On Beyond Hatcher!

The delooping problem

Eric Peterson

November 1, 2012

Administrivia

- Four lectures, four Thursdays in November
  - The delooping problem (+ the Steenrod algebra)
  - Spectra and formal groups (+ the chromatic SS)
  - Computations with the EHP spectral sequence (+ $\pi_* L_{K(1)} S$)
  - Hints at globalization (+ even more formal geometry)
- Homework readings
  - 1-2 pages after each lecture
  - Optional — not connected to the main stream of lectures
  - Largely computational, reasonably detailed
  - Idea: even if you don't end up understanding the computation, trying to read through it and looking up relevant words (e.g., what's the *Verschiebung*?) will be useful.
- Notes available at http://math.berkeley.edu/~ericp/

$\pi_* S^1$

- Our main goal: the homotopy groups of spheres, $\pi_p S^q = [S^p, S^q]$. These are very hard to compute. The more we can compute, the more we know about topology.

$\pi_* S^1$

- Our main goal: the homotopy groups of spheres, $\pi_p S^q = [S^p, S^q]$. These are very hard to compute. The more we can compute, the more we know about topology.
- $\pi_0 X$ is the connected components of X.

$\pi_* S^1$

- Our main goal: the homotopy groups of spheres, $\pi_p S^q = [S^p, S^q]$. These are very hard to compute. The more we can compute, the more we know about topology.
- $\pi_0 X$ is the connected components of X.
- Fibrations (/ fiber sequences) yield lexseqs on homotopy groups.

$\pi_* S^1$

- Our main goal: the homotopy groups of spheres, $\pi_p S^q = [S^p, S^q]$. These are very hard to compute. The more we can compute, the more we know about topology.
- $\pi_0 X$ is the connected components of X.
- Fibrations (/ fiber sequences) yield lexseqs on homotopy groups.
- Group quotient $\mathbb{Z} \to \mathbb{R} \to S^1$ yields $\pi_* S^1 = \{0, \mathbb{Z}, 0, 0, \ldots\}$, since $\mathbb{R}$ is contractible and $\mathbb{Z}$ is a discrete space.

Part 1: Spectral sequences

Eilenberg-Steenrod axioms

The other tool available to use is "homology." What's homology?

Eilenberg-Steenrod axioms

The other tool available to use is "homology." What's homology? It's a functor $H_* : \textbf{Spaces} \to \textbf{GradedGroups}$ satisfying...

- H_* is homotopy invariant.
- $H_* S^0$ is known, often abbreviated to H_*.
- $H_* \bigvee_\alpha X_\alpha \cong \bigoplus_\alpha H_* X_\alpha$.
- H_* is "locally determined".

Locality

The locality axiom asserts that a cofiber sequence $A \to X \to X/A$ yields a lexseq

$$\cdots \to H_{n+1}X/A \to H_nA \to H_nX \to H_nX/A \to H_{n-1}A \to \cdots .$$

Locality

The locality axiom asserts that a cofiber sequence $A \to X \to X/A$ yields a lexseq

$$\cdots \to H_{n+1}X/A \to H_nA \to H_nX \to H_nX/A \to H_{n-1}A \to \cdots .$$

This can be written compactly like so:

$$\begin{array}{ccc} A & \longrightarrow & X \\ & & \downarrow \\ & & X/A \end{array} \quad \text{yields} \quad \begin{array}{ccc} H_*A & \longrightarrow & H_*X \\ & \nwarrow^{[-1]} & \downarrow \\ & & H_*X/A. \end{array}$$

Filtration spectral sequence

Most of the time, spaces are decomposed into many pieces, not just two. Consider the following filtration of X...

$$\mathrm{pt} \longrightarrow F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow \cdots \longrightarrow X$$

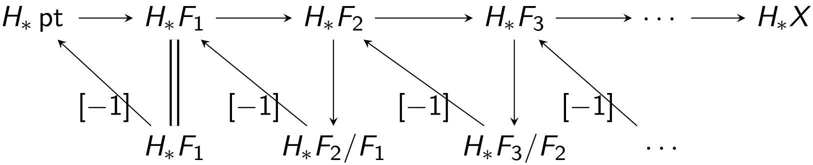
Filtration spectral sequence

… extend each inclusion to a cofiber sequence …

$$\begin{array}{ccccccccc}
\mathrm{pt} & \longrightarrow & F_1 & \longrightarrow & F_2 & \longrightarrow & F_3 & \longrightarrow & \cdots & \longrightarrow & X \\
& & \| & & \downarrow & & \downarrow & & & & \\
& & F_1 & & F_2/F_1 & & F_3/F_2 & & \cdots & &
\end{array}$$

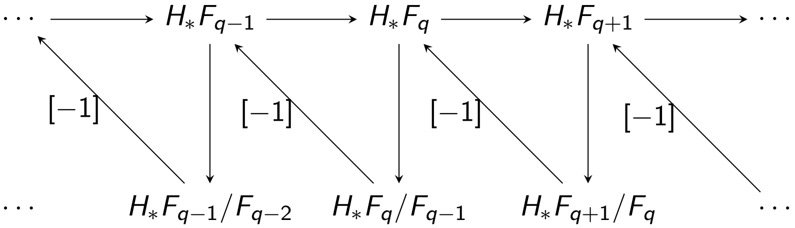
Filtration spectral sequence

… and apply the homology functor H_*.



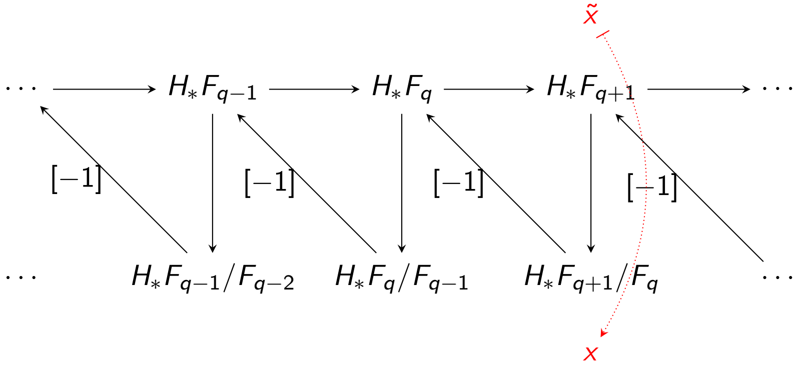
Filtration spectral sequence

Our goal now is to recover H_*X from H_*F_q/F_{q-1}.



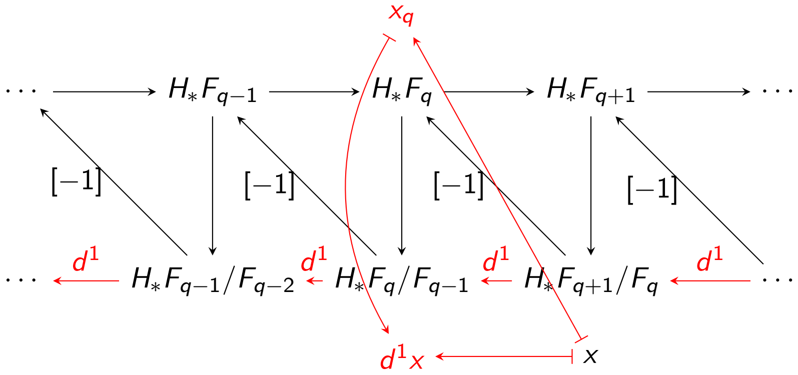
Filtration spectral sequence

Our goal now is to recover H_*X from H_*F_q/F_{q-1}.



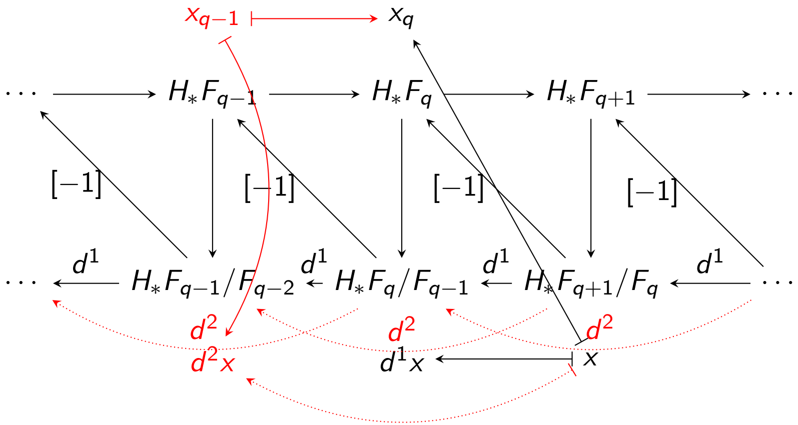
Filtration spectral sequence

Our goal now is to recover H_*X from H_*F_q/F_{q-1}.



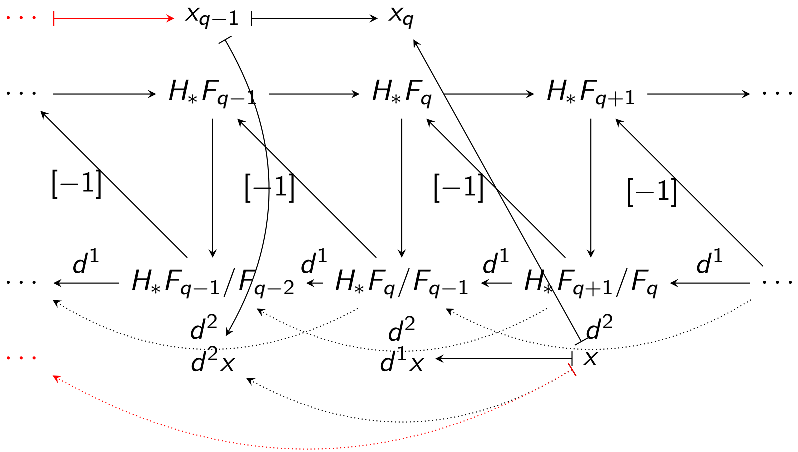
Filtration spectral sequence

Our goal now is to recover $H_* X$ from $H_* F_q / F_{q-1}$.



Filtration spectral sequence

Our goal now is to recover H_*X from H_*F_q/F_{q-1}.



Filtration spectral sequence

- This process defines groups $E^r_{p,q}$ and differentials $d^r : E^r_{p,q} \to E^r_{p-1,q-r}$. The cohomology of the rth page yields the $(r+1)$st, and the first page is

$$E^1_{p,q} = H_p F_q / F_{q-1}.$$

- A class $x \in H_* X$ always lifts to some class x_q in $H_* F_q$ for $q \gg 0$. The smallest such q is the only time x_q images to a nonzero element in the bottom row of the diagram. This smallest q gives a filtration on $H_* X$.
- With some assumptions, the groups $E^\infty_{p,*}$ are isomorphic to the associated graded of this filtration on $H_p X$. "The spectral sequence compares the homology groups of the associated graded to an associated graded of the homology groups."

Atiyah-Hirzebruch spectral sequence

Example: put a cell structure on a connected space X and take $F_q = X^{(q)}$ the q-skeleton.

Atiyah-Hirzebruch spectral sequence

Example: put a cell structure on a connected space X and take $F_q = X^{(q)}$ the q-skeleton.

$$
\begin{array}{ccccccccc}
\mathrm{pt} & \longrightarrow & X^{(1)} & \longrightarrow & X^{(2)} & \longrightarrow & X^{(3)} & \longrightarrow & \cdots & \longrightarrow & X \\
 & & \| & & \downarrow & & \downarrow & & & & \\
 & & \bigvee_\alpha S^1 & & \bigvee_\beta S^2 & & \bigvee_\gamma S^3 & & \cdots & &
\end{array}
$$

Atiyah-Hirzebruch spectral sequence

Example: put a cell structure on a connected space X and take $F_q = X^{(q)}$ the q-skeleton.

$$\begin{array}{ccccccccc} \mathrm{pt} & \longrightarrow & X^{(1)} & \longrightarrow & X^{(2)} & \longrightarrow & X^{(3)} & \longrightarrow & \cdots & \longrightarrow & X \\ & & \| & & \downarrow & & \downarrow & & & & \\ & & \bigvee_\alpha S^1 & & \bigvee_\beta S^2 & & \bigvee_\gamma S^3 & & \cdots & & \end{array}$$

Apply E_*. The d^1-differential detects the attaching degree, so

$$E^2_{p,q} = H_q^{\mathrm{cell}}(X; E_p) \Rightarrow E_p X.$$

Atiyah-Hirzebruch spectral sequence

Example: put a cell structure on a connected space X and take $F_q = X^{(q)}$ the q-skeleton.

$$\begin{array}{ccccccccc}
\mathrm{pt} & \longrightarrow & X^{(1)} & \longrightarrow & X^{(2)} & \longrightarrow & X^{(3)} & \longrightarrow \cdots \longrightarrow & X \\
 & & \Vert & & \downarrow & & \downarrow & & \\
 & & \bigvee_\alpha S^1 & & \bigvee_\beta S^2 & & \bigvee_\gamma S^3 & \cdots &
\end{array}$$

Apply E_*. The d^1-differential detects the attaching degree, so

$$E^2_{p,q} = H^{\mathrm{cell}}_q(X; E_p) \Rightarrow E_p X.$$

Basic exercise: Compute $H_*(\mathbb{R}\mathrm{P}^n; \mathbb{Z})$ and $H_*(\mathbb{R}\mathrm{P}^n; \mathbb{F}_2)$ this way.

Part 2: Spectra, operads, and delooping

Spectra

- Recall that the functor $H^n(-; G)$ is *representable*:

$$H^n(X; G) = [X, K(G, n)].$$

- Brown representability says this is not an accident: for any cohomology theory E^* there are homotopy types $\underline{E}_*$ with

$$E^n X = [X, \underline{E}_n].$$

[1] N.B.: The Brown construction is not functorial.

Spectra

- Recall that the functor $H^n(-; G)$ is *representable*:

$$H^n(X; G) = [X, K(G, n)].$$

- Brown representability says this is not an accident: for any cohomology theory E^* there are homotopy types $\underline{E}_*$ with

$$E^n X = [X, \underline{E}_n].$$

- A good thing to do with a functor[1] is to try to turn it into an equivalence: how do we need to restrict / augment the target to make this happen?

Question: What structure do these spaces $\underline{E}_*$ have?

[1] N.B.: The Brown construction is not functorial.

Spectra

Question: What structure do these spaces $\underline{E}_*$ have?

- The Eilenberg-Steenrod axioms yield a natural isomorphism $E^nX = E^{n+1}\Sigma X$.

Spectra

Question: What structure do these spaces $\underline{E}_*$ have?

- The Eilenberg-Steenrod axioms yield a natural isomorphism $E^n X = E^{n+1} \Sigma X$.

$$[X, \underline{E}_n] = E^n X$$

Spectra

Question: What structure do these spaces $\underline{E}_*$ have?

- The Eilenberg-Steenrod axioms yield a natural isomorphism $E^n X = E^{n+1} \Sigma X$.

$$[X, \underline{E}_n] = E^n X = E^{n+1} \Sigma X$$

Spectra

Question: What structure do these spaces $\underline{E}_*$ have?

- The Eilenberg-Steenrod axioms yield a natural isomorphism $E^n X = E^{n+1} \Sigma X$.

$$[X, \underline{E}_n] = E^n X = E^{n+1} \Sigma X = [\Sigma X, \underline{E}_{n+1}]$$

Spectra

Question: What structure do these spaces $\underline{E}_*$ have?

- The Eilenberg-Steenrod axioms yield a natural isomorphism $E^n X = E^{n+1} \Sigma X$.

$$[X, \underline{E}_n] = E^n X = E^{n+1} \Sigma X = [\Sigma X, \underline{E}_{n+1}] = [X, \Omega \underline{E}_{n+1}].$$

Spectra

Question: What structure do these spaces $\underline{E}_*$ have?

- The Eilenberg-Steenrod axioms yield a natural isomorphism $E^n X = E^{n+1}\Sigma X$.

$$[X, \underline{E}_n] = E^n X = E^{n+1}\Sigma X = [\Sigma X, \underline{E}_{n+1}] = [X, \Omega \underline{E}_{n+1}].$$

- The Yoneda lemma says there are weak equivalences

$$\underline{E}_n \xrightarrow{\simeq} \Omega \underline{E}_{n+1}.$$

Spectra

Question: What structure do these spaces $\underline{E}_*$ have?

- The Eilenberg-Steenrod axioms yield a natural isomorphism $E^n X = E^{n+1}\Sigma X$.

$$[X, \underline{E}_n] = E^n X = E^{n+1}\Sigma X = [\Sigma X, \underline{E}_{n+1}] = [X, \Omega \underline{E}_{n+1}].$$

- The Yoneda lemma says there are weak equivalences

$$\underline{E}_n \xrightarrow{\simeq} \Omega \underline{E}_{n+1}.$$

- This is a complete characterization: every sequence of spaces $\underline{E}_n$ with connecting maps $\underline{E}_n \xrightarrow{\simeq} \Omega \underline{E}_{n+1}$ yields a *spectrum* E.

The ∞-category of **Spectra**

- **Spaces** is more than just a category: you can suspend spaces, take the homotopy category, build the smash product, and so on.

The ∞-category of **Spectra**

- **Spaces** is more than just a category: you can suspend spaces, take the homotopy category, build the smash product, and so on.
- We've built **Spectra** in terms of spaces, and so it would be nice to be able to perform these operations on spectra too.

The ∞-category of **Spectra**

- **Spaces** is more than just a category: you can suspend spaces, take the homotopy category, build the smash product, and so on.
- We've built **Spectra** in terms of spaces, and so it would be nice to be able to perform these operations on spectra too.
- Theorem: You can! "**Spectra** is a stable ∞-category with a symmetric monoidal product $\wedge$ and adjoint functors

$$\Sigma^\infty : \textbf{Spaces} \leftrightarrows \textbf{Spectra} : \Omega^\infty."$$

The ∞-category of **Spectra**

- **Spaces** is more than just a category: you can suspend spaces, take the homotopy category, build the smash product, and so on.
- We've built **Spectra** in terms of spaces, and so it would be nice to be able to perform these operations on spectra too.
- Theorem: You can! "**Spectra** is a stable ∞-category with a symmetric monoidal product $\wedge$ and adjoint functors

$$\Sigma^\infty : \mathbf{Spaces} \leftrightarrows \mathbf{Spectra} : \Omega^\infty."$$

- Big idea: Work with spectra rather than cohomology theories, and study spectra using techniques from algebraic topology.

The ∞-category of **Spectra**

- "Stability" means Σ : **Spectra** $\to$ **Spectra** is an equivalence.
- It is given by the shift operator: $\underline{\Sigma E}_n = \underline{E}_{n+1}$. To get the inverse, $\underline{\Sigma^{-1} E}_n = \underline{\Omega E}_n = \Omega \underline{E}_n$.

The ∞-category of **Spectra**

- "Stability" means Σ : **Spectra** $\to$ **Spectra** is an equivalence.
- It is given by the shift operator: $\underline{\Sigma E}_n = \underline{E}_{n+1}$. To get the inverse, $\underline{\Sigma^{-1} E}_n = \underline{\Omega E}_n = \Omega \underline{E}_n$.
- In particular, this yields "negative-dimensional spheres":

$$\mathbb{S}^{-n} = \Sigma^{-n}\mathbb{S}^0.$$

The homotopy groups $\pi_* E$ of a spectrum are $\mathbb{Z}$-indexed.

Motivation: K-theory

- The space $BU \times \mathbb{Z}$ classifies stable vector bundles. The functor it represents is called $K(-)$.
- For a cofiber sequence $A \to X \to X/A$, the sequence $K(A) \leftarrow K(X) \leftarrow K(X/A)$ is exact in the middle.

Motivation: K-theory

- The space $BU \times \mathbb{Z}$ classifies stable vector bundles. The functor it represents is called $K(-)$.
- For a cofiber sequence $A \to X \to X/A$, the sequence $K(A) \leftarrow K(X) \leftarrow K(X/A)$ is exact in the middle.
- Question: Can $K(-)$ be extended to a cohomology theory with $K(-) = K^0(-)$?

Motivation: K-theory

- The space $BU \times \mathbb{Z}$ classifies stable vector bundles. The functor it represents is called $K(-)$.
- For a cofiber sequence $A \to X \to X/A$, the sequence $K(A) \leftarrow K(X) \leftarrow K(X/A)$ is exact in the middle.
- Question: Can $K(-)$ be extended to a cohomology theory with $K(-) = K^0(-)$?
- $K^{-n}X$ is easy to define:

$$K^{-n}X = K^0\Sigma^n X = K(\Sigma^n X).$$

So, $\underline{K}_{-n} = \Omega^n(BU \times \mathbb{Z})$.

Motivation: K-theory

- The space $BU \times \mathbb{Z}$ classifies stable vector bundles. The functor it represents is called $K(-)$.
- For a cofiber sequence $A \to X \to X/A$, the sequence $K(A) \leftarrow K(X) \leftarrow K(X/A)$ is exact in the middle.
- Question: Can $K(-)$ be extended to a cohomology theory with $K(-) = K^0(-)$?
- $K^{-n}X$ is easy to define:

$$K^{-n}X = K^0\Sigma^n X = K(\Sigma^n X).$$

  So, $\underline{K}_{-n} = \Omega^n(BU \times \mathbb{Z})$.
- Refined question: What about K^n? Can we find spaces $\underline{K}_n$ with $\Omega^n \underline{K}_n = BU \times \mathbb{Z}$? What conditions does $BU \times \mathbb{Z} = \underline{K}_0$ need to satisfy?

The structure of loopspaces

- We know is that $\underline{E}_0$ is a loopspace: $\underline{E}_0 = \Omega \underline{E}_1$. Loopspaces come with a "multiplication" which is not associative or unital, but becomes so after relaxing to the homotopy category.

The structure of loopspaces

- We know is that $\underline{E}_0$ is a loopspace: $\underline{E}_0 = \Omega \underline{E}_1$. Loopspaces come with a "multiplication" which is not associative or unital, but becomes so after relaxing to the homotopy category.
- Pick disjoint subintervals $\{[a_i, b_i] \subseteq [0,1]\}_{i=1}^n$ and loops $\{\gamma_i \in \Omega X\}_{i=1}^n$. Then there is a product

$$\Gamma(t) = \begin{cases} \gamma_1\left(\frac{t-a_1}{b_1-a_1}\right), & t \in [a_1, b_1], \\ \vdots & \vdots \\ \gamma_n\left(\frac{t-a_n}{b_n-a_n}\right), & t \in [a_n, b_n], \\ * & \text{otherwise.} \end{cases}$$

The structure of loopspaces

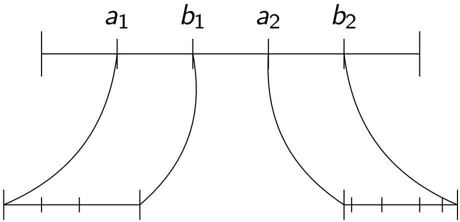
- We know is that $\underline{E}_0$ is a loopspace: $\underline{E}_0 = \Omega \underline{E}_1$. Loopspaces come with a "multiplication" which is not associative or unital, but becomes so after relaxing to the homotopy category.
- Pick disjoint subintervals $\{[a_i, b_i] \subseteq [0,1]\}_{i=1}^n$ and loops $\{\gamma_i \in \Omega X\}_{i=1}^n$. Then there is a product

$$\Gamma(t) = \begin{cases} \gamma_1\left(\frac{t-a_1}{b_1-a_1}\right), & t \in [a_1, b_1], \\ \vdots & \vdots \\ \gamma_n\left(\frac{t-a_n}{b_n-a_n}\right), & t \in [a_n, b_n], \\ * & \text{otherwise.} \end{cases}$$

- This is a family of products $(\Omega X)^{\times n} \times A_n \to \Omega X$.

The structure of loopspaces

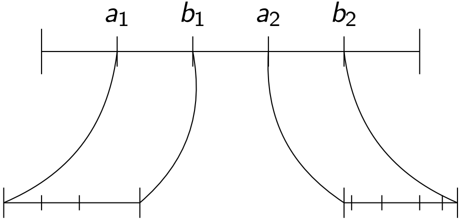
- Subintervals can be nested:



This gives maps $(A_{n_1} \times \cdots \times A_{n_r}) \times A_r \to A_{n_+}$.

The structure of loopspaces

- Subintervals can be nested:

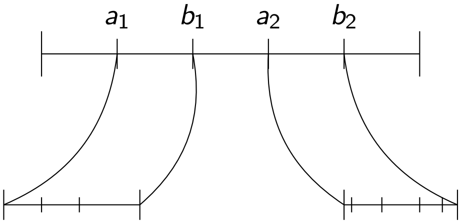


This gives maps $(A_{n_1} \times \cdots \times A_{n_r}) \times A_r \to A_{n_+}$.

- Spaces together with compatible maps like these are called an *operad*. This one is called the A_∞ operad.

The structure of loopspaces

- Subintervals can be nested:



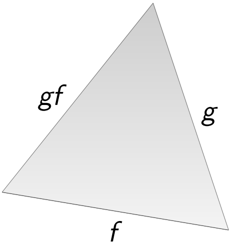
This gives maps $(A_{n_1} \times \cdots \times A_{n_r}) \times A_r \to A_{n_+}$.

- Spaces together with compatible maps like these are called an *operad*. This one is called the A_∞ operad.
- Spaces like ΩX with multiplications parameterized by an operad are called *algebras* over that operad. ΩX is "an A_∞-algebra in spaces" or "an A_∞-space."

Delooping A_∞-spaces

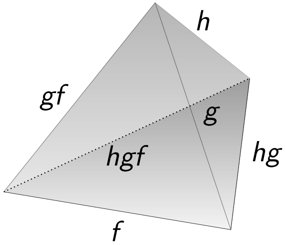
Back to our example HG with $\underline{HG}_0 = G$ and $\underline{HG}_1 = BG$.
Hatcher's model for BG is a simplicial complex with ...

- ... one 0-simplex.
- ... a 1-simplex for each $g \in G$.
- ... a 2-simplex for each pair $(f, g) \in G^2$, with edges labeled by f, g, and gf.



Delooping A_∞-spaces

- ... a 3-simplex for each triple $(f, g, h) \in G^3$ with edges labeled to encode the triple project $f \cdot g \cdot h$.



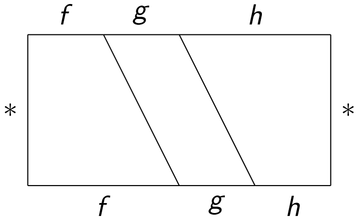
- ... and so on.

Delooping A_∞-spaces

We can mimic this construction for an A_∞-space X; for example, we'll show how to produce the 3-simplices in BX.

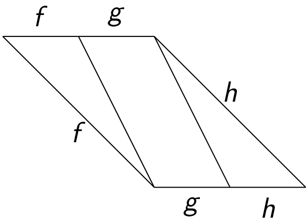
Delooping A_∞-spaces

We can mimic this construction for an A_∞-space X; for example, we'll show how to produce the 3-simplices in BX. Start with a path in A_3 showing homotopy-associativity of $f \cdot (g \cdot h) \sim (f \cdot g) \cdot h$:



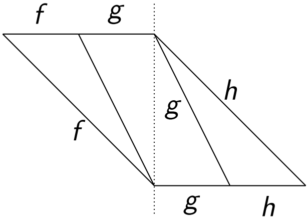
Delooping A_∞-spaces

Now, contract the constant edges labeled $*$ to get



Delooping A_∞-spaces

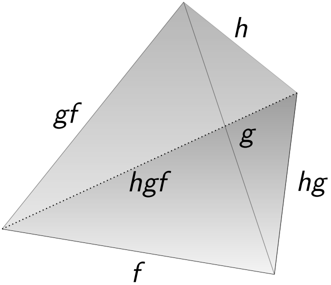
Now, contract the constant edges labeled $*$ to get



Then, if you fold along g and distend, you'll get...

Delooping A_∞-spaces

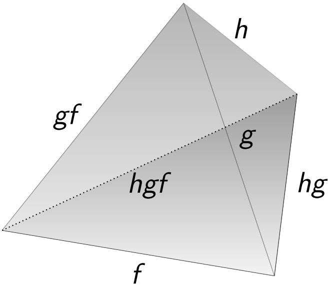
(Fold along g and distend to get...)



The other simplices work similarly.

Delooping A_∞-spaces

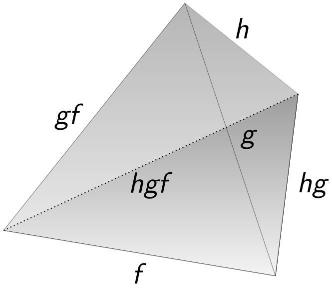
(Fold along g and distend to get...)



The other simplices work similarly. Then, there is a theorem that states if $\pi_0 \underline{E}_1 = 0$, then $B\underline{E}_0 \simeq \underline{E}_1$

Delooping A_∞-spaces

(Fold along g and distend to get...)



The other simplices work similarly. Then, there is a theorem that states if $\pi_0 \underline{E}_1 = 0$, then $B\underline{E}_0 \simeq \underline{E}_1$ — so the loopspace product is enough to deloop $\underline{E}_0$!

E_∞-spaces and spectra

- We can iterate this story: because $\underline{E}_0 = \Omega^2 \underline{E}_2$, it is also an algebra for the operad of sub-squares in $[0,1] \times [0,1]$. This is called the E_2-operad.

E_∞-spaces and spectra

- We can iterate this story: because $\underline{E}_0 = \Omega^2 \underline{E}_2$, it is also an algebra for the operad of sub-squares in $[0, 1] \times [0, 1]$. This is called the E_2-operad.
- More generally, $\underline{E}_0 = \Omega^n \underline{E}_n$, so it is an algebra for the operad of sub-n-cubes in $[0, 1]^{\times n}$, called the E_n-operad.

E_∞-spaces and spectra

- We can iterate this story: because $\underline{E}_0 = \Omega^2 \underline{E}_2$, it is also an algebra for the operad of sub-squares in $[0,1] \times [0,1]$. This is called the E_2-operad.
- More generally, $\underline{E}_0 = \Omega^n \underline{E}_n$, so it is an algebra for the operad of sub-n-cubes in $[0,1]^{\times n}$, called the E_n-operad.
- Because it's an algebra for the E_n-operad for each n, it's called an E_∞-space.

E_∞-spaces and spectra

- You've seen sub-squares before in the proof that $\pi_2 X$ is abelian.
- This means that a nonabelian G cannot be an E_2-space, since $\pi_2 B^2 G = \pi_0 \Omega^2 B^2 G = \pi_0 G = G$ must be abelian.

E_∞-spaces and spectra

- You've seen sub-squares before in the proof that $\pi_2 X$ is abelian.
- This means that a nonabelian G cannot be an E_2-space, since $\pi_2 B^2 G = \pi_0 \Omega^2 B^2 G = \pi_0 G = G$ must be abelian.
- An E_1-space is homotopy-associative, an E_2-space is homotopy-commutative, and E_n-spaces generalize this in the following sense:

  Applying B to an E_n-space results in an E_{n-1}-space.

E_∞-spaces and spectra

- You've seen sub-squares before in the proof that $\pi_2 X$ is abelian.
- This means that a nonabelian G cannot be an E_2-space, since $\pi_2 B^2 G = \pi_0 \Omega^2 B^2 G = \pi_0 G = G$ must be abelian.
- An E_1-space is homotopy-associative, an E_2-space is homotopy-commutative, and E_n-spaces generalize this in the following sense:

  Applying B to an E_n-space results in an E_{n-1}-space.

  An E_∞-space has an infinite sequence of deloopings.

In fact, the assignment $E \mapsto \underline{E}_0$ is an equivalence between connective spectra ($\pi_{*<0} E = 0$) and E_∞-spaces.

A couple extra facts

- The functor $\Omega^\infty \Sigma^\infty X = \operatorname{colim}_n \Omega^n \Sigma^n X$ can be thought of as replacing X with various E_n-space approximations as n grows, yielding in the limit an E_∞-space / connective spectrum.

A couple extra facts

- The functor $\Omega^\infty\Sigma^\infty X = \operatorname{colim}_n \Omega^n\Sigma^n X$ can be thought of as replacing X with various E_n-space approximations as n grows, yielding in the limit an E_∞-space / connective spectrum.
- An A_∞-space is exactly the structure that arises when you replace a topological group G with a homotopy equivalent space X and try to transport the product structure to X.

A couple extra facts

- The functor $\Omega^\infty \Sigma^\infty X = \operatorname{colim}_n \Omega^n \Sigma^n X$ can be thought of as replacing X with various E_n-space approximations as n grows, yielding in the limit an E_∞-space / connective spectrum.
- An A_∞-space is exactly the structure that arises when you replace a topological group G with a homotopy equivalent space X and try to transport the product structure to X. In fact, every A_∞-space has a strictly associative model.

A couple extra facts

- The functor $\Omega^\infty\Sigma^\infty X = \operatorname{colim}_n \Omega^n\Sigma^n X$ can be thought of as replacing X with various E_n-space approximations as n grows, yielding in the limit an E_∞-space / connective spectrum.
- An A_∞-space is exactly the structure that arises when you replace a topological group G with a homotopy equivalent space X and try to transport the product structure to X. In fact, every A_∞-space has a strictly associative model.
- Curiously, this is **not** true for E_∞-spaces: there are E_∞-spaces with no strictly commutative model. This is part of what makes spectra interesting.

Homework

Do the homework reading! You'll learn to compute $H_*(K(\mathbb{F}_2, *); \mathbb{F}_2)$, called the (dual, unstable) Steenrod algebra. With spectral sequences and deloopings, you have all the tools you'll need.

http://math.berkeley.edu/~ericp/