ON BEYOND HATCHER!

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In the Fall semester of 2012, Constantin Teleman encouraged me to run a short seminar which would teach attendees about advanced algebraic topology. UC-Berkeley runs a pair of graduate courses in algebraic topology which more or less go through Allen Hatcher's *Algebraic Topology*, but there is a long road from there to the forefront of the field. Getting students research-ready inside of four seminar talks is an impossible task, of course, and so instead my goal is to sketch a picture of some of the major components of the field, so that students know enough of the flavor of the subject to at least identify whether they're intrigued by its questions and its methods, and then further to know where to look to learn more. To reinforce the impression that there's a lot going on in algebraic topology, each day will be very distinctly flavored from the others, and no day will require absolutely understanding any day that came before it. In the same vein, few things will be completely proven, but I hope to at least define the topological things under discussion. Some knowledge from related fields (algebraic geometry, primarily) will be assumed without hesitation. Each of these talks is meant to last roughly 50 minutes.

The reader should additionally be warned that, due to my limited worldliness, these notes are bent sharply toward what I consider interesting. This means that they are highly algebraic, and they make no mention of exciting geometric things in stable homotopy theory — like string topology, for one example of many.

There are too many people to thank and acknowledge for having taught me enough to write these notes — the topology community is full of vibrant, welcoming personalities. Any errors in the notes are mine, whether by miscopying or misunderstanding. I'm happy to incorporate corrections by email at ericp@math.berkeley.edu. The title of these notes is an allusion to a Dr. Seuss book, *On Beyond Zebra!* — appropriate for the strange and wonderful world of algebraic topology.

0.1. **Conventions and style.** The notes are meant to include definitions and to be largely consistent with the general standards of the field. Even so, if you find something puzzling, maybe this will help clear things up:

- Limits and colimits will always be interpreted to be ∞-categorical / homotopical whenever context allows.
- In this draft version of the notes, footnotes are primarily used for comments and corrections that others and I have suggested, but which I haven't yet incorporated into the main body of the text.
- Oftentimes (but perhaps not always), an unadorned "H_*X" denotes the *reduced* homology of X. This is typically denoted $\widetilde{H}_*X$ elsewhere in the literature, but it's too much hassle to carry around tildes the entire length of the notes.

CONTENTS

1. DAY 1: THE DELOOPING PROBLEM

As we work our way through these lectures, we're going to be chiefly concerned with the homotopy groups of spheres, $\pi_p S^q = [S^p, S^q]$. These groups are now-classical objects in algebraic topology that are tremendously difficult to compute; it's widely believed that we'll never have full information about them.[1] On the other hand, thick-headedness and stubborn determination has told us a little bit, and the size of the sector of groups we understand correlates well with how well we understand the field of algebraic topology as a whole. We'll study many tools which aren't directly related to $\pi_* S^*$ on the face of it, but in the end everything we talk about should result in some new information about them.

The most basic observation about homotopy groups is that for a fiber sequence (in particular, for a covering space), there is an induced long exact sequence. Then, the group quotient $\mathbb{Z} \to \mathbb{R} \to S^1$ gives the computation [14, Theorem 1.7]

$$\pi_p S^1 = \begin{cases} \mathbb{Z} & \text{when } p = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Our study begins after this.

1.1. Spectral sequences.

1.1. Spectral sequences. The other major invariant available apart from homotopy groups in algebraic topology are co/homology groups. A homology theory [47, Definition 7.1] is a functor $H_* : \mathsf{Spaces} \to \mathsf{GradedGroups}$ satisfying the Eilenberg-Steenrod axioms:

(1) H_* is homotopy invariant.
(2) $H_* S^0$ is known, often abbreviated to H_*.
(3) $H_* \bigvee_\alpha X_\alpha = \bigoplus_\alpha H_* X_\alpha$.
(4) H_* is "locally determined," to be discussed below.

The most important of these is the locality axiom. It assets that if we apply H_* to a cofiber sequence, i.e., a sequence of spaces

$$\begin{tikzcd} A \arrow[r] & X \arrow[d] \\ & X/A, \end{tikzcd}$$

then we get a long exact sequence

$$\cdots H_n A \to H_n X \to H_n X/A \to H_{n-1} A \to \cdots.$$

Another way to write this is as a triangle of maps

$$\begin{tikzcd} H_* A \arrow[r] & H_* X \arrow[d] \\ & H_* X/A \arrow[ul, "{[-1]}"] \end{tikzcd}$$

which is exact at every node, and where the map marked "$[-1]$" incorporates a shift down in degree by 1. This axiom makes homology an extremely computable invariant; by decomposing a space into two pieces, A and X/A, the homology of the whole space is in this sense recoverable from the homologies of the individual spaces.

Of course, it is often convenient to decompose spaces into many pieces — a filtration of the space — and so one might begin instead with a diagram of the following shape:

$$\begin{tikzcd} \mathrm{pt} \arrow[r] & F_1 \arrow[r] & F_2 \arrow[r] & F_3 \arrow[r] & \cdots \arrow[r] & F_\infty \arrow[r, equal] & X. \end{tikzcd}$$

Now, each of these maps $F_i \to F_{i+1}$ can be extended to a cofiber sequence by considering F_{i+1}/F_i like so:

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[1] See Ravenel [36] for a cute quote about this.

$$\begin{tikzcd}
\mathrm{pt} \arrow[r] & F_1 \arrow[r] \arrow[d, equal] & F_2 \arrow[r] \arrow[d] & F_3 \arrow[r] \arrow[d] & \cdots \arrow[r] & F_\infty \arrow[r, equal] & X \\
& F_1 & F_2/F_1 & F_3/F_2 & \cdots .
\end{tikzcd}$$

Applying H_* to this diagram then yields a sequence of exact sequences

$$\begin{tikzcd}
H_* \mathrm{pt} \arrow[r] & H_*F_1 \arrow[r] \arrow[d, equal] & H_*F_2 \arrow[r] \arrow[d] & H_*F_3 \arrow[r] \arrow[d] & \cdots \arrow[r] & H_*F_\infty \arrow[r, equal] & H_*X \\
& H_*F_1 \arrow[ul, "{[-1]}"] & H_*F_2/F_1 \arrow[ul, "{[-1]}"] & H_*F_3/F_2 \arrow[ul, "{[-1]}"] & \cdots, \arrow[ul, "{[-1]}"]
\end{tikzcd}$$

and one can ask how to recover H_*X only from these groups H_*F_{q+1}/F_q. Suppose we have a class $x \in H_*F_{q+1}/F_q$, and we'd like to produce a lift $\tilde{x}$ of x to H_*F_{q+1} across the projection map. Since these triangles are exact, this is equivalent to asking if the image of x in H_*F_q vanishes — call that element x_q. However, we're only supposed to be studying the groups on the bottom of the diagram, and so we further push x_q down to H_*F_q/F_{q-1} and test whether it is zero there. If it isn't, then there's no hope for x, which we should discard. If it is, however, then that's great — it doesn't necessarily mean that x_q is zero, but it's enough to define x_{q-1}, a preimage of x_q across $H_*F_{q-1} \to H_*F_q$. This process continues until we get all the way down to H_* pt, where we can finally test whether x_0 (and hence x_i for each $i \leq q$) vanishes once and for all.

This process assembles into what is called a spectral sequence [49, Section 5.4] [28, Section 2.2]. The groups H_pF_q/F_{q-1} are notated $E^1_{p,q}$ and called the first page of the spectral sequence. The map $H_*F_{q+1}/F_q \to H_*F_q/F_{q-1}$ is called $d^1 : E^1_{p,q} \to E^1_{p-1,q-1}$, the first differential. Taking homology (in the sense of the chain complexes of homological algebra) yields a new collection of groups, called $E^2_{p,q} = H_*(E^1_{p,q}, d^1)$. This captures the process we described above: we keep only the classes x whose projected classes x_q vanish. The quotient by d^1-boundaries additionally makes the image of x_{q-1} in the bottom row independent of the choice of x_{q-1}, and so this element becomes a new differential d^2x. This fills out the picture above to

$$\begin{tikzcd}
H_* \mathrm{pt} \arrow[r] & H_*F_1 \arrow[r] \arrow[d, equal] & H_*F_2 \arrow[r] \arrow[d] & H_*F_3 \arrow[r] \arrow[d] & \cdots \arrow[r] & H_*F_\infty \arrow[r, equal] & H_*X \\
& H_*F_1 \arrow[ul, "{[-1]}"] & H_*F_2/F_1 \arrow[ul, "{[-1]}"] \arrow[l, "d^1"'] & H_*F_3/F_2 \arrow[ul, "{[-1]}"] \arrow[l, "d^1"'] \arrow[ll, dotted, bend left, "d^2"] & \cdots, \arrow[ul, "{[-1]}"] \arrow[l, "d^1"'] \arrow[ll, dotted, bend left, "d^2"]
\end{tikzcd}$$

In general, the rth page $E^r_{*,*}$ occurs as the homology of the $(r-1)$st page, and it carries a differential $d^r : E^r_{p,q} \to E^r_{p-1,q-r}$. Under some mild conditions on the initial filtration F_*, the limiting groups $E^\infty_{*,*}$ assemble into an associated graded of the desired groups H_*X — that is, the spectral sequence compares the homology groups of the associated graded to the associated graded of the homology groups.[2]

We'll be using spectral sequences heavily in later days, but not really in the second half of today's talk, so let's do a quick example so that it starts to solidify. A common presentation of a space X in algebraic topology is as a cell complex: X is written as an increasing sequence of subspaces $X^{(q)}$, each of which is formed by attaching closed q-disks to $X^{(q-1)}$, the previous subspace [47, Chapter 5]. Together, these give an ascending filtration $F_q = X^{(q)}$ of X whose filtration quotients are bouquets of q-spheres: $F_q/F_{q-1} = \bigvee_\alpha S^{(q)}_\alpha$. Applying H_* to this diagram yields a spectral sequence in which the d^1-differential records the degree map — i.e., the differential in cellular homology. Taking homology against d^1 then yields the cellular homology of X with coefficients in H_*, and the action of the

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[2] A question that came up when delivering this was: can $E^r_{*,*}$ be described in terms of the homology of F_q relative to F_{q-r}? I don't know the answer, and it seems like maybe this is possible for something like $q < 2r$, I feel it would be surprising in general.

rest of the differentials gives the Atiyah-Hirzebruch spectral sequence with signature

$$E^2_{p,q} = H^{\mathrm{cell}}_p(X; H_q) \Rightarrow H_p X, \qquad d^r_{p,q} : E^r_{p,q} \to E^r_{p-1,q-r}.$$

1.2. **Delooping and operads.** Let's let homological algebra alone for a while and get started on topology. Together, all cohomology theories span a full subcategory of the category of all functors Spaces → GradedGroups.[3] The Brown representability theorem [14, Theorem 4E.1] says that E^* is representable by a graded space[4] $\underline{E}_*$, and we would like to characterize the "image" of the Brown representability theorem in the category of sequences of spaces. Throughout this section, a good example to keep in mind are the Eilenberg-MacLane spaces representing ordinary cohomology: $H^n(X;G) = [X, \underline{HG}_n] = [X, K(G,n)]$.

The key observation is that the suspension axiom for cohomology yields a natural isomorphism

$$[X, \underline{E}_n] = E^n X = E^{n+1}\Sigma X = [\Sigma X, \underline{E}_{n+1}] = [X, \Omega \underline{E}_{n+1}].$$

Applying the Yoneda lemma yields an equivalence $\underline{E}_n \xrightarrow{\simeq} \Omega \underline{E}_{n+1}$, and every sequence of spaces $\underline{E}_n$ together with these structure maps $\underline{E}_n \xrightarrow{\simeq} \Omega \underline{E}_{n+1}$ — altogether called a spectrum — represents a cohomology theory [47, Theorem 8.42].[5]

The category Spaces comes with more structure than merely being a category: we can also build fiber sequences, consider the homotopy category, take smash products, and so on. It would be nice to see this structure also appear in Spectra, since it is assembled from certain sorts of spaces. There are many ways to go about doing this, all of which require some work but land you at the same place, realizing Spectra as a stable ∞-category[6] with a symmetric monoidal product $\wedge$, along with adjoint functors

$$\Sigma^\infty : \mathsf{Spaces} \leftrightarrows \mathsf{Spectra} : \Omega^\infty.$$

These functors are so named because of the following composition law: $\Omega^\infty \Sigma^\infty X = \operatorname{colim}_n \Omega^n \Sigma^n X$. The main thing this means is that you should be fearless about doing various homotopical operations on spectra: you can suspend them, build mapping spaces between them — whatever you like. In fact, this is really the point of working with spectra rather than with cohomology theories: the cone on a cohomology theory doesn't make sense, but on a spectrum it does.

Using these homotopical operations, the graded space $\underline{E}_*$ is recoverable from an arbitrary spectrum by the formula $\underline{E}_n = \Omega^\infty \Sigma^n E$. However, it's worth asking what structure the bottom-most space $\underline{E}_0$ has on its own — in particular, when you can tell that a space X is $\underline{E}_0$ for some spectrum E? For instance, when people were first starting to think about K-theory, this was an extremely relevant question: they knew that the space $BU \times \mathbb{Z}$ classified stable complex vector bundles, and they could tell that a cofiber sequence $A \to X \to X/A$ would be sent to a sequence

$$[A, BU \times \mathbb{Z}] \leftarrow [X, BU \times \mathbb{Z}] \leftarrow [X/A, BU \times \mathbb{Z}]$$

which was exact in the middle. This leads naturally to the question: does the functor $[-, BU \times \mathbb{Z}]$ deserve to be called $K^0(-)$ for some spectrum K — that is, is K-theory a cohomology theory? We can define $K^{-n}(X)$ by

$$K^{-n}(X) = [\Sigma^n X, BU \times \mathbb{Z}] = [X, \Omega^n(BU \times \mathbb{Z})],$$

but what about the positive-degree functors K^n? Is it possible to produce spaces $\underline{K}_n$ with $\Omega^n \underline{K}_n = \underline{K}_0$?[7]

The most basic piece of information we have about such a space $\underline{E}_0$ is that $\underline{E}_0 = \Omega \underline{E}_1$, and so we can begin by investigating the structure of loopspaces. Recall that a loopspace carries a sort of product defined by the concatenation of loops, but that this product is neither associative nor unital. Instead, these two properties hold up to homotopy,

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[3] It does turn out that every such natural transformation automatically commutes with the going-around map. This follows from the ability to rotate the triangle $A \to X \to X \cup CA$ to another triangle $X \to X \cup CA \to (X \cup CA) \cup CX \simeq \Sigma A$.

[4] Rather, graded homotopy type.

[5] A simple but good exercise is to run through checks of the Eilenberg-Steenrod axioms listed before.

[6] Include several references with various settings.

[7] As an aside, Bott periodicity answers this question for periodic K-theory, and Bruno Harris [12] gives a really gorgeous proof.

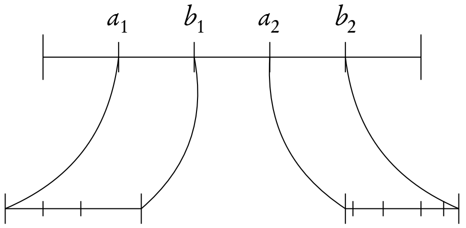

FIGURE 1. Composition in the A_∞-operad.

and their usual proofs [14, Proposition 1.3] involve sliding intervals around — let's try to capture the information that this proof uses. For each collection of n paths $(\gamma_1, \ldots, \gamma_n)$ and n disjoint intervals $([a_1, b_1], \ldots, [a_n, b_n])$ contained in $[0,1]$, we can build the following concatenation:

$$\Gamma(t) = \begin{cases} \gamma_1\left(\frac{t-a_1}{b_1-a_1}\right), & t \in [a_1, b_1], \\ \vdots & \vdots \\ \gamma_n\left(\frac{t-a_n}{b_n-a_n}\right), & t \in [a_n, b_n], \\ * & \text{otherwise.} \end{cases}$$

Writing A_n for the configuration space of such collections of intervals, the fundamental tool in the proof then is that the product on $\Omega\underline{E}_1$ takes the form $A_n \times (\Omega\underline{E}_1)^n \to \Omega\underline{E}_1$ — it's actually a family of products, continuously parametrized by A_n [42]. Moreover, these products are interrelated: writing $n_+ = n_1 + \cdots + n_r$, there are maps $(A_{n_1} \times \cdots \times A_{n_r}) \times A_r \to A_{n_+}$ describing interval nesting, and altogether this collection of spaces A_n together with these maps is called an operad — specifically, the A_∞ operad.[8] The space $\Omega\underline{E}_1$ with its product so parametrized by the various spaces A_n, compatible with the maps between them, is called an A_∞-space or an A_∞-algebra in spaces.

Let's return to the example of HG. You may recall that in this setting, the space $\underline{HG}_1 \simeq K(G,1)$ is given by a certain simplicial complex BG [14, Example 1B.7]. The space BG has a single 0-simplex, a 1-simplex for every point in G, then a 2-simplex joining the edges g, h, and gh to record the relation $(g)\cdot(h) = (g \cdot h)$, and generally we attach in n-simplices labeled by length-n products. It turns out that the structure of an A_∞-algebra on some space X is exactly what is required to build such a space BX; the action of A_n is what defines the n-simplices [42]. Specifically, recall the picture used to prove homotopy associativity of the product $f \cdot g \cdot h$ in ΩX from figure 2. By folding and distending appropriately, we can rearrange this into a 3-simplex with edges f, g, and h — again, the presence of this simplex in the bar complex is meant to record the associativity of the product on X, just as the diagram of homotopies was. Higher dimensional simplices then encode homotopies between these associativity homotopies, and so on.[9] Finally, there is a theorem stating that in the case of $\pi_0\underline{E}_1 = 0$, we get $B\underline{E}_0 = \underline{E}_1$. This is called "delooping" $\underline{E}_0$.

Now, we can iterate this story: not only is it true that $\underline{E}_0 = \Omega\underline{E}_1$, but we can further say that $\underline{E}_0 = \Omega^2\underline{E}_2$. Hence, we can say all these same things again, but with the interval and its subinterval replaced by the square $[0,1] \times [0,1]$ populated with sub-squares. Altogether, these configuration spaces of sub-squares are called the E_2-operad, the configuration spaces of intervals the E_1-operad,[10] and generally sub-n-cubes the E_n-operad. Since $\underline{E}_0 = \Omega^n\underline{E}_n$, we have that $\underline{E}_0$ is an algebra for each of these operads, so it is called an E_∞-space.

You may recognize the idea of sub-squares of $[0,1] \times [0,1]$ from the proof that $\pi_2 X$ is an abelian group [14, pg. 340] — this is not an accident, since double-loopspaces appear in that context via the identity $\pi_2 X = \pi_0\Omega^2 X$!

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[8] More precisely, this is the little intervals model of an A_∞-operad.

[9] To be completely honest about where these simplices come from, this description only holds for discrete G. To handle groups with topologies, you'll want to rephrase all this so that, rather than adding in a simplex for each n-tuple of elements, all n-tuples are handled simultaneously using cartesian products.

[10] So, at the level of our discussion here, there is an equality $E_1 = A_\infty$. More literally, A_∞ was originally defined by Stasheff in terms of combinatorial objects called "associahedra," which is a distinct presentation of the same operad for a suitable notion of operad equivalence.

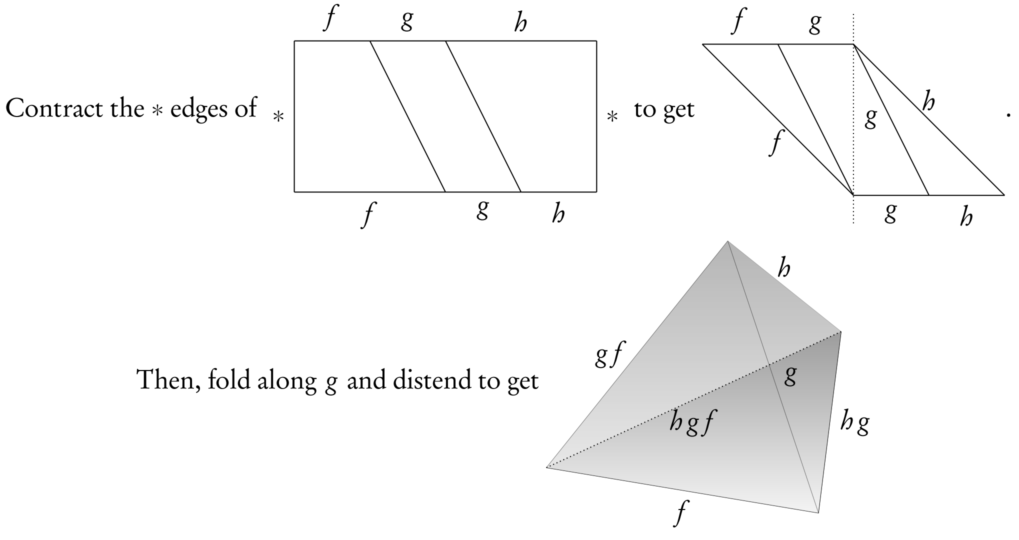

FIGURE 2. A 3-simplex arising from the A_3-action.

What this means for our discussion is that a nonabelian group G, while an E_1-space, cannot be an E_2-space (since then $\pi_2 B^2 G = \pi_0 \Omega^2 B^2 G = G$ is an abelian group). In particular, this means that the group G cannot occur as Ω^∞ of some spectrum E. On the other hand, since abelian groups G appear as $G = \underline{HG}_0$, they do carry an E_∞-space structure. The following statement is an important refinement of this distinction:

Applying B to an E_n-space results in an E_{n-1}-space [27].

The value of an E_∞ space, then, is that it has an infinite sequence of deloopings — after all, $\infty - 1 = \infty$. In fact, by inductively applying this delooping theorem, one sees that the assignment $E \mapsto \underline{E}_0$ is an equivalence of categories between connective spectra and E_∞-spaces. In this light, the functor $\Omega^\infty \Sigma^\infty X = \operatorname{colim}_n \Omega^n \Sigma^n X$ is using the functor Ω to find a sequence of approximations to X which are E_n-spaces as n grows.

Another cool characterization of these spaces is that an A_∞-algebra structure is exactly what arises when replacing a strictly associative group with a homotopy equivalent space, and in fact every A_∞-algebra arises this way — you can always find a strict group in its homotopy equivalence class. Curiously, this is not true for E_∞ spaces; there are E_∞-spaces for which no strictly commutative model exists. However, this is good news: this means that there's something to study in connective spectra that isn't an exact reflection of abelian group theory![11]

2. HOMEWORK: THE DUAL STEENROD ALGEBRA

No construction is worth much unless one can compute with it. In this section, we're going to compute the $H\mathbb{F}_2$-homology of the spaces $(\underline{H\mathbb{F}_2})_n = K(\mathbb{F}_2, n)$ by putting the two ideas from the main lecture — spectral sequences and delooping — next to each other. Altogether, $(H\mathbb{F}_2)_* \underline{H\mathbb{F}_2}_*$ is called the (unstable) dual Steenrod algebra. Our method is based off work of Ravenel and Wilson [38, 50], and there is a separate method due to Serre [41] that's part of every algebraic topologist's education. The odd-primary version of this computation also appears in Wilson's book [50, Theorem 8.5], though it is more complicated than the version here and the exposition is more terse than in the still more complicated Ravenel-Wilson paper [38].[12]

Before we start trying to fit the pieces of this problem together into a computation, we should first take stock of what the pieces even are. Suppose that E is a ring spectrum, meaning that there's a monoid map $E \wedge E \to E$. This

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[11] Two comments that came up in the seminar: the first is that A_∞-structures involve only the multiplication, so really this strict model business is saying we can build a strictly associative monoid. This doesn't tell you anything about how nicely behaved the inverse map is. The other comment is that A_∞-spaces are more than just loopspaces; an A_∞-space X with $\pi_0 X$ a group is a loopspace, but generally it's more complicated than that.

[12] Aaron thinks I should define cohomology operations here.

then induces maps of spaces $\underline{E}_n \wedge \underline{E}_m \to \underline{E}_{m+n}$, so if we further suppose that H_* is a homology theory satisfying $H_*(\underline{E}_n \wedge \underline{E}_m) = H_*\underline{E}_n \otimes_{H_*} H_*\underline{E}_m$, we can produce product maps

$$H_*\underline{E}_n \otimes H_*\underline{E}_m \to H_*(\underline{E}_n \wedge \underline{E}_m) \to H_*\underline{E}_{n+m}.$$

Further supposing that H is a ring spectrum, here's a complete list [38, Section 3] of the algebraic structures this yields on $H_*\underline{E}_*$:

maps on spaces		algebraic structure
multiplication on H	$\rightsquigarrow$	H_* is a ring, and $H_*\underline{E}_n$ is an H_*-module,
diagonal on $\underline{E}_n$	$\rightsquigarrow$	$\psi : H_p\underline{E}_n \to \bigoplus_{i=0}^{p} H_i\underline{E}_n \otimes H_{p-i}\underline{E}_n$,
addition on $\underline{E}_n$	$\rightsquigarrow$	$* : H_p\underline{E}_n \otimes H_q\underline{E}_n \to H_{p+q}\underline{E}_n$,
multiplication on E	$\rightsquigarrow$	$\circ : H_p\underline{E}_n \otimes H_q\underline{E}_m \to H_{p+q}\underline{E}_{n+m}$.

The final three maps are all H_*-module maps, and moreover the $\circ$-product distributes over the $*$-product in the sense that $x \circ (y * z) = \sum (x' * y) \circ (x'' * z)$ for $\psi x = \sum x' \otimes x''$. Altogether, this means that $H_*\underline{E}_*$ is a ring object in the category of H_*-comodules, or "Hopf ring". This greatly rigidifies the shape that $H_*\underline{E}_*$ can take; one important feature is that given just a small handful of explicitly described elements, we can generate a large number of other elements with all the operations we now have access to [50, pg. 46].

Now we set out to use these operations to give a Hopf ring presentation of $H_*\underline{E}_*$. The main tool comes to us as a spectral sequence [38, Section 2]: suppose first that we understand the homology $H_*\underline{E}_n$. If E is connective, then the space $\underline{E}_{n+1}$ can be modeled using the bar construction $B\underline{E}_n$, which comes with a filtration by subspaces $(B\underline{E}_n)_q$ consisting of those simplices with dimension at most q. The filtration quotients $(B\underline{E}_n)_q/(B\underline{E}_n)_{q-1}$ have a nice description: they are bouquets of q-spheres, labeled by q-tuples of points in G, meaning $(B\underline{E}_n)_q/(B\underline{E}_n)_{q-1} \simeq \Sigma^q \underline{E}_n^{\wedge q}$. This gives a filtration spectral sequence of algebras and we can identify its E^2-term as

$$E^2_{*,*} = \operatorname{Tor}^{H_*\underline{E}_n}_{*,*}(H_*, H_*) \Rightarrow H_*\underline{E}_{n+1}, \qquad d^r_{p,q} : E^r_{p,q} \to E^r_{p+r,q-r+1}.$$

Since H_* carries $\wedge$-products of spaces to $\otimes$-products of modules, H_* turns the bar construction on spaces into a bar construction on the algebras. An incredible and non-obvious feature of all this is that the $\circ$-product respects this filtration [38, Section 1], so we can act on the pages of the spectral sequence using it. This means that by writing classes x as $x = x' \circ x''$, we can propagate differentials in the previous spectral sequence acting on x' to new differentials acting on x by the formula $dx = d(x' \circ x'') = (dx') \circ x''$.

From here on out, we will strictly concern ourselves with $H = H\mathbb{F}_2$ and $E = H\mathbb{F}_2$; to remind you, our goal is to compute $H_*\underline{E}_*$. We will proceed by an induction, so we need some initial input. First, we recall the homology of $\mathbb{R}\mathrm{P}^\infty = \underline{E}_1$ from Hatcher [14, Theorem 3.12, Section 4.D]. As graded algebras, there is the description

$$H_*\mathbb{R}\mathrm{P}^\infty = \Gamma[a] = \bigotimes_{i \geq 0} T[a_{(i)}],$$

where $T[x]$ denotes the truncated polynomial algebra $\mathbb{F}_2[x]/\langle x^2 \rangle$, and $\Gamma[x] = \mathbb{F}_2\{1, x^{[1]}, x^{[2]}, \ldots\}$ denotes the divided power algebra with $x^{[i]}x^{[j]} = \binom{i+j}{i} x^{[i+j]}$, and where we write $a_{(i)}$ for $a_{(i)} = a^{[2^i]}$ with degree $|a_{(i)}| = 2^i$. The coproduct on $H_*\mathbb{R}\mathrm{P}^\infty$ is given by $\psi a^{[n]} = \sum_{i=0}^{n} a^{[i]} \otimes a^{[n-i]}$.

We also want to analyze the behavior of the bottom spectral sequence $\operatorname{Tor}^{\mathbb{F}_2[\mathbb{Z}/2]}_{*,*}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow H_*\mathbb{R}\mathrm{P}^\infty$ to be used as base case input to the induction. First, note that there's an isomorphism $\mathbb{F}_2[\mathbb{Z}/2] \cong T[\langle 1 \rangle]$, where $\langle 1 \rangle$ denotes the class $[1] - [0]$. Then, using the bar construction or any other method you like, we can compute $\operatorname{Tor}^{\mathbb{F}_2[\mathbb{Z}/2]}_{*,*}(\mathbb{F}_2, \mathbb{F}_2) \cong \Gamma[\sigma a]$, where σ denotes the suspension map, or $[a]$ in the bar construction. By counting classes, we see that the graded components of this Tor contain exactly as many elements as there are in $H_*\mathbb{R}\mathrm{P}^\infty$, and so it must be that the spectral sequence collapses at E^2, with no differentials.

We are now prepared to begin the inductive step. Our goal is to show that $H_*\underline{E}_{n+1}$ is generated freely by q-fold $\circ$-products. For an index $I = (I_1, \ldots, I_n)$, let's write $a_I = a_{(I_1)} \circ \cdots \circ a_{(I_n)}$, and suppose that we've shown

$$H_*\underline{E}_n = \bigotimes_{|I|=n} T[a_I].$$

The input for the next spectral sequence computing $H_*\underline{E}_{n+1}$ is then

$$\operatorname{Tor}_{*,*}^{H_*\underline{E}_n}(\mathbb{F}_2,\mathbb{F}_2) = \bigotimes_{|I|=n} \operatorname{Tor}^{T[a_I]}(\mathbb{F}_2,\mathbb{F}_2) = \bigotimes_{|I|=n} \Gamma[a_I].$$

We want to use the $\circ$-action to our advantage, which means writing $a_I^{[2^j]}$ in terms of a $\circ$-product. This is most conveniently done with a throw-away tool: define the Verschiebung V as the continuous linear dual of the Frobenius $x \mapsto x^p$ residing on the dual Hopf algebra — in words, V seeks terms of the form $x' \otimes \cdots \otimes x'$ in $\psi^{p-1}x$, i.e., it takes pth roots. This interpretation justifies the computation $V(a^{[p^{j+1}]}) = V(a^{[p^j]})$ and hence $V(a_{(i)}) = V(a_{(i-1)})$. What's more is that it too is compatible with the $\circ$-product: $V(a \circ b) = Va \circ Vb$.

Using these facts together, we can pull apart $a_I^{[2^j]}$ as $a_I^{[2^j]} \equiv a_{\tilde{I}}^{[2^j]} \circ a_{(I_n+j)}$ modulo the kernel of the jth iterate of the Verschiebung, which follows from

$$V^j((a_I)^{[2^j]}) = (a_I) = (a_{\tilde{I}}) \circ a_{I_n} = V^j((a_{\tilde{I}})^{[2^j]}) \circ V^j(a_{(I_n+j)}) = V^j((a_{\tilde{I}})^{[2^j]} \circ a_{(I_n+j)}),$$

where $\tilde{I}$ denotes $\tilde{I} = (I_1, \ldots, I_{n-1})$. Since this rewriting method applies to all the elements in our E^2-page, this means that all differentials in this spectral sequence arise by tacking on $-\circ a_{I_n+j}$ to an old differential[13] — but as the previous spectral sequence had no differentials, this one must not either! Hence, it collapses at E^2, and furthermore there are no additive extension problems since we're working over a field.[14] In all, this means that $H_*\underline{E}_{n+1} = \bigotimes_{|I|=n+1} T[a_I]$.

From the unstable description, we can further read off the stable algebra structure [50, Theorem 8.15]: writing $\xi_n = a_{(n)}$ and following the homology suspension maps shows that $(H\mathbb{F}_2)_* H\mathbb{F}_2 = \mathbb{F}_2[\xi_1, \xi_2, \ldots]$.[15] One thing we can't read off of this presentation is the coproduct structure. The formalism of the spectral sequence respects the coproduct, but we've been silently introducing correction terms to $a_I^{[2^j]}$ by identifying elements modulo the kernel of an iterate of the Verschiebung, so we've lost this information.[16] Even so, it's so important that we are obligated to record it. If you'd like something to puzzle over for next time, meditate on the meaning of this formula:

$$\psi \xi_n = \sum_{i=0}^{n} \xi_{n-i}^{2^i} \otimes \xi_i.$$

Where might you see it in the wild?

3. DAY 2: STABLE HOMOTOPY AND FORMAL LIE GROUPS

The higher homotopy groups of a space have an interesting stabilization property, due to Freudenthal: $\pi_p S^q = \pi_{p+1} S^{q+1}$ for $2q - 1 > p$, with a similar property for a general space X dependent upon its connectivity.[17,18] For comparison, the axioms for homology show that $E_p X = E_{p+1} \Sigma X$ for each p, completely independent of the connectivity of X. Our goal for the time being is to make sense of this early stabilization in homotopy groups, and to compare it to the general theory of homology.

Before we get into the thick of things, it will be helpful to have a second example of a homology theory: the theory of complex bordism, written MU. The way we'll explain it is a bit abstruse, but it's for a good cause — so hang on. Notice first that we can generalize our notion of suspension ΣX of a space X by considering instead a spherical fibration $S^n \to Y \to X$ — this is sort of a suspension of each point in X individually. Then, to make this into a suspension of X collectively, notice that there are two sorts of "coning off" operations we can perform: we can cone off each fiber on its own (yielding a disk bundle over X), and we can cone off the whole bundle at once. Performing both of these operations one after the other yields a new space TY, called the Thom space of

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[13] Inductively, this shows that you can pull the $-^{[2^j]}$ all the way inside, where it becomes a $\circ$-factor of $a_{(j)}$.

[14] There are also no multiplicative extension problems, because the Frobenius vanishes. You can see this just from the map 2 on $K(\mathbb{Z}/2, *)$ vanishing. See pg. 55 of Wilson [50].

[15] This method can be used to compute these algebras at other primes too, though the answers are more complicated. Stably, one finds $(H\mathbb{F}_p)_* H\mathbb{F}_p = \mathbb{F}_p[\xi_1, \xi_2, \ldots] \otimes \Lambda[\tau_1, \tau_2, \ldots]$, where $|\tau_i| = 2p^i - 1$.

[16] Is this true? Are you just being lazy?

[17] Aaron suggests drawing the suspension map.

[18] Cite Kevin's book.

Y [51, Theorem 20.5.4]. In the case that $Y = S^n \times X$, the associated Thom space TY is homeomorphic to the ordinary suspension $S^{n+1}X$, and when E is a more complicated bundle, TY is thought of as a twisted suspension of X. This construction turns out to be useful primarily because there are sources of spherical bundles lying around: for instance, one can take a vector bundle over a compact manifold and put a metric on it, giving a bundle of unit spheres and unit disks.

Last time, we talked about how to produce cohomology theories from spectra by the representability formula $E^nX = [X, \underline{E}_n]$. To produce a homology theory from a spectrum, the formula looks a bit different: $E_{-n}X = \pi_0(E \wedge \mathbb{S}^n \wedge X)$. There is a helpful analogy between the smash product of spectra and the tensor product of modules: applying $- \wedge E$ to a spectrum is like base-changing it to E-coefficients, as in replacing M with $M \otimes R$. To connect with the Thom construction, one can think of $E \wedge \mathbb{S}^n \wedge X$ as a (shifted) untwisted E-line-bundle over X. A spherical bundle Y over X is said to be E-oriented when there is an isomorphism $E \wedge \mathbb{S}^n \wedge X \simeq E \wedge TY$ — literally, E is "untwisting" the twisted suspension specified by the bundle. More generally, E is said to be oriented against C-bundles when it orients every spherical bundle in a collection C, like those coming from complex vector bundles. The spectrum MU is designed to be universal among those ring spectra which are complex-oriented — in case you didn't read the homework, a ring spectrum E is a spectrum which carries a monoid multiplication $\mu : E \wedge E \to E$ and a unit $\eta : \mathbb{S} \to E$. This means that any complex-oriented ring spectrum E receives a ring map $MU \to E$.

3.1. **Simplicial algebraic geometry.** Let's focus on drawing out controlling structure in homology functors, illuminated by the language of algebraic geometry. Suppose that E is a ring spectrum, so that E_* is a ring and E_*X is an E_*-module; the action of $e \in E_n$ on $x \in E_mX$ is given by

$$\mathbb{S}^{n+m} \simeq \mathbb{S}^n \wedge \mathbb{S}^m \xrightarrow{e \wedge x} (E) \wedge (E \wedge X) \xrightarrow{\mu \wedge 1} E \wedge X.$$

To get in the groove of using geometric language, an algebraic geometer would equivalently say that the homology functor $E_*(-)$ takes values in quasicoherent sheaves $\mathscr{E}(-)$ over the affine scheme $\operatorname{Spec} E_*$ [13, pg.111].[19]

That's all well and good, but it appears to buy us very little. In the example $E = H\mathbb{F}_2$, the scheme $\operatorname{Spec} \mathbb{F}_2$ is very small — too small to give us any interesting statements about the structure of its category of sheaves. On the other extreme is $E = MU$, where one computes [32, Theorem 6.5] that $MU_* = \mathbb{Z}[x_1, x_2, \ldots]$ has frighteningly many prime ideals — meaning that the scheme $\operatorname{Spec} MU_*$ is far too large to be tractable, especially when $\operatorname{Spec} \mathbb{Z}[x]$ alone is already an incredibly deep and challenging object in arithmetic geometry.

Both of these problems turn out to be fixed when we take homology cooperations into account. Suppose that X is a space with homology given by $E_*X = \pi_* E \wedge X$. Then, there is a map

$$E \wedge X \xrightarrow{\simeq} \mathbb{S} \wedge E \wedge X \xrightarrow{\eta \wedge 1 \wedge 1} E \wedge E \wedge X,$$

which under a certain flatness condition on E_*E [47, Theorem 17.8] yields on homotopy groups a map

$$E_*X \to \pi_*(E \wedge E \wedge X) \xleftarrow{\cong} E_*E \otimes E_*X,$$

a sort of E_*E-coaction on E_*X. To organize this under algebraic geometry, consider the following table, whose columns we will explain in turn:

$$\begin{array}{ccc}
\mathbb{S} \wedge E \xrightarrow{\eta \wedge 1} E \wedge E, & A \xrightarrow{\eta_L} \Gamma, & X_0 \xleftarrow{\text{dom}} X_1, \\
E \wedge \mathbb{S} \xrightarrow{1 \wedge \eta} E \wedge E, & A \xrightarrow{\eta_R} \Gamma, & X_0 \xleftarrow{\text{cod}} X_1, \\
E \wedge E \xrightarrow{\mu} E, & \Gamma \xrightarrow{\varepsilon} A, & X_1 \xleftarrow{\text{id}} X_0 \\
E \wedge \mathbb{S} \wedge E \xrightarrow{1 \wedge \eta \wedge 1} E \wedge E \wedge E, & \Gamma \xrightarrow{\Delta} \Gamma \otimes_A \Gamma, & X_1 \xleftarrow{\circ} X_1 \times_{X_0} X_1, \\
E \wedge E \xrightarrow{\tau} E \wedge E, & \Gamma \xrightarrow{\chi} \Gamma, & X_1 \xleftarrow{(-)^{-1}} X_1.
\end{array}$$

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[19] Aaron suggests explaining the correspondence between modules and quasicoherent sheaves.

The left column contains a small garden of maps of spectra we can write down using the ring structure on E. Then, setting $A = E_* = \pi_* E$ and $\Gamma = E_* E = \pi_* E \wedge E$, the maps in the middle column come from applying the functor π_* to the left column; the vague flatness we were assuming ago is that η_R is a flat map for $E_* E$ considered via η_L as an E_*-module.[20] Finally, setting $X_0 = \operatorname{Spec} A$ and $X_1 = \operatorname{Spec} \Gamma$, the right column are the same maps viewed under the functor Spec. This last column is where the most familiar structure is visible: these are exactly the maps you would want to make the pair (X_0, X_1) into a groupoid object called $\mathscr{M}_E$, and indeed this turns out to be the case. Dually, the structure in the middle column is said to make (A, Γ) into a "Hopf algebroid" [35, Appendix A.1]. As for our sheaf $\mathscr{E}(X)$ over $\operatorname{Spec} E_* = X_0$, the extra information in X_1 gives us a means to move around in X_0 by following arrows in the groupoid. The Γ-coaction on $E_* X$ is precisely the data needed to make $\mathscr{E}(X)$ compatible with this movement: it makes $\mathscr{E}(X)$ an X_1-equivariant sheaf.[21]

What turns out to be really interesting is $\mathscr{M}_{MU}$; it's sort of the example to rule all other examples. It turns out[22] that complex orientability of E is equivalent to requesting that there exist an isomorphism $E^*\mathbb{CP}^\infty \cong E^*[\![c_1]\!]$ for some generator $c_1 \in E^2\mathbb{CP}^\infty$, a generalized first Chern class. The Yoneda lemma says that the tensor product of line bundles induces a multiplication $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$, which in turn induces another multiplication $G_E := \operatorname{Spf} E^*\mathbb{CP}^\infty \cong \hat{\mathbb{A}}^1$ [45, Example 8.18]. This scheme G_E is called a (commutative, 1-dimensional) formal Lie group.[23] This gets its name from considering the Taylor expansion at the identity of a classical Lie group: one gets a power series, i.e., a function on a formal neighborhood of the identity, satisfying the various group axioms. One can construct a moduli $\mathscr{M}_{\mathrm{fg}}$ of formal Lie groups, and since MU itself is complex-oriented, there's a map $\mathscr{M}_{MU} \to \mathscr{M}_{\mathrm{fg}}$. A primal theorem in stable homotopy theory is that this is an isomorphism [32, Theorem 6.5]: X_0 for MU is the moduli of formal group laws and X_1 acts by coordinate changes.

This is really remarkable: we've found a functor $\mathscr{M}\mathscr{U}$ from the stable category to sheaves on the moduli of formal groups, which is some well-studied object in arithmetic geometry. One of the real prizes of stable homotopy theory is that we can also move in the other direction. To understand that statement, one of the great things about our definition of an A_∞-space is that we can swap out "space" for various other objects, like "ring spectrum." It turns out that MU is an A_∞-ring spectrum (as is $H\mathbb{F}_2$), meaning that we can form the cosimplicial spectra X and M with n-cosimplices given by

$$X_n = \overbrace{MU \wedge \cdots \wedge MU}^{n+1 \text{ copies}}, \qquad M_n = \overbrace{MU \wedge \cdots \wedge MU}^{n+1 \text{ copies}} \wedge X.$$

Note that X_0 and X_1, along with their cosimplicial structure maps, agree with those above. Applying π_* to the levels of these objects gives a cosimplicial module over a cosimplicial ring; we denote the associated sheaf over the associated simplicial scheme $\mathscr{M}\mathscr{U}(X)$ and $\mathscr{M}_{MU}$ respectively. In this way, we get a spectral sequence[24] — the Adams spectral sequence[25] — again arising from the skeletal filtration:

$$H^*(\mathscr{M}_{MU}; \mathscr{M}\mathscr{U}(X)) \Rightarrow \pi_* X^\wedge_{MU}.$$

Here, $X^\wedge_{MU}$ is the totalization of the cosimplicial spectrum M, called the MU-nilpotent completion of X. To make this useful, here are two more central theorems in stable homotopy theory:

1. One can identify $H^*(\mathscr{M}_{\mathrm{fg}}; \mathscr{M}\mathscr{U}(X))$ with $\operatorname{Ext}^{*,*}_{MU_*MU\text{-comods}}(MU_*, MU_*X)$, allowing computations [18, Proposition 11.6] [35, Appendix A.1].
2. There is a weak equivalence $\mathbb{S}^\wedge_{MU} \simeq \mathbb{S}$ [34, Example 1.16].

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[20] It is a healthy exercise to check this.

[21] Whole rest of this paragraph is COCTALOS.

[22] The primary thing in this argument is the splitting principle, which says that given any (complex) vector bundle $V \to X$, one can build a map $Y \to X$ which is injective on cohomology and along which V pulls back to a sum of line bundles.

[23] Expand this explanation.

[24] Haynes? If not Haynes, then Behrens.

[25] For MU, this is sometimes called the Adams-Novikov spectral sequence.

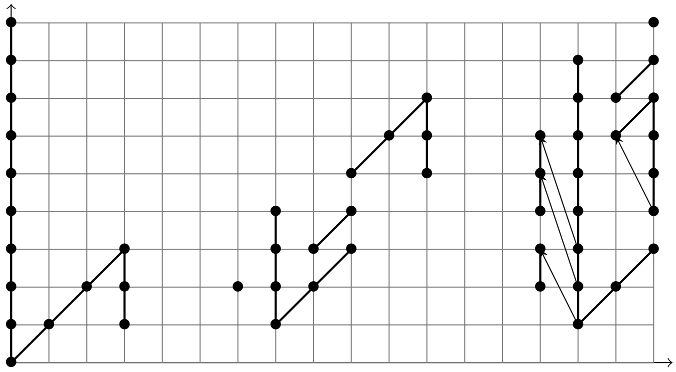

FIGURE 3. E_2-page for the $H\mathbb{F}_2$-Adams spectral sequence. Thin lines denote multiplication by 2 and by η, thin arrows denote differentials. **Label elements? Identify some groups?**

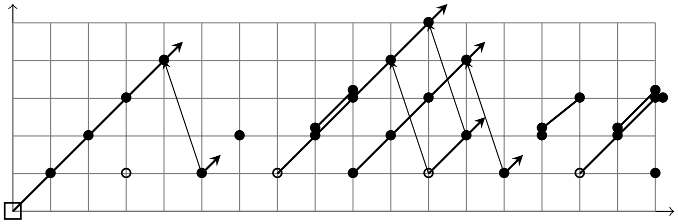

FIGURE 4. E_2-page for the 2-local MU-Adams spectral sequence. Thick lines denote multiplication by η, thin arrows denote differentials. **Label elements?**

Analogues of these facts are also true for $H\mathbb{F}_2$, where $\mathbb{S}^\wedge_{H\mathbb{F}_2} \simeq \mathbb{S}^\wedge_2$ is the 2-adic sphere. To help see what these spectral sequences do, look to figures 3 and 4 for pictures of their E_2-terms [33, pp. 412, 429]. Just their appearances are already fairly striking: the $H\mathbb{F}_2$-Adams spectral sequence

3.2. **Qualitative arithmetic geometry of $\mathscr{M}_{\mathrm{fg}}$.** In order to make serious use of the MU-Adams spectral sequence, we should collect as many facts as we can about $\mathscr{M}_{\mathrm{fg}}$. It will be helpful to work one prime at a time, so from here on localize everything at a prime p — ideally p would be also be odd, but that's not so important at this level of detail. Our first goal is to find a smaller presentation of $\mathscr{M}_{\mathrm{fg}}$; it turns out that the sheaf cohomology $H^*\mathscr{M}_{\mathrm{fg}}$ is sensitive only to the homotopy types of the groupoids output by this groupoid-scheme, and not their actual collections of objects and arrows — this is essentially asserting that we are interested in the stack cohomology of $\mathscr{M}_{\mathrm{fg}}$ [18, Theorem 12.1] [30, Section 1]. So, if we can find smaller presentations of the same groupoids, we'll stand a better chance of computing things with fewer elements flying around.

We approach this by considering the tools used to analyze the groups classified by $\mathscr{M}_{\mathrm{fg}}$. One common tool when working with an ordinary group G at a prime p is to study its p-torsion $G[p]$. For us, this corresponds to producing from a formal group law $x +_G y$ its p-series

$$[p]_G(x) = \overbrace{x +_G \cdots +_G x}^{p \text{ times}},$$

and considering the vanishing locus $G[p] = \operatorname{Spf} R[\![c_1]\!]/[p](c_1)$ of $[p](c_1)$. Over a field of characteristic p, it's possible to solve coefficient-by-coefficient for a coordinate-changing power series γ such that the p-series for $x +_G y$ transported along γ takes the form

$$[p]_{\gamma\cdot G}(x) = px +_G \sum_{q=1}^{\infty}{}_G v_q x^{p^q}.$$

This process is called p-typicalization, and we can use it to find the smaller presentation we want. It turns out that every p-typical curve is realizable as the p-series for some formal group, and so we may take $A = \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$ and Γ the moduli of invertible power series preserving p-typicality as our presentation [15, Theorem 15.2.9].

Of course, both A and Γ in this new presentation are infinitely generated, and so at a glance it may not seem like we've accomplished much of a reduction. In fact, we've gained a lot: let's call the first q such that v_q is nonzero the "height" of G, which roughly encodes the size of $G[p]$. The simultaneous vanishing locus of v_i for $i < q$ determines a closed substack $\mathcal{M}_{\mathrm{fg}}^{\geq q}$ of $\mathcal{M}_{\mathrm{fg}}$ of those formal groups of height at least q, corresponding to the ideal $I_q = \langle p, \ldots, v_{q-1}\rangle$. Despite both our original and our new presentations having an enormous number of prime ideals, it turns out that these ideals I_q are the only ones which are closed under the Γ-coaction — meaning $\mathcal{M}_{\mathrm{fg}}$ has a unique closed substack of codimension q for each q, each contained in the next [50, Theorem 4.9]. This is quite informative as to how the geometry of this object behaves.

3.3. **Reflection in topology.** From here, we can potentially spend our time in two ways. One is to compute as much as we can of $H^*\mathcal{M}_{\mathrm{fg}}$ and embark on a detailed study of the MU-Adams spectral sequence in order to make elementwise statements about the stable homotopy groups of spheres [35, Chapters 4, 5]. This is now possible, given the smaller presentation, but it is very hard and somehow very old school. Of course, it's also very rewarding since we directly get the information we're after, and so we'll spend some time discussing this in the next homework reading. Another approach we might take is to bank on the notion that the stable homotopy category and the category of sheaves on $\mathcal{M}_{\mathrm{fg}}$ are very tightly connected. Following this idea, we might try to reproduce the tools used in the study of $\mathcal{M}_{\mathrm{fg}}$ inside of the stable homotopy category itself.

The first result in this direction is that our smaller presentation has a realization in topology: there is a map $MU \to MU$ selecting a formal group law whose p-series is the universal p-typical p-series given above [31, Theorem 4]. The image of this map is a ring spectrum BP, called Brown-Peterson theory, with $BP_* = A$ and $BP_*BP = \Gamma$ for the smaller A and Γ we found. The next result is the Landweber exact functor theorem, which states that if $\operatorname{Spec} R \to \mathcal{M}_{\mathrm{fg}}$ is a flat map of stacks, then the pullback of the sheaf $\mathcal{MU}(X)$ to $\operatorname{Spec} R$ defines a homology theory (and, in particular, $\operatorname{Spf} R^*\mathbb{C}\mathrm{P}^\infty$ is the classified formal group) [18, Theorem 21.1].[26] Recall that we have at hand some interesting substacks of $\mathcal{M}_{\mathrm{fg}}$:

$$\begin{aligned}
\mathcal{M}_{\mathrm{fg}}^{\geq q} &= (\operatorname{Spec} A/I_q)//\operatorname{Spec}\Gamma, \\
\mathcal{M}_{\mathrm{fg}}^{q} &= (\operatorname{Spec} v_q^{-1}A/I_q)//\operatorname{Spec}\Gamma, \\
\mathcal{M}_{\mathrm{fg}}^{<q} &= (\operatorname{Spec} v_{q-1}^{-1}A/\langle v_q, v_{q+1}, \ldots\rangle)//\operatorname{Spec}\Gamma.
\end{aligned}$$

These stacks are closed, relatively closed, and open respectively.[27] Open immersions are flat, so restriction to $\mathcal{M}_{\mathrm{fg}}^{<q}$ yields a cohomology theory $E(q-1)$, called Johnson-Wilson theory, with coefficient ring $\mathbb{Z}_{(p)}[v_1, \ldots, v_{q-2}, v_{q-1}^{\pm}]$. This open substack is thought of as the complement to its closed partner, in the sense that if a sheaf vanishes on it then the sheaf's support must lie in $\mathcal{M}_{\mathrm{fg}}^{\geq q}$.[28] The substack $\mathcal{M}_{\mathrm{fg}}^{q}$ can also be realized by coning off each of the v_i from $E(q)$ in turn, yielding $K(q)$ with $K(q)_* = \mathbb{F}_p[v_q^{\pm}]$, called Morava K-theory.

As a quick aside, this same program also explains the description of the diagonal on the Steenrod algebra given in the previous homework [18, pp. 22-23]. Since $H\mathbb{F}_2$ is complex oriented, this whole discussion informs us that we should seek out power series preserving the formal group $G_{H\mathbb{F}_2}$, i.e., power series ξ satisfying $\xi(s+t) = \xi(s) + \xi(t)$. These are exactly those power series of the form $\xi(t) = \sum_{t=0}^{\infty} \xi_i t^{2^i}$, with the convention that $\xi_0 = 1$. They obey the

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[26] This is typically stated in terms of "regular sequences": when R is a p-local ring, then this is equivalent to asserting that multiplication by v_n on R/I_n is a monomorphism.

[27] Constantin would like to know why this last one is open.

[28] What exactly is this acyclicity statement? What's a reference?

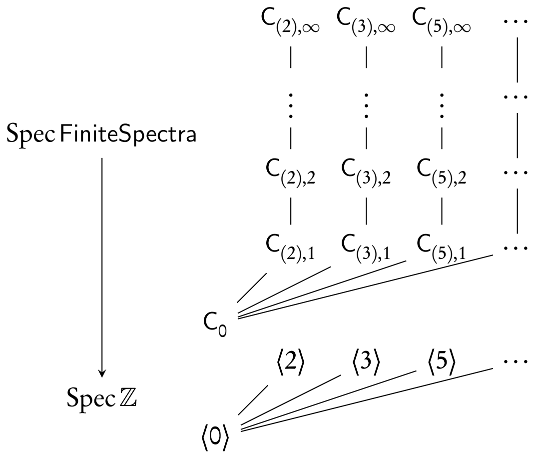

FIGURE 5. A diagram of the points in Spec FiniteSpectra. Lines indicate that the above ideals are contained in the closure of the bottom one.

composition law

$$\xi(\zeta(t)) = \sum_{i=0}^{\infty} \xi_i \left(\sum_{j=0}^{\infty} \zeta_j t^{2^j} \right)^{2^i} = \sum_{i=0}^{\infty} \xi_i \left(\sum_{j=0}^{\infty} \zeta_j^{2^i} t^{2^{i+j}} \right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \xi_k \zeta_{n-k}^{2^k} \right) t^{2^n},$$

and after some rearranging the formula for the Steenrod diagonal appears.

With these cohomology theories in hand, we can also state result mimicking the ideals I_q in the p-local finite stable homotopy category. A full subcategory $\mathsf{C} \subseteq$ FiniteSpectra will be said to be...[29]

- ... thick if it is closed under weak equivalences, retracts, and cofiber sequences.
- ... an ideal if $x \wedge y$ is in C for each $x \in \mathsf{C}$ and $y \in$ FiniteSpectra.
- ... a prime ideal if $x \wedge y \in \mathsf{C}$ forces at least one of x or y to lie in C.

The homology theory $K(q)_*$ determines a functor from FiniteSpectra to the $\otimes$-category of graded vector spaces over the ground field $K(q)_*$. The "kernel" of this functor — the full subcategory spanned by $K(q)$-acyclics — forms a thick prime ideal denoted C_q. It turns out that there is a proper inclusion $\mathsf{C}_{q+1} \subsetneq \mathsf{C}_q$, and that this is a complete enumeration of all such ideals — they all appear as $K(q)$-acyclics for some q. With this, we can draw a picture of the spectrum of FiniteSpectra; see figure 5 [6, Section 9].

There are a number of further characterizations of these categories C_q, all conjectured by Ravenel [34, Section 10] and most proven by Devinatz, Hopkins, and Smith [9, 20, 37], collectively called the nilpotence conjectures. All of them are interesting, but here is the most interesting one [20, Theorem 9]: a finite spectrum F belongs to C_q if and only if F admits a map $v : \Sigma^{2p^N(p^q-1)} F \to F$ satisfying

$$K(m)_* v = \begin{cases} v_q^{p^N} & \text{if } m = q, \\ 0 & \text{otherwise} \end{cases}$$

for some $N \gg 0$. This is the beginning of a deep and interesting periodicity phenomenon in stable homotopy theory, explored more fully in tonight's homework reading.

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[29] I recall an old Demazure reference that described the geometric space associated to a category sufficiently near to a category of modules. That would make a good reference.

4. HOMEWORK: THE CHROMATIC SPECTRAL SEQUENCE

Let's use the descending chain of substacks $\mathcal{M}_{\mathrm{fg}}^{\geq q}$ we found to organize the computation of $H^*\mathcal{M}_{\mathrm{fg}}$. First, we can rephrase this problem in terms of group cohomology, which will be slightly more manageable. Taking global sections of $\mathcal{M}\mathcal{U}(X)$ along with its X_1-action gives a module $M = \mathcal{M}\mathcal{U}(X)(X_0) = MU_*X$ together with an action by $G := X_1(X_0)$. Specifically, the action of $g \in G$ is given by the composite

$$MU_* \xrightarrow{\text{coact}} MU_* \otimes_{MU_*} MU_*MU \xrightarrow{1\otimes g} MU_* \otimes_{MU_*} MU_* \to MU_*.$$

A theorem of Morava [30, Section 1] says that the sheaf cohomology $H^*\mathcal{M}_{\mathrm{fg}}$ is isomorphic to the group cohomology $H^*(G;M)$. Computing group cohomology is of course still very hard, but at least we can say what $H^0(G;M)$ is: the G-invariants in M. However, one quickly shows that this is just $\mathbb{Z}_{(p)}$, which is not much — the only nonunit present is p.

Our goal is to produce all the elements of cohomology that we can in order to populate the spectral sequence. Having found a way to produce p, we can delete it and see if we can find something else. This is accomplished using the short exact sequence

$$M \xrightarrow{\cdot p} M \to M/p.$$

Taking group cohomology, this induces a going-around map $H^0(G;M/p) \to H^1(G;M)$, and so we're tasked with computing more invariants. It turns out that $H^0(G;M/\langle p\rangle) = \mathbb{F}_p[v_1]$, and so we apply the connecting homomorphism to produce potential new elements $\alpha_m = \partial v_1^m$ in $H^1(G;M)$. This pattern continues: there's a sequence

$$M/I_{q-1} \xrightarrow{\cdot v_q} M/I_{q-1} \to M/I_q$$

and a calculation $H^0(G;M/I_q) = \mathbb{F}_p[v_q]$, yielding "Greek letter" elements $\alpha_m^{(q)} = \partial \cdots \partial v_q^m \in H^q(G;M)$, where $\alpha^{(q)}$ is a stand-in for the qth Greek letter. With slightly more generality, we can consider global functions on an infinitesimal thickening of $\mathcal{M}_{\mathrm{fg}}^{\geq q}$ in the parent stack $\mathcal{M}_{\mathrm{fg}}$; these correspond to regular ideals

$$I_q^J = \langle p^{J_0}, v_1^{J_1}, \ldots, v_{q-1}^{J_{q-1}} \rangle.$$

The same procedure with the connecting homomorphisms applies to give elements called $\alpha_{m/J}^{(q)}$.[30] In all, these constructions give us access to a large number of potential elements in $H^*(G;M)$ [35, Theorem 5.1.3, 5.1.19].

Having uncovered this much, we now try to organize our procedure with the hope of incorporating more than the elements $\alpha_{m/J}^{(q)}$. Taking a colimit over J_q in the individual quotient sequences yields a parent sequence

$$M/I_{q-1}^\infty \to v_q^{-1}M/I_{q-1}^\infty \to M/I_q^\infty$$

containing all their information. Applying group cohomology to these new short exact sequences and then splicing the results together yields the chromatic spectral sequence [50, 4.59] [29, Section 3], with signature

$$E_1^{*,*,q} = H_{\mathrm{group}}^{*,*}(G; v_q^{-1}M/I_q^\infty) \Rightarrow H_{\mathrm{group}}^{*,*}(G;M) = H_{\mathrm{sheaf}}^{*,*}\mathcal{M}_{\mathrm{fg}}.$$

To recap, we've carved up the problem of computing $H^*\mathcal{M}_{\mathrm{fg}}$ into computing the cohomologies of various other sheaves and stitching them together via a spectral sequence. It would be nice to better understand exactly what it is we're computing as input to this spectral sequence, i.e., to study the group cohomology appearing at a fixed $q = Q$. It would in particular be nice to see that we have in fact made the problem simpler.

Well, $M_q = v_q^{-1}M/I_q^\infty$ is an ideal sheaf on $\mathcal{M}_{\mathrm{fg}}$ selecting the universal infinitesimal thickening of $\mathcal{M}_{\mathrm{fg}}^{\geq q}$, together with the condition that we consider only points of height exactly q. This turns out to agree with the universal deformation space $(E_q)_*$ of H_q, the Honda formal group over $\operatorname{Spec} \mathbb{F}_{p^q}[v_q^{\pm}]$ characterized by its p-series $[p]_{H_q}(x) = v_q x^{p^q}$.[31] Acting by changes of coordinate corresponds to acting by the automorphisms of H_q: each automorphism of the deformation restricts to an automorphism of H_q, and each such automorphism of H_q extends

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[30] To get a sense of what these elements are, the element $\alpha_{m/j}$ satisfies $p\alpha_{m/j} = \alpha_{m/(j-1)}$.

[31] ???

to an automorphism of $(E_q)_*$ by universality [44, Section 24]. These lifted automorphisms turn out to be easy to characterize: they correspond to the units $\mathbb{S}_q$ of $\mathscr{O}_q$, the noncommutative polynomial ring

$$\mathscr{O}_q = \mathbb{W}_{\mathbb{F}_{p^q}}\langle S\rangle / \left(\begin{array}{c} Sw = w^{\phi}S, \\ S^q = p \end{array} \right),$$

where ϕ denotes a lift of the Frobenius on $\mathbb{F}_{p^q}$ to the ring of Witt vectors $\mathbb{W}\mathbb{F}_{p^q}$ and the element S acts on H_q by $x \mapsto x^p$ [35, Theorem A2.2.17]. This group $\mathbb{S}_q$ is called the qth Morava stabilizer group, and in all gives the identification

$$H^{*,*}(G; v_q^{-1}M/I_q^{\infty}) = H^{*,*}(\mathbb{S}_q; (E_q)_*).$$

This description turns out to be very powerful; the most important thing is that multiplication by v_q in $(E_q)_*$ is invertible, which is the primary fountainhead of periodicity results. Following on from the realization results from the main lecture, the deformation space $(E_q)_*$ is also Landweber exact as an MU-algebra, and so it yields a cohomology theory E_q called Morava E-theory [21, Section 1.1].[32]

5. DAY 3: COMPUTATIONS WITH THE EHP SPECTRAL SEQUENCE

I've been pushing the intricate computational parts of algebraic topology into the reading so far, but because it's such a crucial part of what it feels like to actually do work in this field, I wanted to spend at least one day doing a computation together. In the lectures, I'll be drawing a lot of intermediate stages on the board. For those of you reading at home, it will be extremely helpful to get a sheet of graph paper to keep track of where you are in the computation and which groups are under discussion.

5.1. Basic facts.

Today we'll consider how we can use what we've learned so far to produce unstable homotopy groups of spheres. There's one major fact that we're going to assume: there exists [22] a 2-local fibration

$$S^q \xrightarrow{E} \Omega S^{q+1} \xrightarrow{H} \Omega S^{2q+1}.$$

The map E is adjoint to identity on ΣS^q via the exponential adjunction. The map H denotes something called the Hopf invariant — the exact action of the map won't be so important. We have one of these for each $q \in \mathbb{N}$, and we can put them together into an unrolled exact couple like so:

$$\begin{tikzcd}
\mathrm{pt} \arrow[r] & \Omega S^1 \arrow[r] \arrow[d] & \Omega^2 S^2 \arrow[r] \arrow[d] & \Omega^3 S^3 \arrow[r] \arrow[d] & \cdots \arrow[r] & \Omega^q S^q \arrow[r] \arrow[d] & \Omega^{q+1} S^{q+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & QS^0 \\
 & \Omega S^1 \arrow[ul] & \Omega^2 S^3 \arrow[ul] & \Omega^3 S^5 \arrow[ul] & \cdots \arrow[ul] & \Omega^q S^{2q-1} \arrow[ul] & \Omega^{q+1} S^{2q+1} \arrow[ul] & \cdots . \arrow[ul] &
\end{tikzcd}$$

The going-around map in each triangle is denoted P, for Whitehead product.[33] Applying π_* to this diagram yields a spectral sequence, creatively titled the EHP spectral sequence[34], with signature

$$E^1_{p,q} = \pi_{p+q} S^{2q-1} \Rightarrow \pi^S_p, \qquad\qquad d^r : E^r_{p,q} \to E^r_{p-1,q-r}.$$

A great many facts about the homotopy groups of spheres can be written into the organizational context of this extremely interesting spectral sequence. Let's begin with the most basic one we started with on the very first day: we computed the homotopy groups of S^1. This means that we know about the bottom row of the EHPSS:

$$E^1_{p,1} = \begin{cases} \mathbb{Z} & \text{when } p = 0, \\ 0 & \text{otherwise.} \end{cases}$$

---

[32] There must be a reference that just covers Landweberness...

[33] VFOS notes, maybe.

[34] Sometimes "EHPSS," for maximum compactness.

Putting together H_*S^{2q-1} and the Hurewicz theorem, we also know two more facts:

$$E^1_{p,q} = \begin{cases} 0 & \text{when } p < q-1, \\ \mathbb{Z} & \text{when } p = q. \end{cases}$$

5.2. **The orthogonal spectral sequence.** Now that we have some classes column-adjacent to each other, the next thing to do is to try to produce differentials in this spectral sequence. The easiest way to produce differentials in any spectral sequence is through sparseness — if we know that the spectral sequence has to play out in some particular way, this often forces the existence of a certain pattern of differentials. Our spectral sequence is not at all sparse, but we can import differentials in by finding another spectral sequence with a map to (or, more difficultly, from) this one [35, Theorem 1.5.13].

To produce a map of spectral sequences, we should look for a sequence of maps $X \to \Omega^q S^q$, i.e., some source of based self-maps of S^q. One way to produce *unbased* maps of S^{q-1} is to begin with an orthogonal automorphism $M \in O(q)$ of $\mathbb{R}^q$, then use M to get a map $S(\mathbb{R}^q) \to S(\mathbb{R}^q)$ from the unit sphere to itself. Taking a single suspension of this map gives a based map $S^q \to S^q$, and this process assembles into a continuous map $O(q) \to \Omega^q S^q$. It turns out that this extends to a map of fiber sequences:

$$\begin{tikzcd}
O(q) \arrow[r] \arrow[d] & O(q+1) \arrow[r] \arrow[d] & S^q \arrow[d] \\
\Omega^q S^q \arrow[r] & \Omega^{q+1} S^{q+1} \arrow[r] & \Omega^{q+1} S^{2q+1}.
\end{tikzcd}$$

These top sequences fit together to give an unrolled exact couple, and taking homotopy again induces a spectral sequence, called the orthogonal spectral sequence, with signature

$$E^1_{p,q} = \pi_p S^q \Rightarrow \pi_p O, \qquad\qquad d^r : E^r_{p,q} \to E^r_{p-1,q-r}.$$

The map of spectral sequences converges to a map $J : \pi_n O \to \pi_n \mathbb{S}$, called the J-homomorphism [35, Theorem 1.1.12].

Let's work to produce a differential in the orthogonal spectral sequence. We also have another bunch of commuting squares

$$\begin{tikzcd}
 & & S^q \arrow[dd, equal] \\
\mathbb{R}\mathrm{P}^{q-1} \arrow[r] \arrow[dd] & \mathbb{R}\mathrm{P}^q \arrow[ur] \arrow[dd] & \\
 & & S^q \\
O(q) \arrow[r] & O(q+1), \arrow[ur] &
\end{tikzcd}$$

where the map $\mathbb{R}\mathrm{P}^{q-1} \to O(q)$ is given by sending a point $x \in \mathbb{R}\mathrm{P}^{q-1}$ to the reflection along the corresponding line in $\mathbb{R}^q$. However, the top sequence is a cofiber sequence, so it does not give a spectral sequence in homotopy, and hence we have to be more creative about how we make use of it. Both of the rows can be rotated once to the left, the top sort of by accident and the bottom because it's a fiber sequence, to produce

$$\begin{tikzcd}
S^{q-1} \arrow[dr] \arrow[dd, dotted] & & & S^q \arrow[dd, equal] \\
 & \mathbb{R}\mathrm{P}^{q-1} \arrow[r] \arrow[dd] & \mathbb{R}\mathrm{P}^q \arrow[ur] \arrow[dd] & \\
\Omega S^q \arrow[dr] & & & S^q \\
 & O(q) \arrow[r] & O(q+1). \arrow[ur] &
\end{tikzcd}$$

The miracle is that the dashed map exists, making the square commute: it is the inclusion of the bottom cell. We want to use this to say something about the d^1-differential in the orthogonal spectral sequence, so we stitch in the next rectangle to put everything in one picture:

$$\begin{tikzcd}
 & S^{q-1} \arrow[rr] \arrow[dd] \arrow[dr] & & S^{q-1} \arrow[dd, equal] & & S^q \arrow[dd, equal] \\
\mathbb{RP}^{q-2} \arrow[rr, crossing over] \arrow[dd] & & \mathbb{RP}^{q-1} \arrow[ur] \arrow[rr, crossing over] \arrow[dd] & & \mathbb{RP}^q \arrow[ur] \arrow[dd] & \\
 & \Omega S^q \arrow[rr, crossing over] \arrow[dr] & & S^{q-1} & & S^q \\
O(q-1) \arrow[rr] & & O(q) \arrow[rr] \arrow[ur] & & O(q+1). \arrow[ur] &
\end{tikzcd}$$

The two new horizontal maps in the back are, respectively, $1+(-1)^{q-1}$ and the d^1-differential in the orthogonal spectral sequence. From this, we can see that the action of d^1 in the orthogonal spectral sequence on the group $E^1_{p+1,q} = \pi_{p+1}S^q \to \pi_p S^{q-1} = \pi_p E^1_{p,q-1}$ is given by multiplication by $1+(-1)^{q-1}$ — the degree of the attaching map in the CW presentation of $\mathbb{RP}^\infty$, also called the cellular differential. In turn, this maps to the EHPSS to induce the same pattern of differentials there, finally resulting in the computation $E^2_{2,2} = E^\infty_{2,2} = \mathbb{Z}/2$. There is nothing else in the $q = 2$ column, so this means $\pi^S_2 = \mathbb{Z}/2$.

5.3. **Stabilization feedback.** The next major feature of the EHPSS comes from the Freudenthal suspension theorem, mentioned at the very beginning of the previous talk.[35] This theorem implies that $E^1_{p,q} = \pi^S_{p-q}$ for $p < 3q-3$. In turn, this means that the diagonal of $\mathbb{Z}$ entries in the EHPSS is part of a larger pattern: there is an infinite sequence of diagonals, each beginning slightly farther out than the one previous. We just computed $\pi^S_2 = \mathbb{Z}/2$, and so we can propagate this result along that diagonal to fill out many entries in the spectral sequence. In turn, this completes the $p = 2$ column. One must show that the are no differentials entering this column, but it turns out that there aren't any, which means $\pi^S_3 = \mathbb{Z}/2$, filling out another stable ray.

5.4. **Truncation feedback.** We now come to an important feature of the spectral sequence that allows us some access to unstable information. Rather than considering the full filtration producing the EHPSS as above, we can instead stop the filtration at a selected level Q;[36] this produces a spectral sequence of signature

$$E^1_{p,q} = \begin{cases} \pi_{p+q}S^{2q-1} & \text{when } q \le Q, \\ 0 & \text{when } q > Q, \end{cases} \Rightarrow \pi_{p+Q}S^Q \qquad\qquad d^r : E^r_{p,q} \to E^r_{p-1,q-r}.$$

Setting $Q = 3$ shows that the $p = 2$ column converges to $\pi_5 S^3$ as well, so that $\pi_5 S^3 = \mathbb{Z}/2$. This group appears in the spectral sequence as $E^1_{3,2}$, which completes the $p = 3$ column. Another application is that we can set $Q = 2$; since the row $q = 1$ is largely empty, this means all but the bottom of the unstable groups $\pi_{2+*}S^2$ appear as $E^1_{*,2}$, and so we can begin to read off the unstable homotopy groups of S^2 as we go along.[37] One cool thing that can happen when truncating the spectral sequence is that we can delete the source of a differential, meaning that what was target of the differential can accidentally survive. This accounts for the spurious classes seen in the unstable homotopy groups of spheres, which grow (as Q increases) and shrink (as differentials come into play) before settling down at the stable stems.

Next, using the computation of π^S_3 from the ANSS, one can check two things: first, there are no differentials entering the $p = 3$ column. This means that $\pi_6 S^3$ is an extension of $\mathbb{Z}/2$ by $\mathbb{Z}/2$ and that π^S_3 is an extension of that group by yet another $\mathbb{Z}/2$. The second thing is that $\pi^S_3 = \mathbb{Z}/8$ forces the extension determining $\pi_6 S^3$ to be nontrivial, so that $\pi_6 S^3 = \mathbb{Z}/4$. This gives another group in the $q = 2$ row, $E^1_{4,2}$, and the next stable diagonal ray.

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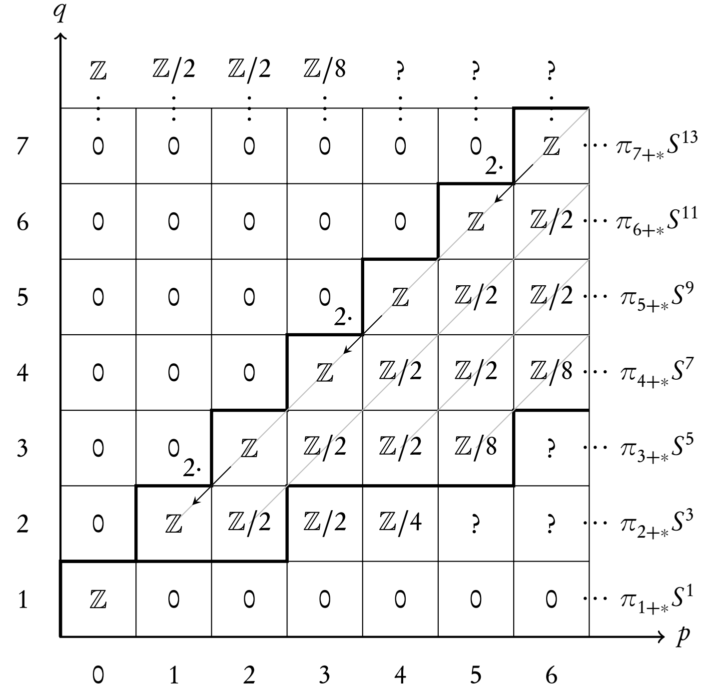
[35] Kevin's book again.

[36] Green book, VFOS

[37] Does the isomorphism $\pi_p S^2 = \pi_p S^3$ force the nonexistence of classes in the EHPSS $q = 2$ line? Or the existence of differentials?



	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8$	?	?	?	
7	0	0	0	0	0	0 2·	$\mathbb{Z}$	$\pi_{7+*}S^{13}$
6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\pi_{6+*}S^{11}$
5	0	0	0	0 2·	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\pi_{5+*}S^{9}$
4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8$	$\pi_{4+*}S^{7}$
3	0	0 2·	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8$	?	$\pi_{3+*}S^{5}$
2	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/4$	?	?	$\pi_{2+*}S^{3}$
1	$\mathbb{Z}$	0	0	0	0	0	0	$\pi_{1+*}S^{1}$
	0	1	2	3	4	5	6	

FIGURE 6. The EHPSS as understood in this discussion.

The story continues on from here; there's even more structure in the EHPSS that we haven't discussed that gives access to, for instance, further differentials. It's worth remarking on two extra things, apart from what we've talked about so far. First is that we actually haven't had to invoke homology computations at any point if we didn't want to. At the very beginning we used Hurewicz to compute $\pi_n S^n$, but we could have instead used the Hopf fibration to compute $\pi_2 S^2 = \mathbb{Z}$. This in turn forces $\pi_3 S^3 = \mathbb{Z}$ in order for the EHPSS truncated at $Q = 2$ to converge to $\pi_{2+*}S^2$ correctly. Then, $\pi_3 S^3 = E^1_{1,2}$ lies in the stable range, which gives us the other classes we desired. The computation of π^S_3 answered an extension problem for us, and this is generally what the stable information is useful for, but somehow all the classes are already present in the spectral sequence, waiting to be discovered.

5.5. **The Serre finiteness theorem.** The second remark is that we have made enough statements about the EHPSS to prove the Serre finiteness theorem: all of (the 2-primary components of) the homotopy groups of spheres are finitely generated, $\pi_n S^n$ and $\pi_{4n-1}S^{2n}$ each contain a single infinite summand, and all other groups are torsion. Noticing that differentials have destroyed all the infinite groups in the first stable diagonal by the E^2-page, this means that the feedback feature of the EHPSS we've been using to fill out the bottom rows will only draw from extensions of the finite groups we've already calculated in the bottom-left. The only exception is the group $\pi_{4n-1}S^{2n}$, which appears when we truncate the spectral sequence across one of the d^1-differentials we calculated.

6. HOMEWORK: THE HOMOTOPY OF THE $K(1)$-LOCAL SPHERE

Last week we discussed the chromatic viewpoint of stable homotopy theory, and in the previous homework reading we constructed the chromatic spectral sequence, used to compute the E_2-page of the MU-Adams spectral sequence. One thing we can do to understand how the figures in chromatic homotopy theory interact with each other is to blank out large parts of the chromatic spectral sequence and play only with what's left over. When we set $E_1^{p,q} = 0$ for all q greater than some fixed $q = Q$, the result is called the homotopy of the "$E(Q)$-local sphere."[38] When we set $E_1^{p,q} = 0$ for every q other than $q = Q$, the result is called the homotopy of the "$K(Q)$-local sphere" [19,

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[38] Cite the CSS for the $E(Q)$-ASS.

Proposition 7.4]. Our goal in this reading is to compute the homotopy of the $K(1)$-local sphere, roughly following Lurie's method [24, Lecture 35].

There is a deep and mysterious theory called Bousfield localization that assigns actual homotopy types to these groups so that we can do various spectrum-level manipulations, but we won't have time to get into it. Instead, we will just take the following two facts for granted:

- *Chromatic fracture squares*: There is a pullback square

$$\begin{tikzcd} L_{E(1)}\mathbb{S} \arrow[r] \arrow[d] & L_{K(1)}\mathbb{S} \arrow[d] \\ L_{E(0)}\mathbb{S} \arrow[r] & L_{E(0)}L_{K(1)}\mathbb{S}. \end{tikzcd}$$

  In fact, replacing 1 with q and 0 with $(q-1)$, there is a fracture square for each q [24, Proposition 23.5].
- *Arithmetic localization*: There is an equivalence $E(0) = H\mathbb{Q}$, and localization at $H\mathbb{Q}$ has the effect of tensoring all homotopy groups with $\mathbb{Q}$ [46, Chapter 2].

The analysis of the homotopy of the $E(1)$-local sphere proceeds in phases. By far the most difficult is to compute $\pi_* L_{K(1)}\mathbb{S}$, and by far the easiest is to compute $\pi_* L_{E(0)}\mathbb{S}$. We know that $\pi_* L_{E(0)}\mathbb{S}$ agrees with the homotopy of the rational sphere, and the full Serre finiteness theorem states that all the stable homotopy groups of spheres above π_0^S are torsion. Hence, we compute that it is $\mathbb{Q}$ concentrated in degree 0 and vanishing otherwise, and that part of the work is finished.

To study the $K(1)$-local sphere via the truncated chromatic spectral sequence, we see that we're tasked with computing the (continuous) group cohomology $H^*(\mathbb{S}_1;(E_1)_*)$. The deformation space $(E_q)_*\mathbb{Z}_p^\wedge[\![u_1,\ldots,u_{q-1}]\!][u^\pm]$ specializes to give $(E_1)_* = \mathbb{Z}_p^\wedge[u^\pm]$. The stabilizer group is less obvious; recall that $\mathbb{S}_q$ appears as the group of units in the noncommutative polynomial ring

$$\mathscr{O}_q = \mathbb{W}_{\mathbb{F}_{p^q}}\langle S\rangle / \begin{pmatrix} Sw = w^\phi S, \\ S^q = p \end{pmatrix}.$$

In the case that $q = 1$, we're left with $\mathbb{S}_1 = \mathbb{G}_m\mathbb{Z}_p^\wedge$. Then, $(E_1)_*$ is made into an $\mathbb{S}_1$-module by the action $g \cdot u^d = g^d u^d$ for $g \in \mathbb{S}_1$ and $u^d \in (E_1)_*$. If we restrict attention to a prime $p > 2$, then there is a splitting $\mathbb{S}_1 = C_{p-1} \times \mathbb{Z}_p^\wedge$, consisting of the $(p-1)^{th}$ roots of unity and those p-adic units for which a logarithm exists. Using the Lyndon-Hochschild-Serre spectral sequence [49, 6.8.2], we can then split this into two problems: first we compute $H^*(C_{p-1};\mathbb{Z}_p^\wedge[u^\pm])$, and then we compute $H^*(\mathbb{Z}_p^\wedge;-)$ of the result. The first of these tasks is easy: because $(p-1)$ and p are coprime [49, Theorem 6.2.2],

$$H^*(C_{p-1};\mathbb{Z}_p^\wedge[u^\pm]) = H^0(C_{p-1};\mathbb{Z}_p^\wedge[u^\pm]) = (\mathbb{Z}_p^\wedge[u^\pm])^{C_{p-1}} = \mathbb{Z}_p^\wedge[u^{\pm(p-1)}] = A.$$

Next, we compute $H^*(\mathbb{Z}_p^\wedge;A)$. Because we're computing the continuous group cohomology, it's OK[39] to consider the cohomology against the dense subgroup $\mathbb{Z} = \langle\gamma\rangle$ for a topological generator, like $\gamma = 1 + p$. Having reduced to computing some group cohomology of $\mathbb{Z}$, we are really just interested in the long exact sequence[40]

$$\cdots \to A_n \xrightarrow{\gamma-1} A_n \to H^n(\mathbb{Z}_p^\wedge;A) \to \cdots.$$

---

[39] ??

[40] **Embed this into the main text. How can you cite it?** The easiest way to see this resolution is by identifying $\mathbb{R}$ as a model for the contractible space $E\mathbb{Z}$, with cell structure given by $E\mathbb{Z}^{(0)} = \mathbb{Z}$ and $E\mathbb{Z}^{(1)} = \mathbb{R}$. Accordingly, we're interested in attaching a copy of A_* to each point in $E\mathbb{Z}^{(0)}$, mapping A_n at the point m to the A_n at $(m+1)$ by $A_n \xrightarrow{\gamma-1} A_n$, and taking a cofiber to compute the group cohomology.

This reduces our remaining task to computing the action of $\gamma - 1$ on $u^{d(p-1)}$ for varying d. To save time on arithmetic, I'll just tell you:

$$\gamma \cdot u^{d(p-1)} - u^{d(p-1)} = \begin{cases} 0 & \text{for } d = 0, \\ w_d d p \cdot u^{d(p-1)} & \text{for } d \neq 0 \text{, where } w_d \text{ is some unit } w_d \in \hat{\mathbb{G}}_m \mathbb{Z}_p^\wedge. \end{cases}$$

[41] Since the Lyndon-Hochschild-Serre spectral sequence is concentrated in $p = 0$, there are no differentials, and this gives $H^*(\mathbb{S}_1; (E_1)_*)$. In turn, there are also no differentials in the Adams spectral sequence for degree reasons, yielding the description

$$\pi_i L_{K(1)}\mathbb{S} = \begin{cases} \mathbb{Z}_p^\wedge, & i = -1, 0, \\ \mathbb{Z}_p^\wedge/(kp), & i = 2k(p-1) - 1 \text{ for } k \in \mathbb{Z} \setminus \{0\}, \\ 0 & \text{otherwise.} \end{cases}$$

To finish the computation, we compute the corner piece, which is the rationalization of these groups:

$$\pi_i L_{E(0)} L_{K(1)}\mathbb{S} = \mathbb{Q} \otimes \pi_i L_{K(1)}\mathbb{S} = \begin{cases} \mathbb{Q}_p, & i = -1, 0, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, to compute the $E(1)$-local homotopy, we must compute the homotopy of a fiber product of spaces, inducing a Mayer-Vietoris long exact sequence on homotopy groups. In the end, this yields

$$\pi_i L_{E(1)}\mathbb{S} = \begin{cases} \mathbb{Q}_p/\mathbb{Z}_p^\wedge = \mathbb{Z}/p^\infty, & i = -2, \\ \mathbb{Q} \cap \mathbb{Z}_p^\wedge = \mathbb{Z}_{(p)}, & i = 0, \\ \mathbb{Z}_p/(kp), & i = 2k(p-1) - 1 \text{ for } k \in \mathbb{Z} \setminus \{0\}, \\ 0 & \text{otherwise.} \end{cases}$$

You can see a couple things emerge in this answer. The first is that previously we had $\pi_0 L_{E(0)}\mathbb{S} = \mathbb{Q}$, which didn't agree with the true stable 0-stem π_0^S. Now, by passing to the $E(1)$-local sphere, we've fixed that: we found that its 0th homotopy group is instead $\mathbb{Z}_{(p)}$, just as we'd desire of the p-local sphere. On the other hand, you can also see essentially all the other groups, while following a nice pattern, are incorrect and do not agree with the stable stems. It turns out that this is a general fact: there is a radius around the 0-stem for which the $E(q)$-local sphere agrees with the actual stable stems, which is both finite and monotonically increasing with q. This goes by the name of chromatic convergence [37, Theorem 7.5.7].

As a closing remark, the $K(2)$-local sphere is substantially more complicated, as the stabilizer group $\mathbb{S}_2$ and the module $(E_2)_*$ are both so much larger. By incredible force of will, this was accomplished in the '90s by Shimomura and collaborators, though just recently Mark Behrens has been making a substantial effort to understand and clarify their brutally difficult work. Uninformed computations for $q > 2$ will almost certainly be impossible for humans to carry out.

Mention the J-homomorphism appearing as the image of $\pi_*\mathbb{S} \to \pi_* L_{K(1)}\mathbb{S}$.

7. DAY 4: HINTS AT GLOBALIZATION

At this point, we've pretty much introduced all the well-understood parts of this story. What people are working on now is a way to make the rest of the computations we know about fit together into a coherent picture, meaning we would like to take small special cases we know about and somehow globalize them.

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[41] State the resulting cohomology $H * (\mathbb{Z}_p^\wedge; A)$ before talking about the spectral sequence?

7.1. **Filtering the Steenrod algebra.**

Trying to understand K-theory is really where all of this exotic topology business got started, and each time we see a pattern we're trying to generalize, K-theory is an example of the bottom nontrivial case. So, in approaches to globalization that we'll discuss, we'll first analyze the K-theory case and then wander outward from there.

Let's first return to the $H\mathbb{F}_2$-Adams spectral sequence, which you'll recall takes as input the stack cohomology of $\mathscr{M}_{H\mathbb{F}_2}$ against the structure sheaf $\mathbb{F}_2$. The reason this cohomology is so enormous is because the action on the structure sheaf is extremely non-free. At the other extreme, the sheaf $\mathscr{H}\mathbb{F}_2(H\mathbb{F}_2)$ carries a free action, and accordingly its higher stack cohomology vanishes. It would be nice to interpolate between these two endpoints with spectra X_n whose sheaves $\mathscr{H}\mathbb{F}_2(X_n)$ carry an action that becomes gradually less free in a controlled way.

Specifically, we can define a sequence of quotient algebras $\mathscr{A}(n)_*$ of the dual Steenrod algebra $\mathscr{A}_*$ by imposing the minimal set of relations to ensure that the power series it classifies are of degree at most 2^{n+1}. For example,

$$\mathscr{A}(0)_* = \mathscr{A}_*/\langle \xi_1^2, \xi_2, \xi_3, \ldots \rangle,$$
$$\mathscr{A}(1)_* = \mathscr{A}_*/\langle \xi_1^4, \xi_2^2, \xi_3, \xi_4, \ldots \rangle.$$

We define the Hopf algebra quotient [47, pg. 497] by the formula

$$\mathscr{A}_*//\mathscr{A}(n)_* = \mathscr{A}_* \otimes_{\mathscr{A}(n)_*} k.$$

There is a "change-of-rings" isomorphism [47, Theorem 19.7]

$$\operatorname{Ext}^{*,*}_{\mathscr{A}_*}(k, \mathscr{A}_*//\mathscr{A}(n)_*) = \operatorname{Ext}^{*,*}_{\mathscr{A}(n)_*}(k,k),$$

meaning that the sheaf $\mathscr{A}_*//\mathscr{A}(n)_*$ is exactly what we were after: it has a free part that swallows part of the stack cohomology and an utterly non-free part that lets some of it escape through.

With this in mind, we might set out on a program to construct spectra X_n realizing $\mathscr{H}\mathbb{F}_2(X_n) = \mathscr{A}_*//\mathscr{A}(n)_*$. It turns out that picking $X_0 = H\mathbb{Z}_{(2)}$ gives $\mathscr{H}\mathbb{F}_2(X_0) = \mathscr{A}_*//\mathscr{A}(0)_*$ [41], and picking $X_1 = ko$ gives $\mathscr{H}\mathbb{F}_2(X_1) = \mathscr{A}_*//\mathscr{A}(1)_*$ [43]. The interesting spectrum *tmf*, which we'll discuss more in a moment, further turns out to be an example of an X_2,[42] indicating that there is some a real pattern to pursue. Unfortunately, there is also a nonexistence result: X_n does not exist as described for $n \geq 3$.[43,44]

Problems: What is the correct weakening so that X_n does exist for $n \geq 3$? Even if X_n can't exist, what if we pretended it did — what are some interesting features of $H^*(\mathscr{M}_{H\mathbb{F}_2}; \mathscr{H}\mathbb{F}_2(X_3))$?

7.2. **Integrality.**

Ordinary K-theory also motivates the spectra $K(q)$ and E_q seen in the chromatic picture, a connection which up until now we've ignored. To pin down the relation, let's first write down the coefficient rings of these theories: the homotopy groups of E_1 are given by $\pi_* E_1 = \mathbb{Z}_p^\wedge[u^\pm]$ with $|u| = 2p-2$, whereas $\pi_* KU = \mathbb{Z}[\beta^\pm]$ with $|\beta| = 2$. At a glance the homotopy of KU appears much denser than that of E_1. It turns out that they are very nearly the same: after p-completing KU, there is a multiplicative splitting $KU_p^\wedge \simeq \bigvee_{n=1}^{p-1} \Sigma^{2n} E_1$. There is a similar splitting for mod-p K-theory: $KU/p \simeq \bigvee_{n=1}^{p-1} \Sigma^{2n} K(1)$.[45]

This indicates how we should think of these spectra: $K(1)$ is the residue field of E_1, and KU is an integral object which p-completes to the special case of E_1. It is easiest to understand the sort of globalization going on here in terms of the $\operatorname{Spec} \mathsf{FiniteSpectra}$ picture: we stratified the spectrum according to primes and to height. We have these spectra $E(\infty) = BP$ which capture all the heights at a particular prime — it covers the information in one of the columns in the grid. On the other hand, KU captures only the information at height 1, but it does so for all primes simultaneously — it covers the information in one of the rows.

So, to generalize, one could ask what ring spectrum X_n has an integral ground ring and which splits upon p-completion to a wedge of suspensions of E_n. It turns out that we have recently produced a candidate for $n = 2$, called topological modular forms and written *TMF*. The algebraic geometry underlying this spectrum arises when

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[42] ...Hopkins?

[43] This is remark 7.50, pg. 89, of Rognes' *Notes on the Adams Spectral Sequence*.

[44] Somewhere in this section we should cite Henriques' drawing as an example of the relative computational complexity of $n = 2$ vs $n = 1$.

[45] These are Landweber flat, so don't require much of a proof. Where can you find this?

considering elliptic curves which are allowed to have nodal degeneracies. The identity on such an elliptic curve is always a smooth point, and its formal completion yields a formal Lie group of height 1 or 2.[46] In turn, the moduli of such elliptic curves has nice properties from the view of formal groups, with height 1 and 2 information appearing, and also from the view of algebraic geometry, since the objects it classifies are projective varieties.

Lifting this business into topology requires quite a lot of work [25]. Consider the moduli of elliptic curves $\overline{\mathcal{M}_{\mathrm{ell}}}$, and let $\operatorname{Spec} R \to \overline{\mathcal{M}_{\mathrm{ell}}}$ be a flat map, i.e., a point in the flat topology. One can check that the map $\overline{\mathcal{M}_{\mathrm{ell}}} \to \mathcal{M}_{\mathrm{fg}}$ is a flat map, and so the pullback of $\mathcal{M}\mathcal{U}(X)$ to $\operatorname{Spec} R$ defines a cohomology theory called "elliptic cohomology (on R with respect to the chosen curve)." Identifying $\overline{\mathcal{M}_{\mathrm{ell}}}$ with its étale site, this gives a map

$$\begin{tikzcd}
\text{HomologyTheories} \arrow[r, "\pi_0"] & \text{Rings} \\
\overline{\mathcal{M}_{\mathrm{ell}}}. \arrow[u] \arrow[ur, "\mathcal{O}"'] &
\end{tikzcd}$$

The Goerss-Hopkins-Miller [10, 39] theorem constructs E_∞-models for the Morava E-theories. We can apply this theorem to sufficiently small points in $\overline{\mathcal{M}_{\mathrm{ell}}}$ to produce a local lift

$$\begin{tikzcd}
E_\infty\text{-RingSpectra} \arrow[r] & \text{HomologyTheories} \arrow[r, "\pi_0"] & \text{Rings} \\
& \overline{\mathcal{M}_{\mathrm{ell}}}. \arrow[ul, dotted, "\mathcal{O}_{top}"] \arrow[u] \arrow[ur, "\mathcal{O}"'] &
\end{tikzcd}$$

Finally, the whole purpose of Lurie's work on derived algebraic geometry [26, 23, ...] is to define what it means to sheafify this locally-defined presheaf of E_∞-ring spectra. The core difficulty is that it is not homotopically reasonable to request that sections of such sheaves patch merely when their overlaps are strictly equal; instead, we should learn to tolerate that they are merely homotopic. In the end, his program is meant to produce a lift

$$\begin{tikzcd}
E_\infty\text{-RingSpectra} \arrow[r] & \text{HomologyTheories} \arrow[r, "\pi_0"] & \text{Rings} \\
& \overline{\mathcal{M}_{\mathrm{ell}}}. \arrow[ul, "\mathcal{O}^\dagger_{top}"] \arrow[u] \arrow[ur, "\mathcal{O}"'] &
\end{tikzcd}$$

Finally, taking global sections of the sheaf $\mathcal{O}^\dagger_{top}$ yields a ring spectrum called *TMF*.[47]

But, once again, for $n \geq 3$ we run out of candidates for X_n — the trouble this time is that there are no 1-dimensional abelian varieties of height greater than 2.[48] Recent work of Behrens and Lawson [7] have used polarized Shimura varieties to produce an example of an integral higher height theory, called *TAF* for "topological automorphic forms." Shimura varieties are higher dimensional abelian varieties, and polarization tags them with additional structure that allows us to canonically split off a 1-dimensional subgroup from their completion at the identity.[49] Little has been done with this new theory, but it's promising that it exists at all.

Problems: Make computations with *TAF*. Make computations with *TMF*, for that matter. There are also loads of problems to solve about Lurie's underlying machine that produces *TMF*. One could also offer alternative approaches to the problem by finding sources of 1-dimensional formal Lie groups other than polarized Shimura varieties; it's not clear that this is the right approach to take, or that the choices made in its construction are sane ones. Szymik has done interesting work in this last direction [48].

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[46] ?

[47] *TMF* is so-named because $TMF_*[\frac{1}{6}]$ is isomorphic to the ring of modular forms.

[48] ?

[49] ?

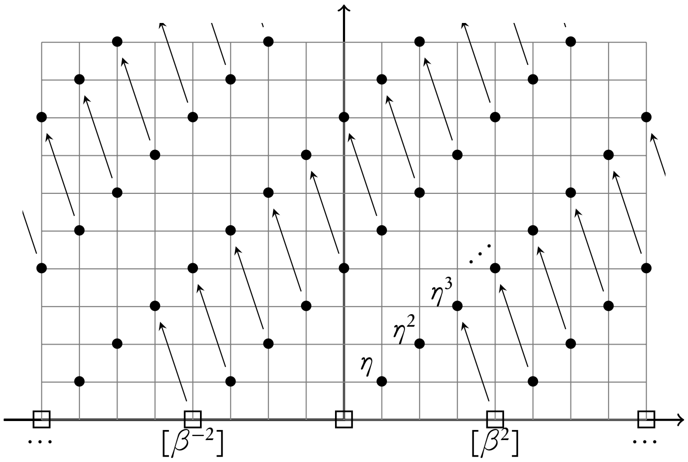

FIGURE 7. The homotopy fixed point spectral sequence for KO. Squares indicate $\mathbb{Z}$, dots indicate $\mathbb{Z}/2$.

7.3. **Equivariance.** Spectra in general are very difficult objects to control, and ring spectra aren't too much more rigid than that — their study contains at least all of classical ring theory, for instance. To find footholds in stable homotopy theory, we spend a lot of effort trying to produce rigid structures, and one example of a particularly rigid structure is an equivariant spectrum. The definition underlying an equivariant spectrum is intricate, but the upshot of the program is that you end up with spectra X carrying G-actions in a way that you can produce fixed-point spectra X^{hG} and orbit spectra X_{hG}, along with all the niceties like spectral sequences comparing the fixed points of their homotopy groups to the homotopy groups of their fixed points [40].

Most of our examples of equivariant spectra come directly from geometry: for instance, the C_2-action on complex vector bundles by complex conjugation rigidifies to make KU into a C_2-equivariant spectrum. You might guess that KU^{hC_2}, the fixed points of the conjugation action, is KO, in the same way that the fixed points of the conjugation action on complex vector bundles are exactly the complexified real vector bundles. This turns out to be true, and while a proof is beyond the scope of this discussion, it's instructive to draw the spectral sequence computing $\pi_* KU^{hC_2}$, pictured in figure 7. Also worth mentioning is the nonequivalence $kU^{hC_2} \not\simeq kO$, visible in the same picture.

Previously, we have discussed the action of the Morava stabilizer group $\mathbb{S}_n$ on the sheaf $\mathscr{E}_n(X)$. This action lifts directly into the world of homotopy theory, via the Goerss-Hopkins-Miller [10, 39] theorem: the spectrum E_n is equivariant against an action of $\mathbb{S}_n$ inducing this action on the level of homotopy groups. There are three further beautiful theorems:

(1) The appearance of group cohomology in the chromatic spectral sequence and the realization of E_n as an $\mathbb{S}_n$-equivariant spectrum are not coincidences. There is an equivalence $E_n^{h\mathbb{S}_n} \simeq L_{K(n)}\mathbb{S}$ [8, Proposition 7.1].
(2) Just as we can produce KO from KU as the fixed points of C_2, we can produce $L_{K(2)}$ *TMF* from E_2 by taking fixed points against a maximal finite subgroup of $\mathbb{S}_2$.[50]
(3) For the last, $\mathbb{S}_1$ contains only the p-adic units, whereas $\mathbb{S}_n$ for $n > 1$ contains many more elements in addition to a subgroup of p-adic units. There is a certain projection map $\det : \mathbb{S}_n \to \mathbb{Z}_p^\times$, the kernel of which is denoted $S\mathbb{S}_n$, and the fixed point spectrum $E_n^{hS\mathbb{S}_n}$ plays the role of a K-theory spectrum in the $K(n)$-local category: for example, it carries an analogue of the image of J [?].

Many of these facts have globalization built into them: they hold for each n. So, we can define many spectra in the vein of KU, KO, and *TMF* by taking $n \geq 3$ and investigating what appears. In particular, the fixed points of E_n against a maximal finite subgroup of $\mathbb{S}_n$ is denoted EO_n, the nth higher real K-theory spectrum. [17, Section

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[50] Talk about 1) the relationship to actual real K-theory, and 2) this approximation the $K(n)$-local sphere.

1][51] These spectra are of immense interest, both for chromatic homotopy theory proper and for applications — for instance, the Kervaire invariant problem was recently solved [16], and the proof arose not from seeking out information about the Kervaire classes but by accident during an attempt to compute things about the homotopy of eo_n. (This last bit is unpublished, but see Hill's thesis [?] or the Hopkins-Mahowald note [17] for related computations.)

In a completely different direction, there is a program connecting height n phenomena for a space X to height $(n-1)$ phenomena in the free loopspace LX, which carries an S^1-action by loop rotation. This manifests in at least two ways:

1. The "chromatic redshift" conjecture attempts to connect algebraic K-theory to chromatic homotopy theory through topological Hochschild homology. For a space X and E_∞-ring spectrum E, one can build the tensor $X \wedge E$, and topological Hochschild homology is defined by $S^1 \otimes E$.[52] The conjecture, more or less, states that $K(E_n) \approx E_{n+1}$. The name comes from increasing the periodicity length (from $|v_n| = 2(p^n - 1)$ to $|v_{n+1}| = 2(p^{n+1} - 1)$), so "redshifting." [1, Section 4][53]
2. The Atiyah-Segal completion theorem states that for G-equivariant K-theory, the map $K^*_G(\mathrm{pt}) \to K^*(BG)$ is a completion. In the specific case of $G = S^1$, $K^*_G(\mathrm{pt}) = \mathbb{Z}[u^\pm]$ is the ring of functions on the algebraic variety $\mathbb{G}_m$, and the map is the inclusion $\hat{\mathbb{G}}_m \to \mathbb{G}_m$.[54] One would like a similar construction of equivariant elliptic cohomology, so that $E\ell\ell^*_{S^1}(\mathrm{pt})$ is the elliptic curve one completes to get $BU(1)_{E\ell\ell}$ [4]. A main problem here is that an elliptic curve is a projective variety, and so has no global functions, but there is a construction that gets around this by building a sheaf-valued equivariant elliptic cohomology [11]. This construction works by locally defining the sheaf to be ordinary S^1-equivariant cohomology, then using the isomorphism of complex elliptic curves to $S^1 \times S^1$ to incorporate the information in the elliptic curve into the gluing twists.
3. **Stolz-Teichner**[55]

In yet another a completely different direction, a program has recently begun to study spectra which are equivariant against all compact Lie groups simultaneously in a compatible way. Many geometrically constructed spectra can be constructed carefully enough to have such a structure, providing many examples of so-called "global spectra."[56] One of the most intriguing artifacts of this approach is that distinct constructions of geometric spectra which once yielded the same nonequivariant homotopy type now produce distinct global spectra — this is something not previously visible that is sure to tell us something new and interesting.[57]

Problems: What can be computed about the homotopy of the spectra EO_n, or their values on various interesting inputs? What other objects familiar from the ordinary stable category have "determinantal analogues," like $E_n^{hS\mathbb{S}_n}$ for KU? Any information about algebraic K-theory would be helpful; the results we know of are very difficult and connect very important objects, and conceptual explanations would be immensely useful. Can Grojnowski's construction be generalized to produce cohomology theories of heights higher than 2? What versions of globally equivariant geometric spectra enjoy which properties? We have understood the nonequivariant stable category through MU and formal groups; what tools are appropriate for understanding the equivariant stable category?

7.4. **Postnikov tower for bordism.**

You'll recall that ordinary singular homology is constructed by studying the continuous maps from n-simplices into a space, modulo those collections of simplices which appear as the hull of an $(n+1)$-simplex. It is very natural to ask what happens when you replace simplices with some other geometric object instead; using manifolds results in a theory called bordism. Specifically, continuous maps from some closed n-manifold to the space replace continuous maps from n-simplices in, and continuous maps from a non-closed $(n+1)$-manifold with boundary a collection of closed n-manifolds replace the hulls of $(n+1)$-simplices. The resulting functor is a homology theory called MO_*, and its coefficients are the unoriented bordism ring [47, Chapter 12].

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[51] There ought to be a better Hopkins-Miller reference, but they haven't published their original work.

[52] For an A_∞-ring, it's defined as $E \wedge_{E \wedge E^{op}} E$, and then this description for E_∞-rings comes from the pushout diagram for $S^1 = \Sigma S^0$.

[53] Aaron suggests talking about blueshift as well: "the Tate spectrum of a v_n-periodic spectrum should be v_{n-1}-periodic."

[54] Lurie survey

[55] Aaron says the draw of S-T is that it connects the dimension filtration to the height filtration.

[56] May, Schwede

[57] Schwede

$$\begin{tikzcd}
MO \arrow[d] & MSO \arrow[l] \arrow[d] & MSpin \arrow[l] \arrow[d] & MString \arrow[l] \arrow[d] & \cdots \arrow[l] & \mathbb{S} \arrow[l] \arrow[dlllll, crossing over] \arrow[dllll, crossing over] \arrow[dlll, crossing over] \arrow[dll] \\
H\mathbb{Z}/2 & H\mathbb{Z} & KO & TMF & \cdots &
\end{tikzcd}$$

FIGURE 8. The bordism tower

The first thing to notice about bordism is that the unoriented setting is very uninteresting: for an unoriented closed manifold M, the manifold $M \times I$ induces the relation $M + M = 0$ in the bordism ring, making MO_* a $\mathbb{Z}/2$-algebra [47, Example 12.24.1]. This statement turns out to be very strong: there is a map $MO \to H\mathbb{Z}/2$ of ring spectra detecting the bordism class of a point, this map turns out to be a split surjection, and it induces a splitting $MO \simeq \bigvee \Sigma^* H\mathbb{Z}/2$ [47, Theorem 20.8]. This means that, as far as stable homotopy is concerned, MO contains no information not already present in $H\mathbb{Z}/2$.

There are two ways to liven this construction up, both of which involve putting extra structure on the manifolds involved [47, Definition 12.19]. The easiest thing to do is to request that the tangent bundle of the n-manifold carry an action not just by $O(n)$ but by G_n for some sequence of groups G_n with compatible maps $G_n \to O(n)$. For instance, An almost-complex structure on the tangent bundle corresponds to a (stable) action by the group of unitary matrices, yielding the bordism theory MU we've been studying all along. The other method of generalization is more subtle: you can impose certain conditions on the characteristic classes of the tangent bundle of the manifold. For instance, MSO is the bordism theory for which the first Stiefel-Whitney class of the tangent bundle vanishes.[58] The most extreme version of this is for the manifolds to vary a stable framing, yielding a bordism theory weakly equivalent to $\mathbb{S}$, the stable sphere.

Interpreting $\mathbb{S}$ as a sort of bordism theory casts the data of a ring spectrum in a new light: the image of the unit map $\eta : \mathbb{S} \to E$ corresponds to the invariants of framed manifolds which are detectable by E-theory. People running with this idea uncovered something surprising about the most familiar unit maps: they factor through the bottom layers of the bordism tower as in Figure 8. For examples of the sort of data these capture, the map $MO \to H\mathbb{Z}/2$ induces a genus $MO_* \to H\mathbb{Z}/2_*$ counting the number of unoriented isolated points in a manifold. This map lifts to $MSO \to H\mathbb{Z}$ inducing a genus $MSO_* \to H\mathbb{Z}_*$ counting the number of oriented isolated points with sign.[59] The next two genera further refine these maps; for instance, if the first and second class both vanish, then the manifold admits a $Spin$ structure, making it a point in $MSpin$, and there is a map $MSpin \to KO$, called the Atiyah-Bott-Shapiro[60] orientation, or $\hat{A}$-genus. This bordism theory and its associated genus become important in the context of Atiyah-Singer index theory[61], which connects the value of the $\hat{A}$-genus[62] to certain analytical properties of the Dirac operator, a square-root of the Laplacian differential operator on the manifold. This is an enormously deep theorem that connects two very different sorts of geometry: homotopy theory and analysis on manifolds.

It would be nice to be able to continue this story as well, and we have produced a partial generalization to height 2 cohomology. Namely, there is bordism theory $MString$ arising from those $Spin$-manifolds whose next Stiefel-Whitney class also vanishes, along with a map called the Witten genus $MString_*[\frac{1}{6}] \to TMF_*[\frac{1}{6}]$.[63] This genus has been shown to enrich to a map of spectra $MString \to TMF$ which connects a large part of both of their homotopy groups [2, 3]. No higher height analogue of the index theorem is presently known, largely because of our lack of understanding of the geometry of *TMF*; bridging this gap is one of the goals of the Stolz-Teichner program [**?**]. Little is known about higher height genera as well; requiring the next Stiefel-Whitney class to vanish does not appear to be the right move, and so we're missing both a higher height source to replace $MString$ and a higher height target to replace *TMF*.

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[58] ?

[59] ?

[60] ABS, MO question cites a thesis

[61] AS papers

[62] Miller

[63] Witten

Problems: Produce higher height candidates for replacing $MString$, say MX_n, and *TMF*, say Y_n. Produce genera $MX_n \to Y_n$. Produce a map $MString \to EO_2$, to help connect *TMF* to EO_2. Find geometric models for Y_n, beginning with *TMF*. The bordism spectra also support E_∞ structures, as do our targets *KO* and *TMF*, and another goal is to check that these orientations are E_∞-maps [5]. If this is true, we would also like higher order results for $MX_n \to Y_n$ for $n \geq 3$.

8. HOMEWORK: THE ALGEBRAIC GEOMETRY OF SPACES

In closing, I wanted to make mention that we've only been using half of the algebraic geometry available to us. This observation is not directly related to globalization, but it's useful to be introduced to this other side. For a space or spectrum X, let X_E denote the scheme $\operatorname{Spf} E^*X$. So far, we've only shown interest in the ground scheme $\mathbb{S}_E$ and in the group "G_E"$= \mathbb{C}\mathrm{P}^\infty_E$. Of course, no matter what space X we give to cohomology, we'll get a ring E^*X in return, and so the geometry of affine schemes has the potential to tell us about a lot of other spaces. Knowing this dictionary between spaces and schemes is very useful for building intuition, so what follows is a partial list of the known correspondences. Working out some of these on your own is an excellent exercise; most of the computations are quite classical, but fitting them into the algebro-geometric framework takes some thought.[64,65]

- Throughout, we'll concern ourselves with complex orientable spectra E. This means that $BU(1)_E = G_E$ is a formal Lie group G.
- The computation of $H\mathbb{Z}^*BU(n)$ as generated by symmetric polynomials in $H\mathbb{Z}^*BU(1)$ generalizes to a computation for any complex oriented E. On the level of schemes, this means $BU(n)_E$ is the scheme $\operatorname{Div}_n^+ G_E$ of positive divisors of degree n on G_E.
- Just as the spaces $BU(n)$ assemble into a directed system by including block matrices into the top-left components, so do these formal schemes. Taking a colimit shows that BU_E is the scheme $\operatorname{Div}_0 G$ of divisors on G of weight 0. This also gives a description of $\underline{KU}_0 = \mathbb{Z} \times BU$: the $\mathbb{Z}$ factor counts divisor weight, so $(\mathbb{Z} \times BU)_E = \operatorname{Div} G$.
- There is a map $BU(1) \to BSp(1)$ corresponding to symplectification of complex line bundles. Analysis of this map shows that $BSp(1)_E = \operatorname{fib}[(G_E \times G_E)_{\Sigma_2} \to G_E]$. Correspondingly, E_*BSp is the free symmetric algebra on $E_*BSp(1)$, so $BSp_E = \operatorname{Div}_0 BSp(1)_E$. The space BO also fits into this framework: $BO_E = BU_E^\chi$ is the fixed-points of BU under the C_2-action given by complex conjugation.
- The quotient $\mathbb{Z}/n \to S^1 \xrightarrow{\cdot n} S^1$ yields a fiber bundle, and delooping it once gives a fiber bundle $B\mathbb{Z}/n \to BU(1) \to BU(1)$. In many cases, this can be shown to induce a short exact sequence of group schemes, so that $B\mathbb{Z}/n_E$ appears as the n-torsion $G_E[n]$ of G_E. This gives an interpretation of BA_E for any finite abelian group A: it is the mapping scheme $\operatorname{Maps}(\underline{A}, G_E)$. It turns out that there is also a theory explaining BG_E for nonabelian groups G, called Hopkins-Kuhn-Ravenel character theory, but it is much more complex.
- Following on from our discussion on the first day, Hopf ring methods can be used to show that $(\underline{H\mathbb{Z}/n}_q)_{K(r)}$ is the qth alternating power of the n-torsion $(B\mathbb{Z}/n)_{K(r)} = G_{K(r)}[n]$ for each Morava K-theory (including $K(\infty) = H\mathbb{F}_p$). For $0 < r < \infty$, the scheme of n-torsion is finite, which means that $(\underline{H\mathbb{Z}/n}_q)_{K(r)}$ vanishes for large enough q. This has several interesting consequences.
- We discussed the filtration of BU by $BU(n)$, but Bott periodicity states that BU is equivalent to ΩSU, which carries its own filtration by $\Omega SU(n)$. The scheme $\mathbb{C}\mathrm{P}^n_E$ corresponds to the divisor of nth order points, and since $E_*\Omega SU(n)$ is the free symmetric algebra on $E_*\mathbb{C}\mathrm{P}^n$, the scheme $\Omega SU(n)_E$ is the "free group-scheme" on the divisor $\mathbb{C}\mathrm{P}^n_E$. In turn, S^{2n}_E is the quotient $\mathbb{C}\mathrm{P}^n_E/\mathbb{C}\mathrm{P}^{n-1}_E$, and ΩS^{2n+1} is the free group-scheme on S^{2n}_E.
- Lastly, if X is an H-space with E_*X torsion-free, then $X^E = \operatorname{Spec} E_*X$ satisfies $X^E = \operatorname{Hom}(X_E, \mathbb{G}_m)$. Using this, various homology-schemes for BU and so on can be determined as certain sorts of mapping spaces. The most interesting is that MU^E is the subscheme of functions $BU(1)_E \to \hat{\mathbb{A}}^1$ which are coordinates, i.e.,

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[64] Sort through FSFG, FPFP, Wilson, HKR, and RW for references.

[65] We may want to include a discussion of the scheme of subgroups, $(B\Sigma_{p^n})_E$, and power operations. If so, cite Strickland and Matt's thesis.

isomorphisms of formal schemes. In turn, for another complex-oriented spectrum F, F^E can sometimes be expressed as the scheme of isomorphisms $\mathrm{Iso}(G_E, G_F)$.

Problems: What sort of formal supergeometry appears when E^*X is not even-concentrated? Can we produce examples of spaces that have a description in this language which do *not* come from a connection to $\mathbb{C}\mathrm{P}^\infty$?

REFERENCES

[1] *Guido's book of conjectures*, volume 40 of *Monographies de L'Enseignement Mathématique [Monographs of L'Enseignement Mathématique]*. L'Enseignement Mathématique, Geneva, 2008. A gift to Guido Mislin on the occasion of his retirement from ETHZ June 2006, Collected by Indira Chatterji.

[2] M. Ando, M. J. Hopkins, and N. P. Strickland. Elliptic spectra, the Witten genus and the theorem of the cube. *Invent. Math.*, 146(3):595–687, 2001.

[3] M. Ando and N. P. Strickland. Weil pairings and Morava K-theory. *Topology*, 40(1):127–156, 2001.

[4] Matthew Ando. Power operations in elliptic cohomology and representations of loop groups. *Trans. Amer. Math. Soc.*, 352(12):5619–5666, 2000.

[5] Matthew Ando, Michael J. Hopkins, and Neil P. Strickland. The sigma orientation is an H_∞ map. *Amer. J. Math.*, 126(2):247–334, 2004.

[6] Paul Balmer. Spectra, spectra, spectra—tensor triangular spectra versus Zariski spectra of endomorphism rings. *Algebr. Geom. Topol.*, 10(3):1521–1563, 2010.

[7] Mark Behrens and Tyler Lawson. Topological automorphic forms. *Mem. Amer. Math. Soc.*, 204(958):xxiv+141, 2010.

[8] Ethan S. Devinatz and Michael J. Hopkins. Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups. *Topology*, 43(1):1–47, 2004.

[9] Ethan S. Devinatz, Michael J. Hopkins, and Jeffrey H. Smith. Nilpotence and stable homotopy theory. I. *Ann. of Math. (2)*, 128(2):207–241, 1988.

[10] P. G. Goerss and M. J. Hopkins. Moduli problems for structured ring spectra. In *DANIEL DUGGER AND BROOKE*, 2005.

[11] I. Grojnowski. Delocalised equivariant elliptic cohomology. In *Elliptic cohomology*, volume 342 of *London Math. Soc. Lecture Note Ser.*, pages 114–121. Cambridge Univ. Press, Cambridge, 2007.

[12] Bruno Harris. Bott periodicity via simplicial spaces. *J. Algebra*, 62(2):450–454, 1980.

[13] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[14] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[15] Michiel Hazewinkel. *Formal groups and applications*, volume 78 of *Pure and Applied Mathematics*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1978.

[16] Michael Hill, Michael Hopkins, and Douglas Ravenel. On the non-existence of elements of kervaire invariant one. page 158.

[17] Michael Hopkins and Mark Mahowald. From elliptic curves to homotopy theory. page 30.

[18] Michael J. Hopkins. *Complex oriented cohomology theories and the language of stacks*. 1998.

[19] Michael J. Hopkins, Mark Mahowald, and Hal Sadofsky. Constructions of elements in Picard groups. In *Topology and representation theory (Evanston, IL, 1992)*, volume 158 of *Contemp. Math.*, pages 89–126. Amer. Math. Soc., Providence, RI, 1994.

[20] Michael J. Hopkins and Jeffrey H. Smith. Nilpotence and stable homotopy theory. II. *Ann. of Math. (2)*, 148(1):1–49, 1998.

[21] Mark Hovey and Neil P. Strickland. Morava K-theories and localisation. *Mem. Amer. Math. Soc.*, 139(666):viii+100, 1999.

[22] I. M. James. On the suspension sequence. *Ann. of Math. (2)*, 65:74–107, 1957.

[23] Jacob Lurie. *Higher algebra*.

[24] Jacob Lurie. *Lectures on chromatic homotopy theory (252x)*.

[25] Jacob Lurie. *A survey of elliptic cohomology*.

[26] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[27] J. P. May. *The geometry of iterated loop spaces*. Springer-Verlag, Berlin, 1972. Lectures Notes in Mathematics, Vol. 271.

[28] John McCleary. *A user's guide to spectral sequences*, volume 58 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2001.

[29] Haynes R. Miller, Douglas C. Ravenel, and W. Stephen Wilson. Periodic phenomena in the Adams-Novikov spectral sequence. *Ann. of Math. (2)*, 106(3):469–516, 1977.

[30] Jack Morava. Noetherian localisations of categories of cobordism comodules. *Ann. of Math. (2)*, 121(1):1–39, 1985.

[31] Daniel Quillen. On the formal group laws of unoriented and complex cobordism theory. *Bull. Amer. Math. Soc.*, 75:1293–1298, 1969.

[32] Daniel Quillen. Elementary proofs of some results of cobordism theory using Steenrod operations. *Advances in Math.*, 7:29–56 (1971), 1971.

[33] Douglas C. Ravenel. A novice's guide to the Adams-Novikov spectral sequence. In *Geometric applications of homotopy theory (Proc. Conf., Evanston, Ill., 1977), II*, volume 658 of *Lecture Notes in Math.*, pages 404–475. Springer, Berlin, 1978.

[34] Douglas C. Ravenel. Localization with respect to certain periodic homology theories. *Amer. J. Math.*, 106(2):351–414, 1984.

[35] Douglas C. Ravenel. *Complex cobordism and stable homotopy groups of spheres*, volume 121 of *Pure and Applied Mathematics*. Academic Press Inc., Orlando, FL, 1986.

[36] Douglas C. Ravenel. Localization and periodicity in homotopy theory. In *Homotopy theory (Durham, 1985)*, volume 117 of *London Math. Soc. Lecture Note Ser.*, pages 175–194. Cambridge Univ. Press, Cambridge, 1987.

[37] Douglas C. Ravenel. *Nilpotence and periodicity in stable homotopy theory*, volume 128 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1992. Appendix C by Jeff Smith.

[38] Douglas C. Ravenel and W. Stephen Wilson. The Morava K-theories of Eilenberg-Mac Lane space and the Conner-Floyd conjecture. *Amer. J. Math.*, 102(4):691–748, 1980.

[39] Charles Rezk. Notes on the Hopkins-Miller theorem. In *Homotopy theory via algebraic geometry and group representations (Evanston, IL, 1997)*, volume 220 of *Contemp. Math.*, pages 313–366. Amer. Math. Soc., Providence, RI, 1998.

[40] Stefan Schwede. *Lectures on equivariant stable homotopy theory*.

[41] Jean-Pierre Serre. Cohomologie modulo 2 des complexes d'Eilenberg-MacLane. *Comment. Math. Helv.*, 27:198–232, 1953.

[42] James Dillon Stasheff. Homotopy associativity of H-spaces. I, II. *Trans. Amer. Math. Soc. 108 (1963), 275-292; ibid.*, 108:293–312, 1963.

[43] Robert E. Stong. Determination of $H^*(\mathrm{BO}(k,\cdots,\infty),Z_2)$ and $H^*(\mathrm{BU}(k,\cdots,\infty),Z_2)$. *Trans. Amer. Math. Soc.*, 107:526–544, 1963.

[44] Neil P. Strickland. *Functorial philosophy for formal phenomena*.

[45] Neil P. Strickland. Formal schemes and formal groups. In *Homotopy invariant algebraic structures (Baltimore, MD, 1998)*, volume 239 of *Contemp. Math.*, pages 263–352. Amer. Math. Soc., Providence, RI, 1999.

[46] Dennis P. Sullivan. *Geometric topology: localization, periodicity and Galois symmetry*, volume 8 of *K-Monographs in Mathematics*. Springer, Dordrecht, 2005. The 1970 MIT notes, Edited and with a preface by Andrew Ranicki.

[47] Robert M. Switzer. *Algebraic topology—homotopy and homology*. Classics in Mathematics. Springer-Verlag, Berlin, 2002. Reprint of the 1975 original [Springer, New York; MR0385836 (52 #6695)].

[48] Markus Szymik. $K3$ spectra. *Bull. Lond. Math. Soc.*, 42(1):137–148, 2010.

[49] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

[50] W. Stephen Wilson. *Brown-Peterson homology: an introduction and sampler*, volume 48 of *CBMS Regional Conference Series in Mathematics*. Conference Board of the Mathematical Sciences, Washington, D.C., 1982.

[51] W. Stephen Wilson. *Brown-Peterson homology: an introduction and sampler*, volume 48 of *CBMS Regional Conference Series in Mathematics*. Conference Board of the Mathematical Sciences, Washington, D.C., 1982.

Add many, many more references.