

# On Beyond Hatcher!

## Computations in unstable homotopy groups

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- There is a map

$$S^q \xrightarrow{E} \Omega S^{q+1}$$

- There is a map, which is the fiber of a 2-local fibration

$$S^q \xrightarrow{E} \Omega S^{q+1} \xrightarrow{H} \Omega S^{2q+1}.$$

$E$  is for *Einhängung*,  $H$  for the *Hopf invariant*.

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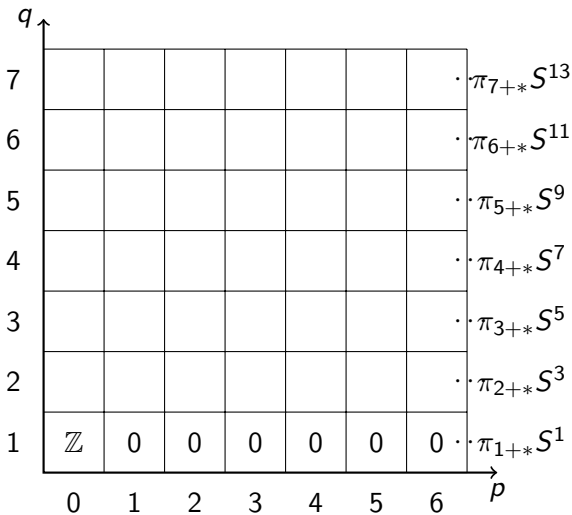
- Varying  $q$ , we get a spectral sequence

$$\begin{array}{ccccccccccc}
 \Omega S^1 & \rightarrow & \Omega^2 S^2 & \rightarrow & \Omega^3 S^3 & \rightarrow & \dots & \rightarrow & \Omega^q S^q & \rightarrow & \Omega^{q+1} S^{q+1} & \rightarrow & \dots \\
 \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & & & \downarrow & \nearrow & \downarrow & \nearrow & \\
 \Omega S^1 & & \Omega^2 S^3 & & \Omega^3 S^5 & & \dots & & \Omega^q S^{2q-1} & & \Omega^{q+1} S^{2q+1} & & \dots
 \end{array}$$

converging to  $\pi_* \Omega^\infty \Sigma^\infty S^0 = \pi_* \mathbb{S}$ .

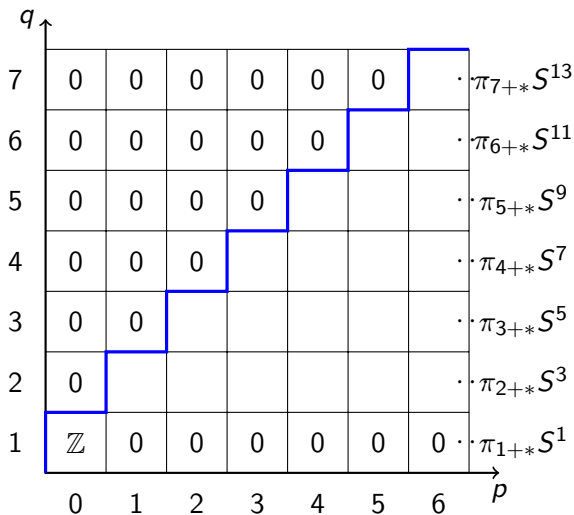
# Three basic facts

The goal is to fill this spectral sequence. First:  $\pi_* S^1$  is known from the very first day.



# Three basic facts

Next, cell decomposition shows  $\pi_p S^q = 0$  for  $p < q$ :



# Three basic facts

The Hurewicz isomorphism  $H_q S^q = \pi_q S^q$  fills the diagonal:

7	0	0	0	0	0	0	$\mathbb{Z} \cdot \pi_{7+*} S^{13}$
6	0	0	0	0	0	$\mathbb{Z}$	$\cdot \pi_{6+*} S^{11}$
5	0	0	0	0	$\mathbb{Z}$		$\cdot \pi_{5+*} S^9$
4	0	0	0	$\mathbb{Z}$			$\cdot \pi_{4+*} S^7$
3	0	0	$\mathbb{Z}$				$\cdot \pi_{3+*} S^5$
2	0	$\mathbb{Z}$					$\cdot \pi_{2+*} S^3$
1	$\mathbb{Z}$	0	0	0	0	0	$\cdot \pi_{1+*} S^1$
	0	1	2	3	4	5	6

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- One such source is the orthogonal groups.

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$$\begin{array}{ccccc} O(q) & \longrightarrow & O(q+1) & \longrightarrow & S^q & \text{fiber sequence} \\ \downarrow & & \downarrow & & \downarrow & \\ \Omega^q S^q & \longrightarrow & \Omega^{q+1} S^{q+1} & \longrightarrow & \Omega^{q+1} S^{2q+1} & \text{fiber sequence} \end{array}$$

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$$\begin{array}{ccccc} \mathbb{R}P^{q-1} & \longrightarrow & \mathbb{R}P^q & \longrightarrow & S^q & \text{cofiber sequence} \\ \downarrow & & \downarrow & & \downarrow & \\ O(q) & \longrightarrow & O(q+1) & \longrightarrow & S^q & \text{fiber sequence} \\ \downarrow & & \downarrow & & \downarrow & \\ \Omega^q S^q & \longrightarrow & \Omega^{q+1} S^{q+1} & \longrightarrow & \Omega^{q+1} S^{2q+1} & \text{fiber sequence} \end{array}$$

# The orthogonal spectral sequence

Now, we'll work to understand the  $d^1$ -differential in the orthogonal spectral sequence.

$$\begin{array}{ccc} \mathbb{R}P^{q-1} & \longrightarrow & \mathbb{R}P^q \\ \downarrow & & \downarrow \\ O(q) & \longrightarrow & O(q+1) \end{array} \begin{array}{c} \nearrow S^q \\ \parallel \\ \searrow S^q \end{array}$$

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$$\begin{array}{ccccc} S^{q-1} & & & & S^q \\ \downarrow & \searrow & & & \parallel \\ \Omega S^q & & \mathbb{R}P^{q-1} & \longrightarrow & \mathbb{R}P^q & \longrightarrow & S^q \\ & & \downarrow & & \downarrow & & \downarrow \\ & & O(q) & \longrightarrow & O(q+1) & & S^q \end{array}$$



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 & S^{q-1} & & S^{q-1} & & S^q \\
 & \downarrow & \searrow & \nearrow & \parallel & \nearrow \\
 \mathbb{R}P^{q-2} & \longrightarrow & \mathbb{R}P^{q-1} & \longrightarrow & \mathbb{R}P^q & \longrightarrow & S^q \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
 & \Omega S^q & & S^{q-1} & & S^q & \\
 & \downarrow & \searrow & \nearrow & \nearrow & \nearrow & \nearrow \\
 O(q-1) & \longrightarrow & O(q) & \longrightarrow & O(q+1) & & 
 \end{array}$$

# The orthogonal spectral sequence

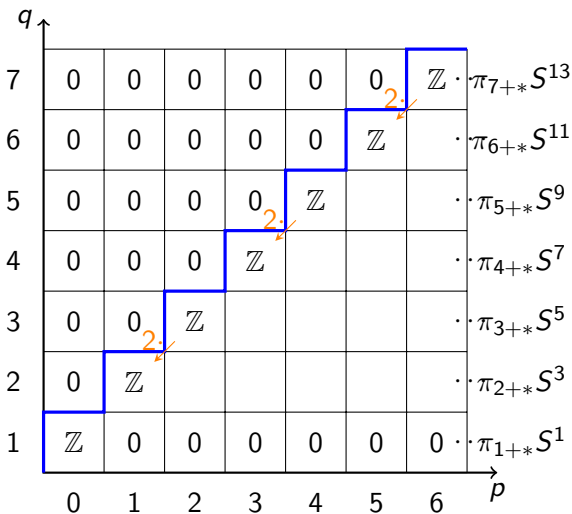
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$$\begin{array}{ccccc}
 S^{q-1} & \longrightarrow & S^{q-1} & & S^q \\
 \downarrow & \searrow & \nearrow & \parallel & \parallel \\
 \mathbb{R}P^{q-2} & \longrightarrow & \mathbb{R}P^{q-1} & \longrightarrow & \mathbb{R}P^q \\
 \downarrow & \searrow & \downarrow & \parallel & \parallel \\
 \Omega S^q & \longrightarrow & S^{q-1} & & S^q \\
 \downarrow & \searrow & \downarrow & \nearrow & \nearrow \\
 O(q-1) & \longrightarrow & O(q) & \longrightarrow & O(q+1)
 \end{array}$$

The back maps are  $d_{\text{cell}}^1$  and  $d_O^1$ .

# Differentials in the EHP spectral sequence

This gives the following differentials.



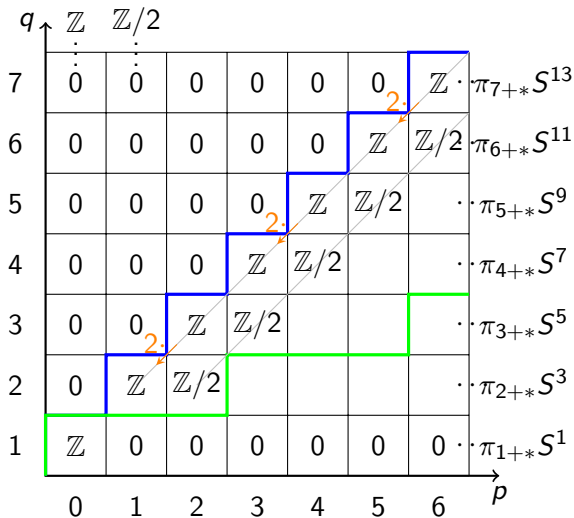
# Convergence and stabilization

The first two columns are mostly empty, so convergence is easy.

$q \uparrow$	$\mathbb{Z}$	$\mathbb{Z}/2$					
7	0	0	0	0	0	0	$\mathbb{Z} \cdot \pi_{7+*} S^{13}$
6	0	0	0	0	0	$\mathbb{Z}$	$\cdot \pi_{6+*} S^{11}$
5	0	0	0	0	$\mathbb{Z}$		$\cdot \pi_{5+*} S^9$
4	0	0	0	$\mathbb{Z}$			$\cdot \pi_{4+*} S^7$
3	0	0	$\mathbb{Z}$				$\cdot \pi_{3+*} S^5$
2	0	$\mathbb{Z}$					$\cdot \pi_{2+*} S^3$
1	$\mathbb{Z}$	0	0	0	0	0	$\cdot \pi_{1+*} S^1$
	0	1	2	3	4	5	6 $\rightarrow p$

# Convergence and stabilization

Freudenthal's theorem gives a whole stable range.



# Convergence and stabilization

There are no differentials entering the  $p = 2$  column.

$q \uparrow$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$				
7	0	0	0	0	0	0	$\mathbb{Z} \cdot \pi_{7+*} S^{13}$
6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2 \cdot \pi_{6+*} S^{11}$
5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2 \cdot \pi_{5+*} S^9$
4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\cdot \pi_{4+*} S^7$
3	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$		$\cdot \pi_{3+*} S^5$
2	0	$\mathbb{Z}$	$\mathbb{Z}/2$				$\cdot \pi_{2+*} S^3$
1	$\mathbb{Z}$	0	0	0	0	0	$\cdot \pi_{1+*} S^1$
	0	1	2	3	4	5	6

$p \rightarrow$

- The  $p = 3$  column is missing  $E_{3,2}^1 = \pi_5 S^3$ .

# Truncation

- The  $p = 3$  column is missing  $E_{3,2}^1 = \pi_5 S^3$ .
- We can compute  $\pi_* \Omega^3 S^3$  by truncating the filtration:

$$\begin{array}{ccccc} \Omega S^1 & \rightarrow & \Omega^2 S^2 & \rightarrow & \Omega^3 S^3 \\ \downarrow & \swarrow & \downarrow & \swarrow & \downarrow \\ \Omega S^1 & & \Omega^2 S^3 & & \Omega^3 S^5. \end{array}$$

This has the effect of blanking out the EHPSS above  $q = 3$ .



# Truncation

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This has the effect of blanking out the EHPSS above  $q = 3$ .

- Generally, truncating above  $q = Q$  computes  $\pi_{*+Q} S^Q$ .

# Truncation

So, we blank out above  $Q = 3...$

$q$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$					
7	0	0	0	0	0	0	$\mathbb{Z}$	$\pi_{7+*} S^{13}$
6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\pi_{6+*} S^{11}$
5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\pi_{5+*} S^9$
4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$		$\pi_{4+*} S^7$
3	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$			$\pi_{3+*} S^5$
2	0	$\mathbb{Z}$	$\mathbb{Z}/2$					$\pi_{2+*} S^3$
1	$\mathbb{Z}$	0	0	0	0	0	0	$\pi_{1+*} S^1$
	0	1	2	3	4	5	6	$p$

Diagram illustrating a grid of cells representing a spectral sequence. The vertical axis is labeled  $q$  and the horizontal axis is labeled  $p$ . The grid contains entries from  $\mathbb{Z}$  and  $\mathbb{Z}/2$ , with some cells being zero. A diagonal line is drawn from the bottom-left to the top-right, with orange arrows and the number 2 indicating a differential. Blue boxes highlight the diagonal entries, and green boxes highlight the entries below the diagonal. The right side of the grid is labeled with terms like  $\pi_{q+*} S^p$ .

# Truncation

... which has no effect at  $p = 2$ . So,  $\pi_5 S^3 = \mathbb{Z}/2$ .

$q \uparrow$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$					
7	0	0	0	0	0	0	0	...
6	0	0	0	0	0	0	0	...
5	0	0	0	0	0	0	0	...
4	0	0	0	0	0	0	0	...
3	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$			$\cdots \pi_{3+*} S^5$
2	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$				$\cdots \pi_{2+*} S^3$
1	$\mathbb{Z}$	0	0	0	0	0	0	$\cdots \pi_{1+*} S^1$
	0	1	2	3	4	5	6	$p \rightarrow$

# Stabilization as input

This completes the  $p = 3$  column. What does it converge to?

$q \uparrow$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	?			
7	0	0	0	0	0	0	$\mathbb{Z} \cdot \pi_{7+*} S^{13}$
6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2 \cdot \pi_{6+*} S^{11}$
5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2 \cdot \pi_{5+*} S^9$
4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\cdot \pi_{4+*} S^7$
3	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$		$\cdot \pi_{3+*} S^5$
2	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$			$\cdot \pi_{2+*} S^3$
1	$\mathbb{Z}$	0	0	0	0	0	$\cdot \pi_{1+*} S^1$
	0	1	2	3	4	5	6

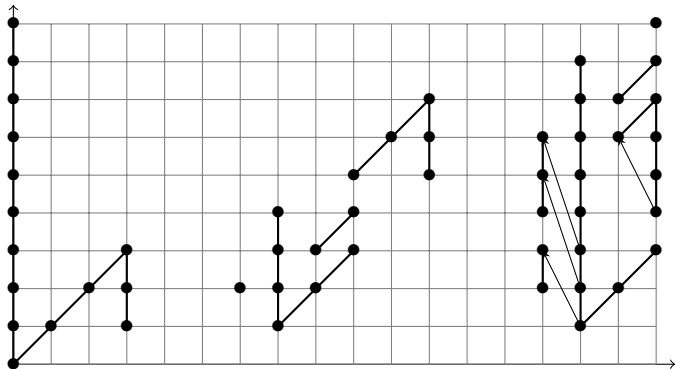
$p \rightarrow$

# Stabilization as input

We have already used the EHPSS to compute  $\pi_{0,1,2}\mathbb{S}$ .

# Stabilization as input

We have already used the EHPSS to compute  $\pi_{0,1,2}\mathbb{S}$ . In the other direction, knowing  $\pi_p\mathbb{S}$  can sort out the structure lost by the associated graded.



Adams SS:  $\pi_3\mathbb{S} = \mathbb{Z}/8$ .

# Stabilization as input

This lets us fill out a new ray, ...

$q \uparrow$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8$			
7	0	0	0	0	0	0	$\mathbb{Z} \cdot \pi_{7+*} S^{13}$
6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2 \cdot \pi_{6+*} S^{11}$
5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2 \cdot \pi_{5+*} S^9$
4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8 \cdot \pi_{4+*} S^7$
3	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8$	$\cdot \pi_{3+*} S^5$
2	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$			$\cdot \pi_{2+*} S^3$
1	$\mathbb{Z}$	0	0	0	0	0	$\cdot \pi_{1+*} S^1$
	0	1	2	3	4	5	6

$p \rightarrow$

# Stabilization as input

... describes a new part of the truncation, ...

$q$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/4$				
7	0	0	0	0	0	0	0	...
6	0	0	0	0	0	0	0	...
5	0	0	0	0	0	0	0	...
4	0	0	0	0	0	0	0	...
3	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8$		$\cdots \pi_{3+*} S^5$
2	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/4$			$\cdots \pi_{2+*} S^3$
1	$\mathbb{Z}$	0	0	0	0	0	0	$\cdots \pi_{1+*} S^1$
	0	1	2	3	4	5	6	$p$

The grid contains red '0's in the upper right and red  $\mathbb{Z}/4$ 's in the lower left. Blue boxes highlight a staircase pattern of cells from (3,3) to (7,7). Green boxes highlight a path from (1,1) to (3,3). An orange arrow labeled '2' points from (2,1) to (3,2).



# Stabilization as input

... and completes the  $p = 4$  column.

$q \uparrow$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8$			
7	0	0	0	0	0	0	$\mathbb{Z} \cdot \pi_{7+*} S^{13}$
6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2 \cdot \pi_{6+*} S^{11}$
5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2 \cdot \pi_{5+*} S^9$
4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8 \cdot \pi_{4+*} S^7$
3	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8$	$? \cdot \pi_{3+*} S^5$
2	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/4$	$?$	$? \cdot \pi_{2+*} S^3$
1	$\mathbb{Z}$	0	0	0	0	0	$0 \cdot \pi_{1+*} S^1$
	0	1	2	3	4	5	6

$p \rightarrow$

- This story continues indefinitely, and there is more unexplored structure in the EHPSS.

- This story continues indefinitely, and there is more unexplored structure in the EHPSS.
- What we have is enough to prove Serre finiteness:  
 $\pi_p S^q$  is finite, except for  $\pi_q S^q = \mathbb{Z}$  and  $\mathbb{Z} \subseteq \pi_{4q-1} S^{2q}$ .

# Serre finiteness: the exceptions

There's a  $\mathbb{Z}$  in the corner...

$q \uparrow$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8$			
7	0	0	0	0	0	0	$\mathbb{Z} \cdot \pi_{7+*} S^{13}$
6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2 \cdot \pi_{6+*} S^{11}$
5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2 \cdot \pi_{5+*} S^9$
4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8 \cdot \pi_{4+*} S^7$
3	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8$	$? \cdot \pi_{3+*} S^5$
2	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/4$	$?$	$? \cdot \pi_{2+*} S^3$
1	$\mathbb{Z}$	0	0	0	0	0	$0 \cdot \pi_{1+*} S^1$
	0	1	2	3	4	5	6 $\rightarrow p$

Diagram illustrating the Serre finiteness exceptions. The grid shows the structure of the homotopy groups  $\pi_{q+*} S^p$  for  $q \leq 7$  and  $p \leq 6$ . The entries are:

- $\pi_{7+*} S^{13}$ :  $\mathbb{Z}$  (at  $p=6$ )
- $\pi_{6+*} S^{11}$ :  $\mathbb{Z}$  (at  $p=5$ ),  $\mathbb{Z}/2$  (at  $p=6$ )
- $\pi_{5+*} S^9$ :  $\mathbb{Z}$  (at  $p=4$ ),  $\mathbb{Z}/2$  (at  $p=5$ ),  $\mathbb{Z}/2$  (at  $p=6$ )
- $\pi_{4+*} S^7$ :  $\mathbb{Z}$  (at  $p=3$ ),  $\mathbb{Z}/2$  (at  $p=4$ ),  $\mathbb{Z}/2$  (at  $p=5$ ),  $\mathbb{Z}/8$  (at  $p=6$ )
- $\pi_{3+*} S^5$ :  $\mathbb{Z}$  (at  $p=2$ ),  $\mathbb{Z}/2$  (at  $p=3$ ),  $\mathbb{Z}/2$  (at  $p=4$ ),  $\mathbb{Z}/8$  (at  $p=5$ ),  $?$  (at  $p=6$ )
- $\pi_{2+*} S^3$ :  $\mathbb{Z}$  (at  $p=1$ ),  $\mathbb{Z}/2$  (at  $p=2$ ),  $\mathbb{Z}/2$  (at  $p=3$ ),  $\mathbb{Z}/4$  (at  $p=4$ ),  $?$  (at  $p=5$ ),  $?$  (at  $p=6$ )
- $\pi_{1+*} S^1$ :  $\mathbb{Z}$  (at  $p=0$ )

Blue boxes highlight the diagonal elements  $\mathbb{Z}$  and  $\mathbb{Z}/2$ . Green boxes highlight the elements  $\mathbb{Z}$  and  $\mathbb{Z}/2$  in the  $q=2$  row. Orange arrows labeled "2" point to the  $\mathbb{Z}$  elements in the  $q=6$  and  $q=5$  rows.

# Serre finiteness: the exceptions

... and truncating at an even  $Q$  loses a  $d^1$ .

$q \uparrow$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z} \times \mathbb{Z}/4$				
7	0	0	0	0	0	0	0	$\pi_{7+*} S^{13}$
6	0	0	0	0	0	0	0	$\pi_{6+*} S^{11}$
5	0	0	0	0	0	0	0	$\pi_{5+*} S^9$
4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8$	$\pi_{4+*} S^7$
3	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8$	?	$\pi_{3+*} S^5$
2	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/4$	?	?	$\pi_{2+*} S^3$
1	$\mathbb{Z}$	0	0	0	0	0	0	$\pi_{1+*} S^1$
	0	1	2	3	4	5	6	$p \rightarrow$

Diagram illustrating the Serre finiteness exceptions. The grid shows the relationship between the degree  $q$  (vertical axis) and the dimension  $p$  (horizontal axis). The entries represent the homotopy groups of spheres, with some entries highlighted in blue, green, and red. A diagonal line is drawn through the grid, and a blue arrow points to the entry  $\mathbb{Z}$  at  $(p=1, q=2)$ .

# Serre finiteness: the exceptions

In the middle, there are extensions of finite groups.

$q \uparrow$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8$				
7	$\vdots$	$\vdots$	$\vdots$	$\vdots$	0	0	$\mathbb{Z} \cdot \pi_{7+*} S^{13}$	
6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2 \cdot \pi_{6+*} S^{11}$	
5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2 \cdot \pi_{5+*} S^9$	
4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8 \cdot \pi_{4+*} S^7$	
3	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8$	$? \cdot \pi_{3+*} S^5$	
2	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/4$	$?$	$? \cdot \pi_{2+*} S^3$	
1	$\mathbb{Z}$	0	0	0	0	0	$0 \cdot \pi_{1+*} S^1$	
		0	1	2	3	4	5	6
								$p \rightarrow$

The grid contains several highlighted regions:
 

- A blue path of boxes from (p=5, q=6) to (p=3, q=4).
- A green path of boxes from (p=2, q=3) to (p=1, q=2).
- Red text for the entries  $\mathbb{Z}/2$  and  $\mathbb{Z}/8$  in the rightmost column.
- Orange arrows labeled "2:" pointing from (p=5, q=6) to (p=4, q=5) and from (p=3, q=4) to (p=2, q=3).

The homework reading this week details a different computation: the homotopy of the sphere, lensed through the first Morava  $K$ -theory  $K(1)$ . This should give you a sense of what the  $K$ -theories are detecting, and is also emblematic a different sort of computation in homotopy theory.

<http://math.berkeley.edu/~ericp/>