On Beyond Hatcher!

Computations in unstable homotopy groups

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Set-up

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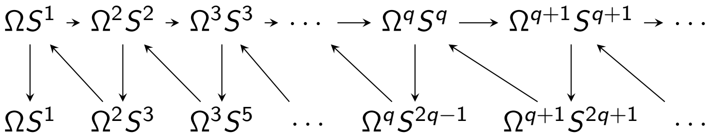
  E is for *Einhängung*, H for the *Hopf invariant*.
- Varying q, we get a spectral sequence



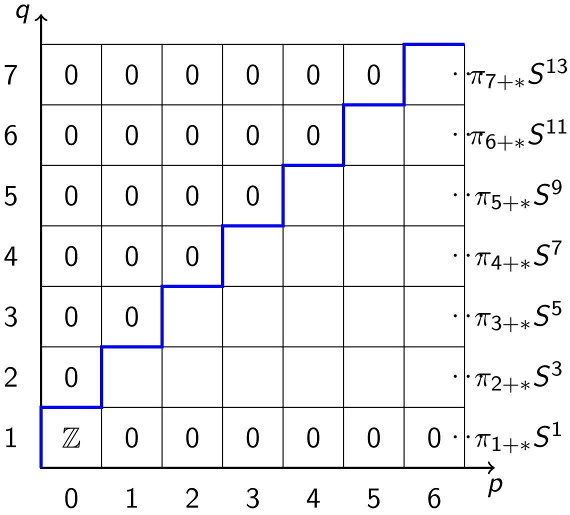
  converging to $\pi_*\Omega^\infty\Sigma^\infty S^0 = \pi_*\mathbb{S}$.

Three basic facts

The goal is to fill this spectral sequence. First: $\pi_* S^1$ is known from the very first day.

q \ p	0	1	2	3	4	5	6	
7								$\cdots \pi_{7+*} S^{13}$
6								$\cdots \pi_{6+*} S^{11}$
5								$\cdots \pi_{5+*} S^{9}$
4								$\cdots \pi_{4+*} S^{7}$
3								$\cdots \pi_{3+*} S^{5}$
2								$\cdots \pi_{2+*} S^{3}$
1	$\mathbb{Z}$	0	0	0	0	0	0	$\cdots \pi_{1+*} S^{1}$

Next, cell decomposition shows $\pi_p S^q = 0$ for $p < q$:



Three basic facts

The Hurewicz isomorphism $H_q S^q = \pi_q S^q$ fills the diagonal:

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5	0	0	0	0	$\mathbb{Z}$			$\pi_{5+*}S^{9}$
4	0	0	0	$\mathbb{Z}$				$\pi_{4+*}S^{7}$
3	0	0	$\mathbb{Z}$					$\pi_{3+*}S^{5}$
2	0	$\mathbb{Z}$						$\pi_{2+*}S^{3}$
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$$\begin{array}{ccccc} \mathbb{R}P^{q-1} & \longrightarrow & \mathbb{R}P^{q} & \longrightarrow & S^q \\ \downarrow & & \downarrow & & \downarrow \\ O(q) & \longrightarrow & O(q+1) & \longrightarrow & S^q \\ \downarrow & & \downarrow & & \downarrow \\ \Omega^q S^q & \longrightarrow & \Omega^{q+1} S^{q+1} & \to & \Omega^{q+1} S^{2q+1} \end{array}$$

cofiber sequence

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Now, we'll work to understand the d^1-differential in the orthogonal spectral sequence.

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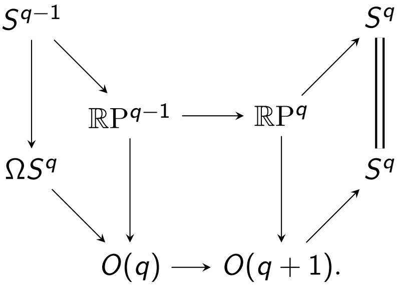

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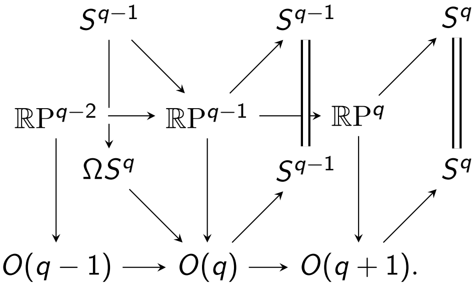

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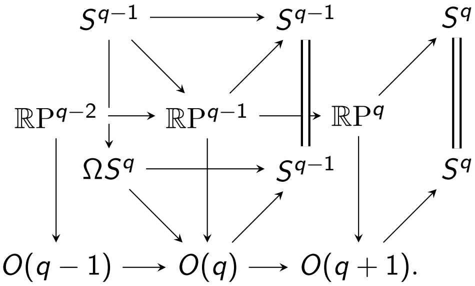
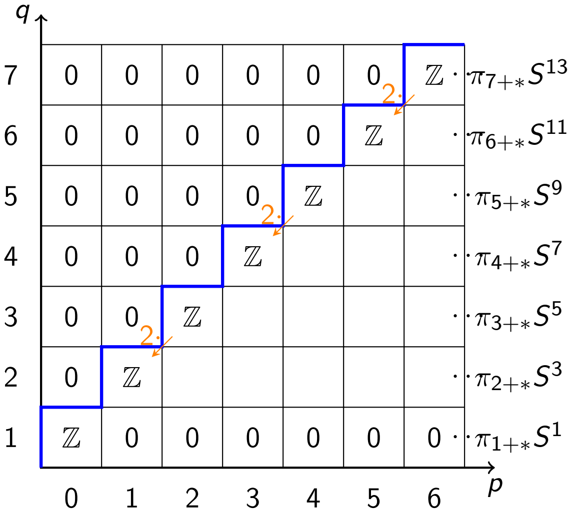


The back maps are d^1_{cell} and d^1_O.

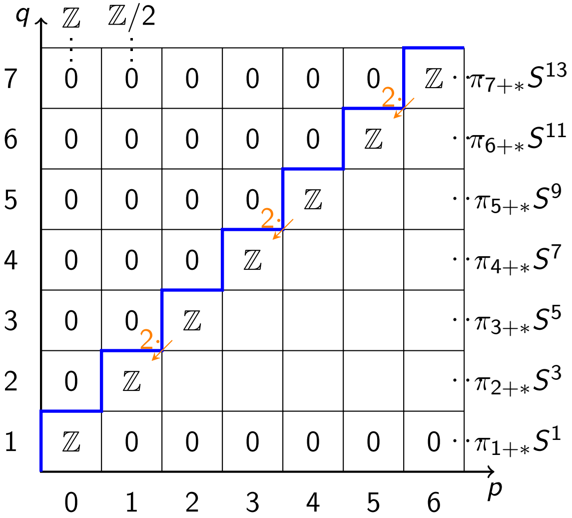
Differentials in the EHP spectral sequence

This gives the following differentials.



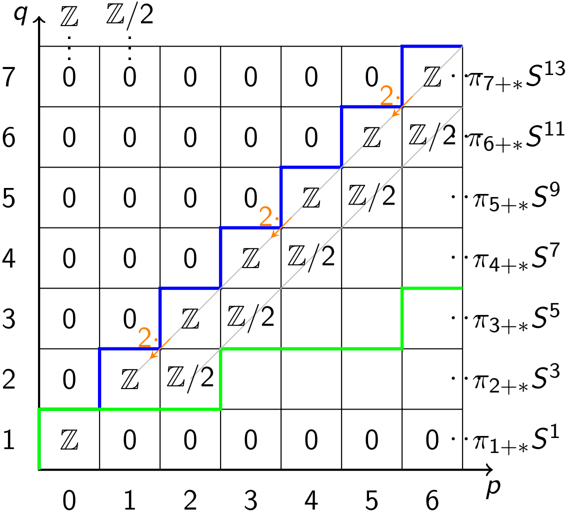
Convergence and stabilization

The first two columns are mostly empty, so convergence is easy.

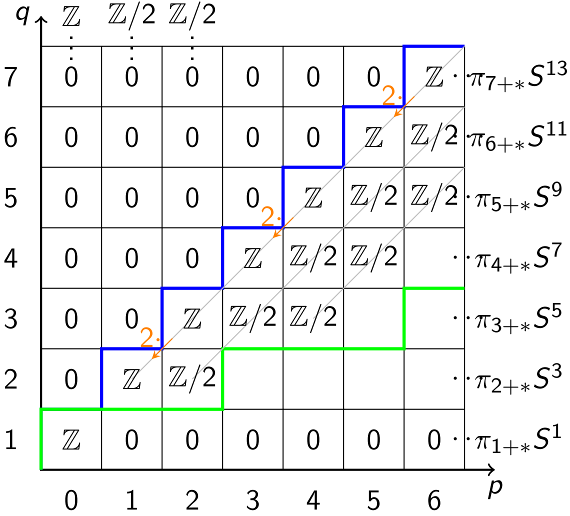


Convergence and stabilization

Freudenthal's theorem gives a whole stable range.



There are no differentials entering the $p = 2$ column.



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\Omega S^1 & \to & \Omega^2 S^2 & \to & \Omega^3 S^3 \\
\downarrow & \nwarrow & \downarrow & \nwarrow & \downarrow \\
\Omega S^1 & & \Omega^2 S^3 & & \Omega^3 S^5.
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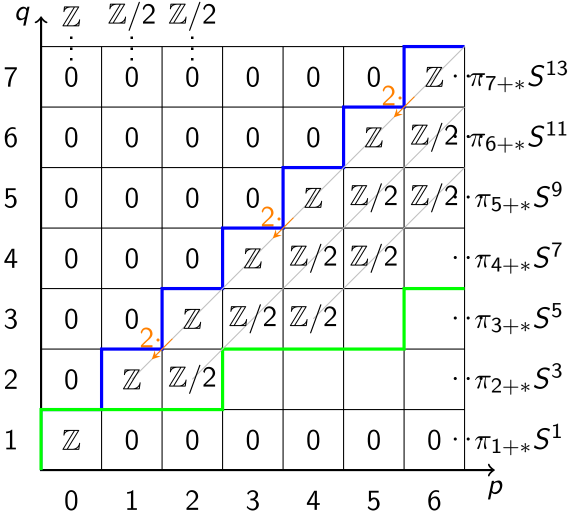
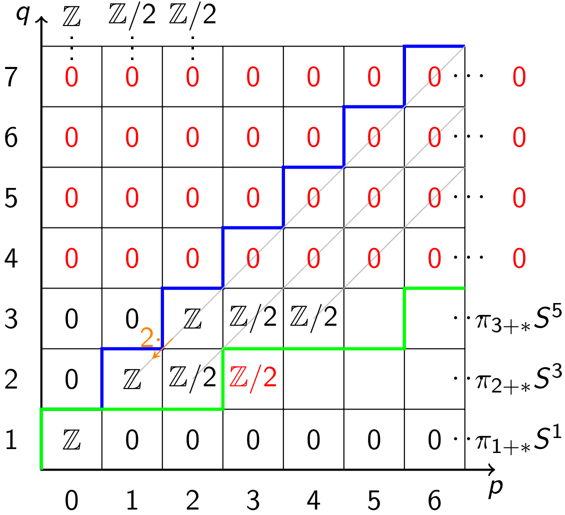
  This has the effect of blanking out the EHPSS above $q = 3$.
- Generally, truncating above $q = Q$ computes $\pi_{*+Q} S^Q$.

Truncation

So, we blank out above $Q = 3$...

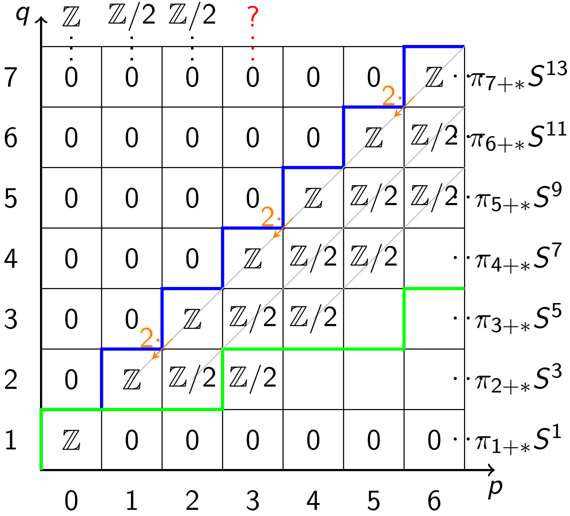


... which has no effect at $p = 2$. So, $\pi_5 S^3 = \mathbb{Z}/2$.



Stabilization as input

This completes the $p = 3$ column. What does it converge to?



Stabilization as input

We have already used the EHPSS to compute $\pi_{0,1,2}\mathbb{S}$.

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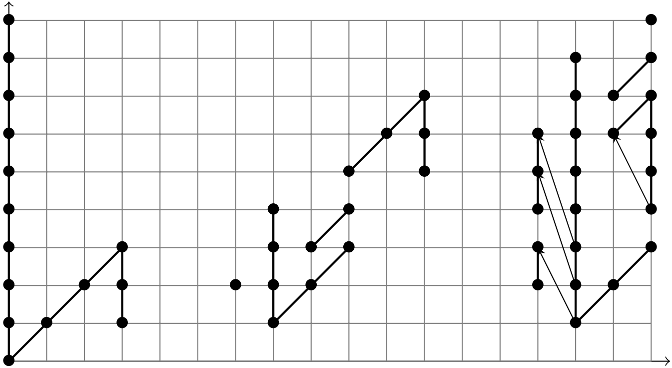
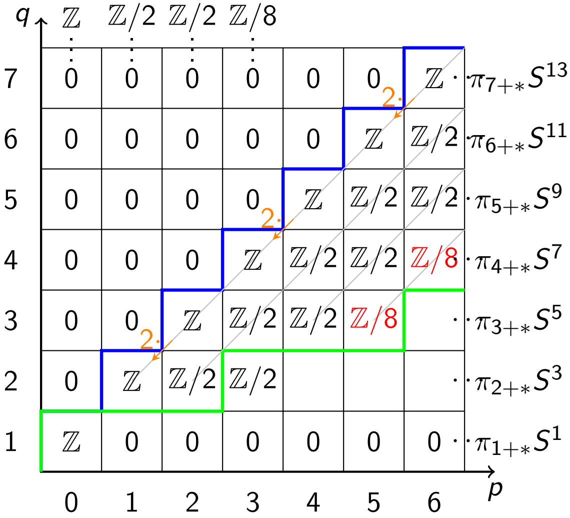


Adams SS: $\pi_3\mathbb{S} = \mathbb{Z}/8$.

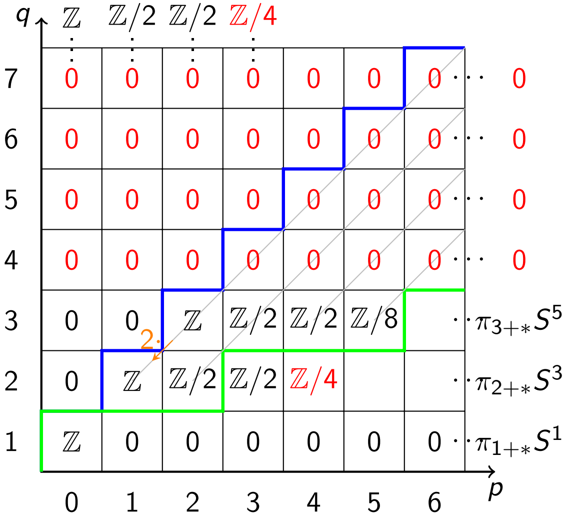
Stabilization as input

This lets us fill out a new ray, ...



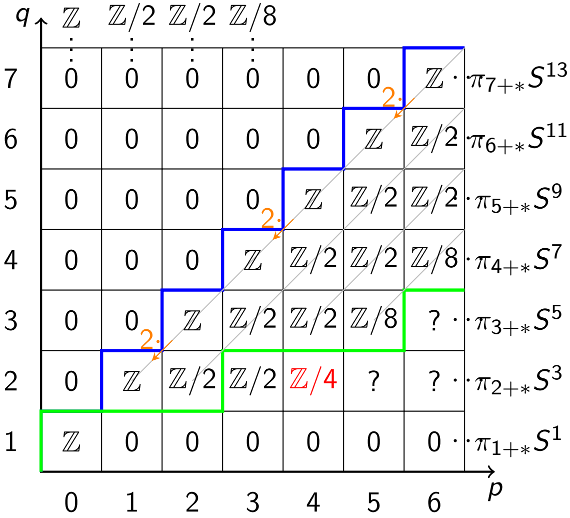
Stabilization as input

... describes a new part of the truncation, ...



Stabilization as input

… and completes the $p = 4$ column.



Serre finiteness

- This story continues indefinitely, and there is more unexplored structure in the EHPSS.

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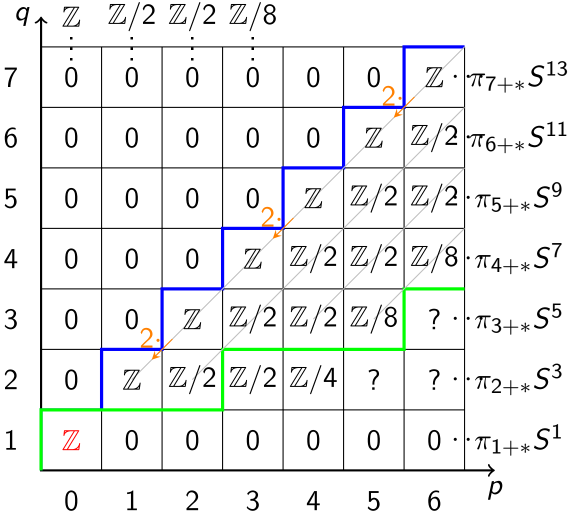
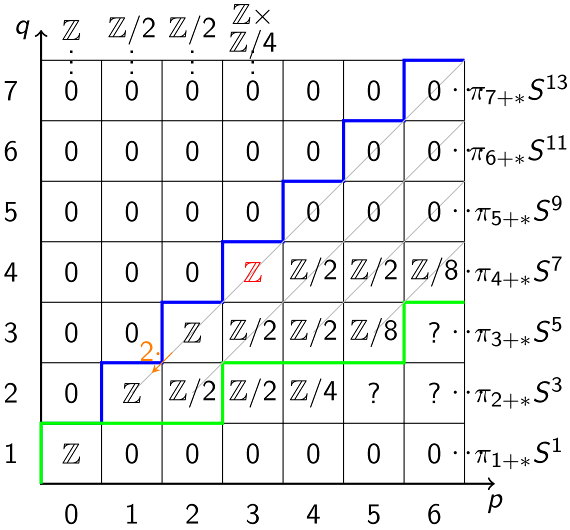
- This story continues indefinitely, and there is more unexplored structure in the EHPSS.
- What we have is enough to prove Serre finiteness:

$$\pi_p S^q \text{ is finite, except for } \pi_q S^q = \mathbb{Z} \text{ and } \mathbb{Z} \subseteq \pi_{4q-1} S^{2q}.$$

There's a $\mathbb{Z}$ in the corner...

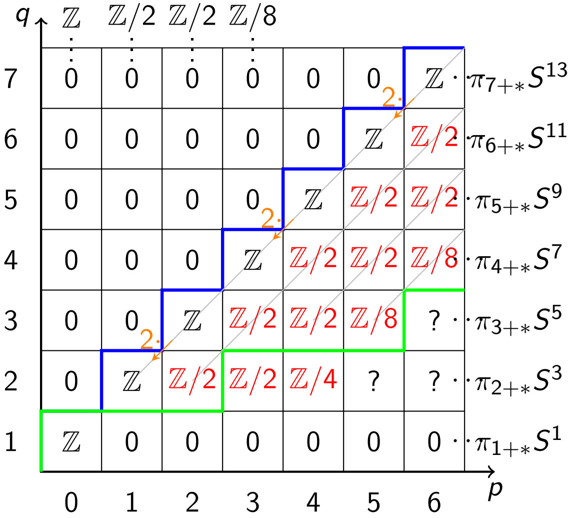


… and truncating at an even Q loses a d^1.



Serre finiteness: the exceptions

In the middle, there are extensions of finite groups.



Homework

The homework reading this week details a different computation: the homotopy of the sphere, lensed through the first Morava K-theory $K(1)$. This should give you a sense of what the K-theories are detecting, and is also emblematic a different sort of computation in homotopy theory.

http://math.berkeley.edu/~ericp/