'll use the notation $\log x$ to denote the base 2 logarithm of x. Generally, for $b > 1$ we write $\log_b x$ to denote the base b logarithm of x, and we reserve $\ln x$ for the natural logarithm of x.

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Definition

The factorial function $f : \mathbb{N} \to \mathbb{Z}^+$ is denoted by $f(n) = n!$ and defined by the pair of formulas

$$0! = 1, \qquad n! = n \cdot (n-1)!.$$

Some important functions

Remark

Stirling's formula tells us

$$n! \sim \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n,$$

where $f \sim g$ denotes the sentence

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1.$$

The symbol $\sim$ is read "is asymptotic to".

2.4: Sequences and Summations

Sequences

Definition

A sequence is a function from a subset of the set of integers (usually either $\{0, 1, 2, \ldots\}$ or $\{1, 2, 3, \ldots\}$) to a set S. We often write a_n, called a *term*, to denote the image of n. To denote the sequence as a whole, we often write $\{a_n\}$ or (a_n). (Note that $\{a_n\}$ unfortunately conflicts with the notation for sets introduced earlier.)

Sequences

Definition

A *geometric progression* is a sequence

$$a, ar, ar^2, ar^3, \ldots,$$

where a is the *initial term* and r is the *common ratio*.

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Definition

A *arithmetic progression* is a sequence

$$a, a+d, a+2d, a+3d, \ldots,$$

where a is the *initial term* and d is the *common difference*.

Recurrence relations

Definition

A *recurrence relation* for a sequence $\{a_n\}$ is an equation which expression a_n in terms of the previous terms $a_0, \ldots, a_{n-1}$, for $n \geq n_0$ for some fixed index n_0. A *solution* of the recurrence relation is a sequence satisfying the given equations.

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Example

The Fibonacci sequence $f_0, f_1, f_2, \ldots$ is determined by the equations

$$\begin{aligned} f_0 &= 0, \\ f_1 &= 1, \\ f_n &= f_{n-1} + f_{n-2} \end{aligned}$$

for $n = 2, 3, 4, \ldots$.

Recurrence relations: an example

Consider the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 2$.

- Does $a_n = 3n$ satisfy this relation?

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$$\begin{aligned} a_n &= 2a_{n-1} - a_{n-2} \\ &= 2(3(n-1)) - 3(n-2) \\ &= 6n - 6 - 3n + 6 \\ &= 3n. \qquad \text{OK!} \end{aligned}$$

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- Does $a_n = 2^n$ satisfy this relation?

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- Does $a_n = 2^n$ satisfy this relation?

$$\begin{aligned} a_0 &= 1, \\ a_1 &= 2, \\ a_2 &= 4, \\ a_2 &= 2a_1 - a_0 \\ &= 4 - 1 = 3. \qquad \text{(Not OK.)} \end{aligned}$$

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$$\begin{aligned} a_n &= 2a_{n-1} - a_{n-2} \\ &= 2 \cdot 5 - 5 \\ &= 5. \qquad \text{OK!} \end{aligned}$$

Summations

Definition

Pick a sequence $\{a_n\}$. We use the notations

$$\sum_{j=m}^{n} a_j, \qquad \textstyle\sum_{j=m}^{n} a_j, \qquad \text{or} \textstyle\sum_{m \le j \le n} a_j$$

(read as "the sum of a_j with j ranging from m to n") to denote

$$a_m + a_{m+1} + \cdots + a_{n-1} + a_n.$$

The variable j is called *the index of summation*.

Summations

Remark

The choice of the letter j is entirely arbitrary — we can replace it with any unbound symbol we please. For instance:

$$\sum_{j=m}^{n} a_j = \sum_{i=m}^{n} a_i = \sum_{k=m}^{n} a_k.$$

Summations

Theorem: Geometric summation

If a and r are real numbers with $r \neq 0$,

$$\sum_{j=0}^{n} ar^j = \begin{cases} \frac{ar^{n+1}-a}{r-1} & \text{if } r \neq 1, \\ (n+1)a & \text{if } r = 1. \end{cases}$$

Summations

Here are some other summation formulas:

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2.5: Cardinality of Sets

Cardinalities

Definition

The sets A and B have the same *cardinality* if there is a one-to-one correspondence $A \to B$. In this case, we write $|A| = |B|$.

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Definition

When there is an injection $A \to B$, we write $|A| \leq |B|$, so that the cardinality of B is at least that of A. If A and B also have distinct cardinalities, we write $|A| < |B|$.

Cardinalities

Definition

A set which is either finite or the same cardinality as the integers is called *countable*, and *uncountable* otherwise. When an infinite set is countable, we write its cardinality as $\aleph_0$.

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Show that the set of odd integers is countable.

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\frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \cdots & \\
\frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \cdots & \\
\frac{1}{5} & \vdots & \vdots & \vdots & & \\
\vdots & & & & &
\end{array}$$

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$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{3}$	$\frac{4}{3}$	$\cdots$	
$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$	$\cdots$	
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$\frac{1}{2}$	$\frac{2}{2}$	$\frac{3}{2}$	$\frac{4}{2}$	$\cdots$	
$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{3}$	$\frac{4}{3}$	$\cdots$	
$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$	$\cdots$	
$\frac{1}{5}$	$\vdots$	$\vdots$	$\vdots$		
$\vdots$					

Cardinalities

Example

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$$
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r_1 &= 0.d_{1,1}d_{1,2}d_{1,3}d_{1,4}\cdots\ , \\
r_2 &= 0.d_{2,1}d_{2,2}d_{2,3}d_{2,4}\cdots\ , \\
r_3 &= 0.d_{3,1}d_{3,2}d_{3,3}d_{3,4}\cdots\ , \\
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We will produce a number in $[0, 1)$ not in this list.

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We will produce a number in $[0, 1)$ not in this list. Define a number $x = 0.x_1x_2x_3x_4\cdots$ whose digits x_i are given by the rule

$$x_i = \begin{cases} 4 & \text{if } d_{i,i} = 5, \\ 5 & \text{otherwise.} \end{cases}$$

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The number x does not appear in the enumeration. If it did, it would be equal to r_i for some i. But the ith digit x_i is designed not to match the ith digit $d_{i,i}$ of r_i. Hence, it cannot be equal to r_i for any i. So, $[0, 1)$ cannot be enumerated. So, $\mathbb{R}$ cannot be enumerated.

The union of a pair of countable sets is countable

Now, we'll show that when A and B are countable sets, $A \cup B$ is also countable.

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2. We will proceed by cases. We need not consider both the case where A is infinite and B is finite as well as the case where A is finite and B is infinite. In the latter case, set $A' = B$ and $B' = A$; then $A' \cup B' = B \cup A = A \cup B$.

The union of a pair of countable sets is countable

Case 1: A finite, B finite

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If A and B are both finite, then $|A \cup B| = |A| + |B|$ is also finite. Every finite set is automatically countable.

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Case 2: A infinite, B finite

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Case 2: A infinite, B finite

We can list the elements of B as $b_1, b_2, \ldots, b_n$. We can also list the elements of A as $a_1, a_2, \ldots$.

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Case 2: A infinite, B finite

We can list the elements of B as $b_1, b_2, \ldots, b_n$. We can also list the elements of A as $a_1, a_2, \ldots$. The elements of $A \cup B$ can be listed by concatenating the two lists:

$$b_1, b_2, \ldots, b_n, a_1, a_2, \ldots.$$

The union of a pair of countable sets is countable

Case 2: A infinite, B finite

We can list the elements of B as $b_1, b_2, \ldots, b_n$. We can also list the elements of A as $a_1, a_2, \ldots$. The elements of $A \cup B$ can be listed by concatenating the two lists:

$$b_1, b_2, \ldots, b_n, a_1, a_2, \ldots.$$

Since the elements of $A \cup B$ can be listed, it is countable.

The union of a pair of countable sets is countable

Case 3: A infinite, B infinite

The union of a pair of countable sets is countable

Case 3: A infinite, B infinite

We can list the elements of A as $a_1, a_2, \ldots$ and the elements of B as $b_1, b_2, \ldots$.

The union of a pair of countable sets is countable

Case 3: A infinite, B infinite

We can list the elements of A as $a_1, a_2, \ldots$ and the elements of B as $b_1, b_2, \ldots$. The elements of $A \cup B$ can be listed by interleaving the two lists:

$$a_1, b_1, a_2, b_2, \ldots, a_n, b_n, \ldots.$$

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Case 3: A infinite, B infinite

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Cardinalities

Theorem: (Cantor–)Schröder–Bernstein

If A and B are two sets with $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$. (That is, if there are injections $f : A \to B$ and $g : B \to A$, then there is guaranteed to be a bijection $h : A \to B$.)

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Example

Show that $|(0, 1)|$ and $|(0, 1]|$ are equal.