On Beyond Hatcher!

Stable homotopy and formal Lie groups

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- These limiting groups for $q \gg 0$ appear as $\pi_{p-q}\mathbb{S}$. Also, stable homotopy π_* is a homology functor.
- General goal for today: study the relation between stable homotopy and more understandable homology theories.

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- When $Y = S^n \times X$, $TY = S^{n+1}X$. Otherwise, TY is a *twisted suspension*.

Generalized homology

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- The operation $- \otimes R$ is a base-change functor; think of $- \wedge E$ similarly. $E \wedge \mathbb{S}^n \wedge X$ is a base change of the shift $\mathbb{S}^n \wedge X$ to E-coefficients.
- Y is said to be E*-oriented* when there is an isomorphism

  $$E \wedge TY \simeq E \wedge \mathbb{S}^n \wedge X.$$

  E untwists Y, and this is a spectral Thom isomorphism.

Complex bordism, *MU*

- Sphere bundles are in ample supply: every complex vector bundle V on a manifold admits a metric, giving a sphere bundle SV and a disk bundle DV.
- Again thinking of E like a module, E is a *ring spectrum* when given a monoid structure

$$E \wedge E \xrightarrow{\mu} E, \; \mathbb{S} \xrightarrow{\eta} E.$$

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MU is defined to be the universal ring spectrum for which every such sphere bundle is oriented. Every other ring spectrum E which is oriented for complex vector bundles receives a homotopy-unique ring spectrum map $MU \to E$.[1]

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Part 1: Simplicial algebraic geometry

Homology theories are sheaf-valued

- When E is a ring spectrum, E_* is a ring and E_*X is an E_*-module:

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- E_*X an E_*-module $\leftrightarrow$ $\mathcal{E}(X)$ a quasicoherent sheaf on Spec E_*.
- Problem: $(H\mathbb{F}_2)_* = \mathbb{F}_2$ and $MU_* = \mathbb{Z}[x_1, x_2, \ldots]$. These are too small and too large respectively.

What other structure can we write down on homology?

Homology theories are sheaf-valued

- We used the multiplication $E \wedge E \xrightarrow{\mu} E$ to produce the E_*-action. What use is the unit $\mathbb{S} \xrightarrow{\eta} E$?
- We can build a map

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With some flatness, this gives

$$E_* X \to \pi_*(E \wedge E \wedge X) \xleftarrow{\cong} E_* E \otimes_{E_*} E_* X.$$

This is some sort of coaction map.

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Ring spectrum

$$\mathbb{S} \wedge E \xrightarrow{\eta \wedge 1} E \wedge E,$$

$$E \wedge \mathbb{S} \xrightarrow{1 \wedge \eta} E \wedge E,$$

$$E \wedge E \xrightarrow{\mu} E,$$

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$$\mathbb{S} \wedge E \xrightarrow{\eta \wedge 1} E \wedge E, \qquad A \xrightarrow{\eta_L} \Gamma,$$

$$E \wedge \mathbb{S} \xrightarrow{1 \wedge \eta} E \wedge E, \qquad A \xrightarrow{\eta_R} \Gamma,$$

$$E \wedge E \xrightarrow{\mu} E, \qquad \Gamma \xrightarrow{\varepsilon} A,$$

$$E \wedge \mathbb{S} \wedge E \xrightarrow{1 \wedge \eta \wedge 1} E \wedge E \wedge E, \qquad \Gamma \xrightarrow{\Delta} \Gamma \otimes_A \Gamma,$$

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Homology theories are sheaf-valued

Set $A = E_*$, $\Gamma = E_* E$, $X_0 = \operatorname{Spec} A$, and $X_1 = \operatorname{Spec} \Gamma$.

Ring spectrum

$$\begin{array}{lll}
\mathbb{S} \wedge E \xrightarrow{\eta \wedge 1} E \wedge E, & A \xrightarrow{\eta_L} \Gamma, & X_0 \xleftarrow{\text{dom}} X_1, \\
E \wedge \mathbb{S} \xrightarrow{1 \wedge \eta} E \wedge E, & A \xrightarrow{\eta_R} \Gamma, & X_0 \xleftarrow{\text{cod}} X_1, \\
E \wedge E \xrightarrow{\mu} E, & \Gamma \xrightarrow{\varepsilon} A, & X_1 \xleftarrow{\text{id}} X_0 \\
E \wedge \mathbb{S} \wedge E \xrightarrow{1 \wedge \eta \wedge 1} E \wedge E \wedge E, & \Gamma \xrightarrow{\Delta} \Gamma \otimes_A \Gamma, & X_1 \xleftarrow{\circ} X_1 \times_{X_0} X_1, \\
E \wedge E \xrightarrow{\tau} E \wedge E, & \Gamma \xrightarrow{\chi} \Gamma, & X_1 \xleftarrow{(-)^{-1}} X_1.
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The Γ-coaction on E_*X makes $\mathcal{E}(X)$ into an X_1-equivariant sheaf.

Back to bordism

E is complex oriented iff there's an isomorphism $E^*\mathbb{C}\mathrm{P}^\infty \cong E^*[\![c_1]\!]$.
Algebraic geometry sees Spf $E^*[\![c_1]\!] = \hat{\mathbb{A}}^1$ as a *formal affine line*.

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$$\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty \xrightarrow{\otimes} \mathbb{C}\mathrm{P}^\infty$$
$$E^*\mathbb{C}\mathrm{P}^\infty \otimes_{E_*} E^*\mathbb{C}\mathrm{P}^\infty \leftarrow E^*\mathbb{C}\mathrm{P}^\infty$$
$$G_E \times G_E \to G_E.$$

This is called a formal Lie group. To see why, Taylor expand a Lie group's multiplication at the origin.

Back to bordism

- There is a moduli of formal Lie groups: $\mathcal{M}_{\mathbf{fg}}$.
- Write $\mathcal{M}_E$ for the groupoid scheme (X_0, X_1) from earlier. When E is complex oriented, there is a map $\mathcal{M}_E \to \mathcal{M}_{\mathbf{fg}}$.
- Important theorem: $\mathcal{M}_{MU} \to \mathcal{M}_{\mathbf{fg}}$ is an isomorphism.

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- Important theorem: $\mathcal{M}_{MU} \to \mathcal{M}_{\mathbf{fg}}$ is an isomorphism.
- Summary: We have a functor $\mathcal{MU}(-)$ from **Spectra** to quasicoherent sheaves on the moduli of formal groups.

How can we use $\mathcal{M}_{\mathbf{fg}}$ to learn about spectra?

Adams spectral sequence

- "A_∞-space" from last time generalizes to "A_∞ ring spectrum". $H\mathbb{F}_2$ and MU are examples of A_∞ ring spectra.
- A space Y gives cosimplicial spectra X_* and M_* with

$$X_n = \overbrace{E \wedge \cdots \wedge E}^{n+1 \text{ copies}}, \qquad M_n = X_n \wedge Y.$$

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- Applying π_* to the skeletal filtration gives the E-Adams spectral sequence

$$H^*(\mathcal{M}_E; \mathcal{E}(Y)) \cong \operatorname{Ext}^{*,*}_{\Gamma\text{-comods}}(A, E_* Y) \Rightarrow \pi_* Y^\wedge_E.$$

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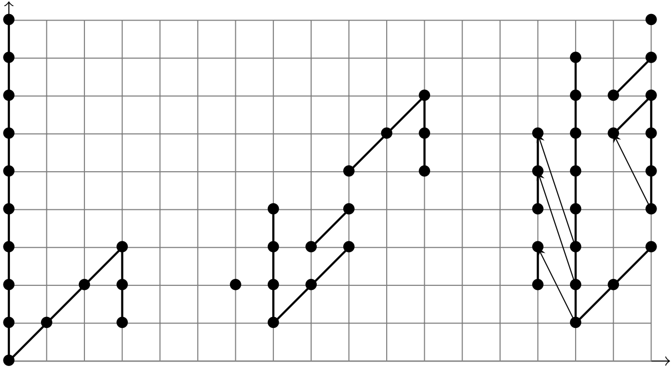
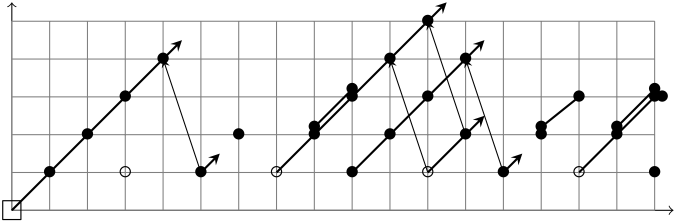
- There is an equivalence $\mathbb{S}^\wedge_{MU} \simeq \mathbb{S}$, so the MU-ASS has signature $H^*\mathcal{M}_{\mathbf{fg}} \Rightarrow \pi_*\mathbb{S}$.

The $H\mathbb{F}_2$-Adams E_2-term



The *MU*-Adams E_2 term localized at 2



Part 2: The geometry of Spec **FiniteSpectra**

Working up to weak equivalence

- The MU-Adams spectral sequence transforms statements about $H^*\mathcal{M}_{\mathbf{fg}}$ into statements about $\pi_*\mathbb{S}$.
- The cohomology $H^*\mathcal{M}_{\mathbf{fg}}$ is only sensitive to the homotopy type of the groupoids $(X_0, X_1)(R)$, not on the sets $X_0(R)$ and $X_1(R)$ themselves.
- We can improve our computation by finding a smaller A and Γ with the same homotopy type of (X_0, X_1).
- From here on, fix a prime p and work p-locally.

p-typicality

- Useful in classical group theory: the p-torsion $G[p]$.
- Useful in formal group theory: the p-series and p-torsion

$$[p]_G(x) = \overbrace{x +_G \cdots +_G x}^{p \text{ times}}, \quad G[p] = \operatorname{Spf} R[\![c_1]\!]/\langle [p]_G(c_1) \rangle.$$

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- Over a field, we can find a coordinate γ for which

$$[p]_{\gamma \cdot G}(c_1) = px +_G \sum_{q=1}^{\infty}{}_G\, v_q x^{p^q}$$

for some coefficients v_q. This coordinate is *p-typical*.

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- Every p-typical curve arises as the p-series of some formal group. So, $A = \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$ gives a smaller presentation of $\mathcal{M}_{\mathbf{fg}}$.
- The index of the first nonzero v_q is the *height* of G, an isomorphism invariant encoding the size of $G[p]$. There is a closed substack $\mathcal{M}_{\mathbf{fg}}^{\geq q} = V(p, v_1, \ldots, v_{q-1})$ of $\mathcal{M}_{\mathbf{fg}}$.

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- This list is complete: $\mathcal{M}_{\mathbf{fg}}$ has a unique closed substack of codimension q for each q, each contained in the next.
- In the homework reading, you'll use this filtration to organize the computation of $H^* \mathcal{M}_{\mathbf{fg}}$.

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- Johnson-Wilson theory: $v_{q-1}^{-1} BP_*/\langle v_q, v_{q+1}, \ldots \rangle$ determines an open substack $\mathcal{M}_{\mathbf{fg}}^{<q}$ complementary to $\mathcal{M}_{\mathbf{fg}}^{\geq q}$. This gives a homology theory $E(q-1)$.

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- Coning off each v_i for $i < q$ in $E(q)$ yields the qth Morava K-theory $K(q)$, realizing the relative open $\mathcal{M}_{\mathbf{fg}}^{=q}$. Its ground ring is $K(q)_* = \mathbb{F}_p[v_q^{\pm}]$.

Spectral realizations and the Steenrod algebra

- $H\mathbb{F}_2$ is complex oriented with formal group law $x +_{G_{H\mathbb{F}_2}} y = x + y$ over $\mathbb{F}_2$.
- So, think of $\mathrm{Spec}(H\mathbb{F}_2)_* H\mathbb{F}_2$ as "automorphisms of $G_{H\mathbb{F}_2}$," which are power series $\xi(t)$ with $\xi(s+t) = \xi(s) + \xi(t)$.

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- These must be of the form $\xi(s) = \sum_{n=0}^{\infty} \xi_n s^{2^n}$, which compose like

$$\xi(\zeta(s)) = \sum_{i=0}^{\infty} \xi_i \left(\sum_{j=0}^{\infty} \zeta_j s^{2^j} \right)^{2^i} = \sum_{i=0}^{\infty} \xi_i \sum_{j=0}^{\infty} \zeta_j^{2^i} s^{2^{i+j}}$$
$$= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} \xi_i \zeta_{n-i}^{2^i} \right) s^{2^n}.$$

The spectrum of a monoidal category

Let's mimic the ideals $I_q = \langle p, v_1, \ldots, v_{q-1} \rangle$ of BP_* for p-local spectra. A full subcategory $\mathbf{C} \subseteq \mathbf{FiniteSpectra}$ is...

- ... *thick* if it's closed under weak equivalences, retracts, and cofiber sequences.

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Define the geometric space of **FiniteSpectra** to be its collection of thick prime ideals.

Spec **FiniteSpectra**

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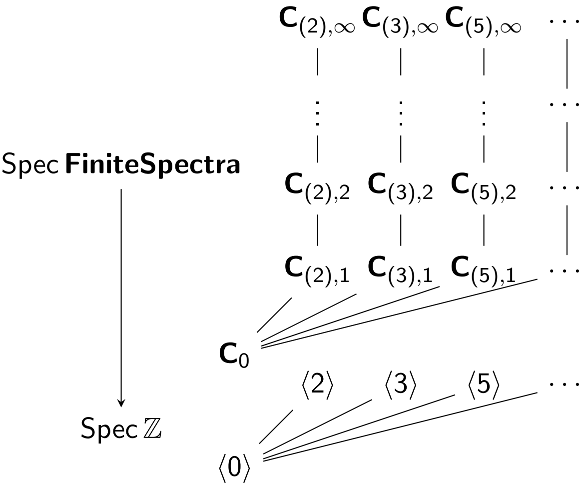
- The $K(q)$-acyclics give a thick prime ideal $\mathbf{C}_q$.
- There is a proper inclusion $\mathbf{C}_{q+1} \subsetneq \mathbf{C}_q$ and this is all such thick prime ideals.

This lets us draw a picture...

Spec **FiniteSpectra**



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- This lifts the v_q-periodicity in $K(q)_*$ to something at the spectrum level...

Homework

The homework this week instructs you how to build the "Chromatic spectral sequence," which gives a framework for computing $H^*\mathcal{M}_{\mathbf{fg}}$. It also highlights the highly periodic nature of the MU-Adams E_2-page, which we just brushed against on the previous slide.

http://math.berkeley.edu/~ericp/