On Beyond Hatcher!

Patterns in fairytales

Eric Peterson

November 29, 2012

Part 1: The geometry of Spec **FiniteSpectra**

Summary of Day 2

- $MU_*(-)$ takes values $\mathcal{MU}(-)$ in q.coh. sheaves on $\mathcal{M}_{\mathbf{fg}}$.
- $\mathcal{M}_{\mathbf{fg}}$ classifies group structures on the formal affine line $\hat{\mathbb{A}}^1 = \operatorname{Spf} R[\![c_1]\!]$.
- MU is some fancy ring spectrum as yet to be described.
- The MU-Adams spectral sequence transforms statements about $H^*\mathcal{M}_{\mathbf{fg}}$ into statements about $\pi_*\mathbb{S}$.

p-typicality

- From here on, pick a prime p and work p-locally.
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- Useful in formal group theory: the p-series and p-torsion

$$[p]_G(x) = \overbrace{x +_G \cdots +_G x}^{p \text{ times}}, \quad G[p] = \operatorname{Spf} R[\![c_1]\!]/\langle [p]_G(c_1) \rangle.$$

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- Over a field, we can find a change of coordinates γ for which

$$[p]_{\gamma \cdot G}(c_1) = \sum_{q=1}^{\infty} v_q x^{p^q}$$

for some coefficients v_q. This coordinate is *p-typical*.

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- The index of the first nonzero v_q is the *height* of G, an isomorphism invariant encoding the size of $G[p]$. There is a closed substack $\mathcal{M}_{\mathbf{fg}}^{\geq q} = V(p, v_1, \ldots, v_{q-1})$ of $\mathcal{M}_{\mathbf{fg}}$.

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- This list is complete: $\mathcal{M}_{\mathbf{fg}}$ has a unique closed substack of codimension q for each q, each contained in the next. (Viewed as a descending filtration, this gives the "chromatic spectral sequence" computing $H^*\mathcal{M}_{\mathbf{fg}}$.)

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- Coning off each v_i for $i < q$ in $E(q)$ yields the qth Morava K-theory $K(q)$, realizing the relative open $\mathcal{M}_{\mathbf{fg}}^{=q}$. Its ground ring is $K(q)_* = \mathbb{F}_p[v_q^{\pm}]$.

The spectrum of a monoidal category

Let's mimic the ideals $I_q = \langle p, v_1, \ldots, v_{q-1} \rangle$ of BP_* for p-local spectra. A full subcategory $\mathbf{C} \subseteq$ **FiniteSpectra** is...

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Define the geometric space of **FiniteSpectra** to be its collection of thick prime ideals.

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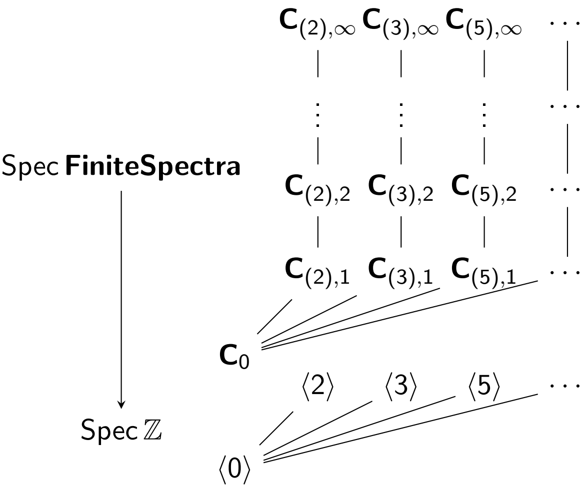
- The $K(q)$-acyclics give a thick prime ideal $\mathbf{C}_q$.
- There is a proper inclusion $\mathbf{C}_{q+1} \subsetneq \mathbf{C}_q$ and *this is all such thick prime ideals*.

This lets us draw a picture...

Spec **FiniteSpectra**



Part 2: Extrapolation

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- What other horizontal generalizations can we find?

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- Solution: study the problem locally and define a *sheaf of ring spectra* on $\mathcal{M}_{\mathbf{ell}}$. Its global sections is TMF, and it has the property $L_{K(2)}\,\mathrm{TMF} \simeq \bigvee_{x \in \mathcal{M}_{\mathbf{ell}}^{ss}} E_2$.

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- Question: What next?

The Steenrod algebra

- $H\mathbb{F}_2$ is complex oriented with formal group law $x +_{G_{H\mathbb{F}_2}} y = x + y$ over $\mathbb{F}_2$.
- So, think of $\mathrm{Spec}(H\mathbb{F}_2)_* H\mathbb{F}_2$ as "automorphisms of $G_{H\mathbb{F}_2}$," which are power series $\xi(t)$ with $\xi(s+t) = \xi(s) + \xi(t)$.

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$$\xi(\zeta(s)) = \sum_{i=0}^{\infty} \xi_i \left(\sum_{j=0}^{\infty} \zeta_j s^{2^j} \right)^{2^i} = \sum_{i=0}^{\infty} \xi_i \sum_{j=0}^{\infty} \zeta_j^{2^i} s^{2^{i+j}}$$
$$= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} \xi_i \zeta_{n-i}^{2^i} \right) s^{2^n}.$$

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- Question: What about $q \geq 3$?

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$$\mathbb{S} \longrightarrow \cdots \longrightarrow MString \to MSpin \longrightarrow MSO \longrightarrow MO$$

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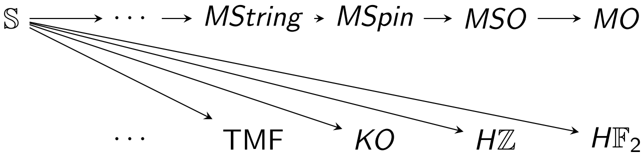

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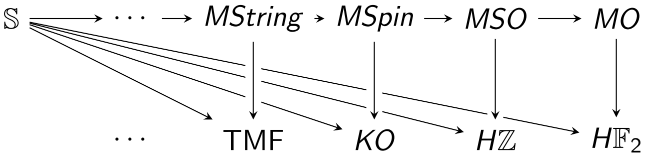

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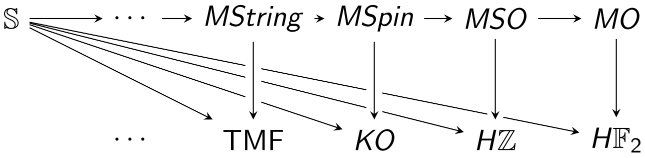

- Question: What comes after $MString$, or after TMF?

Homework

The final homework reading is about the “other half” of algebraic geometry in this picture. When E is a ring spectrum, we've used that E_* is a ring, but also E^*X is a ring for any space X. Taking E^*X to be the ring of functions on a scheme turns out to be profitable for many spaces X.

http://math.berkeley.edu/~ericp/