Math 1A Discussion, Practice Midterm 2 Name: SOLUTIONS

There are 10 questions. Each question is worth 3 marks. Show your work.

1. Find an equation of the line tangent to the curve $y = \frac{2e^x}{x}$ at the point $(2, e^2)$.

$$y' = \frac{x(2e^x) - (2e^x)(1)}{x^2} = \frac{2xe^x - 2e^x}{x^2}.$$

At $(2, e^2)$,

$$y' = \frac{2(2)e^2 - 2e^2}{2^2} = \frac{4e^2 - 2e^2}{2^2} = \frac{2e^2}{4} = \frac{e^2}{2}.$$

So, the tangent line is given by

$$\text{slope} = \frac{\Delta y}{\Delta x} \qquad y - e^2 = \frac{e^2}{2}(x-2)$$

$$\frac{e^2}{2} = \frac{y - e^2}{x - 2} \qquad y - e^2 = \frac{e^2}{2}x - e^2$$

$$\boxed{y = \frac{e^2}{2}x.}$$

2. The size $P(t)$ of a bacterial population is assumed to grow exponentially as a function of time t. Given that $P(0) = 0.1$ and $P(2) = 3.2$, find an equation for $P(t)$.

$P(t) = Ce^{kt}$ (exponential growth)

$P(0) = 0.1$

$Ce^{k(0)} = 0.1$

$C = 0.1$

$P(2) = 3.2$

$Ce^{k(2)} = 3.2$

$(0.1)e^{2k} = 3.2$

$e^{2k} = 32$

$2k = \ln 32$

$k = \frac{\ln 32}{2}.$

$$\boxed{P(t) = 0.1e^{\frac{\ln 32}{2}t} = 0.1(32^{\frac{t}{2}}).}$$

3. Differentiate $x^{\sinh(x)}$.

$$y = x^{\sinh x}$$

$$\ln y = \ln(x^{\sinh x})$$

$$\ln y = \sinh x \ln x$$

$$\frac{y'}{y} = \sinh x\left(\frac{1}{x}\right) + \ln x\,(\cosh x)$$

$$y' = y\left(\frac{\sinh x}{x} + \cosh x \ln x\right) = \boxed{x^{\sinh x}\left(\frac{\sinh x}{x} + \cosh x \ln x\right).}$$

4. Show that the equation $e^x = 5 - x^2$ has exactly 2 solutions.

Let $f(x) = e^x - 5 + x^2$. $f(x)$ is continuous, and

$$f(-3) = e^{-3} - 5 + (-3)^2 = \frac{1}{e^3} - 5 + 9 > 0,$$

$$f(0) = e^0 - 5 + 0^2 = 1 - 5 = -4 < 0,$$

$$f(3) = e^3 - 5 + 3^2 = e^3 - 5 + 9 > 0,$$

so by the intermediate value theorem, $f(x) = 0$ has a solution in the interval $(-3, 0)$ and a solution in the interval $(0, 3)$. Thus $f(x) = 0$ has at least two solutions.

To see that $f(x)$ has at most two solutions, suppose that this did not happen. Then $f(x) = 0$ would have three solutions, so by Rolle's theorem $f'(x) = 0$ would have two solutions, so by Rolle's theorem again $f''(x) = 0$ would have one solution. But

$$f'(x) = e^x + 2x, \qquad f''(x) = e^x + 2 > 0,$$

so $f''(x) = 0$ has no solutions. This is a contradiction, and it shows that $f(x) = 0$ has at most two solutions.

5. Use differentials or a linear approximation to estimate $\sqrt[3]{7.7}$.

Differentials

$y = x^{\frac{1}{3}}$

$\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}}$

$dy = \frac{1}{3}x^{-\frac{2}{3}}dx$

$8^{-\frac{2}{3}} = \frac{1}{8^{\frac{2}{3}}} = \frac{1}{(\sqrt[3]{8})^2} = \frac{1}{2^2} = \frac{1}{4}$

With $x=8$ and $dx=-0.3$,

$dy = \frac{1}{3}8^{-\frac{2}{3}}(-0.3)$

$= \frac{1}{3}\left(\frac{1}{4}\right)(-0.3) = -\frac{0.1}{4} = -0.025$

So,

$\sqrt[3]{7.7} \approx 8^{\frac{1}{3}} + dy = 2 - 0.025 = \boxed{1.975}$

Linear approximation

Let $f(x) = x^{\frac{1}{3}}$. Then

$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$.

At $x=8$,

slope $= \frac{1}{3}8^{-\frac{2}{3}} = \frac{1}{3}\left(\frac{1}{4}\right) = \frac{1}{12}$,

point $= (8,2)$.

So, the tangent line at $x=8$ is given by

slope $= \frac{\Delta y}{\Delta x}$

$\frac{1}{12} = \frac{y-2}{x-8}$ $\quad y-2 = \frac{1}{12}(x-8)$

$y = \frac{1}{12}(x-8)+2$

So $\sqrt[3]{7.7} = f(7.7) \approx \frac{1}{12}(7.7-8)+2$

$= \frac{1}{12}(-0.3)+2 = -0.025+2$

$\boxed{=1.975.}$

6. Find the 29th derivative of e^{-3x}.

Derivative	
0	e^{-3x}
1	$-3e^{-3x}$
2	3^2e^{-3x}
3	-3^3e^{-3x}
4	3^4e^{-3x}
5	-3^5e^{-3x}
$\vdots$	
29	$\boxed{-3^{29}e^{-3x}.}$

7. Find the absolute maximum and minimum values (and where they are attained) of

$$f(x) = x^5 - 5x + 5$$

on the interval $[0, 2]$.

$f'(x) = 5x^4 - 5 = 5(x^4 - 1)$.

The only critical point in the interval $[0,2]$ is $x=1$.

Checking the values of $f(x)$ at the critical points and the endpoints:

$f(1) = 1^5 - 5(1) + 5 = 1 - 5 + 5 = 1$

$f(0) = 0^5 - 5(0) + 5 = 5$

$f(2) = 2^5 - 5(2) + 5 = 32 - 10 + 5 = 27$

So, the absolute maximum of $f(x)$ on $[0,2]$ is 27 at $x=2$, and the absolute minimum is 1 at $x=1$.

8. Find $\lim_{x \to 0} \frac{\sec x - 1}{x^2}$.

indeterminate of the form $\frac{0}{0}$, so L'Hôpital's rule applies

$$= \lim_{x \to 0} \frac{\sec x \tan x}{2x}$$

$$= \lim_{x \to 0} \frac{\sec x(\sec^2 x) + \tan x(\sec x \tan x)}{2}$$

$$= \frac{\sec 0(\sec^2 0) + \tan 0(\sec 0 \tan 0)}{2} = \frac{1+0}{2} = \boxed{\frac{1}{2}}.$$

9. Sketch the curve $y^3 = x^2 - 1$.

$y = (x^2-1)^{\frac{1}{3}}$ zeros at $x = \pm 1$

$y' = \frac{1}{3}(x^2-1)^{-\frac{2}{3}}(2x) = \dfrac{2x}{3(x^2-1)^{\frac{2}{3}}}$ $y' = 0$ at $x = 0$

y' is undefined at $x = \pm 1$. Since $\lim_{x \to \pm 1} \dfrac{2x}{3(x^2-1)^{\frac{2}{3}}} = \pm\infty$, there are vertical tangent lines at $x = \pm 1$

y'' = too complicated

		-1		0		1	
y'	$-$		$-$		$+$		$+$
y	decreasing		decreasing		increasing		increasing

Plug in values:

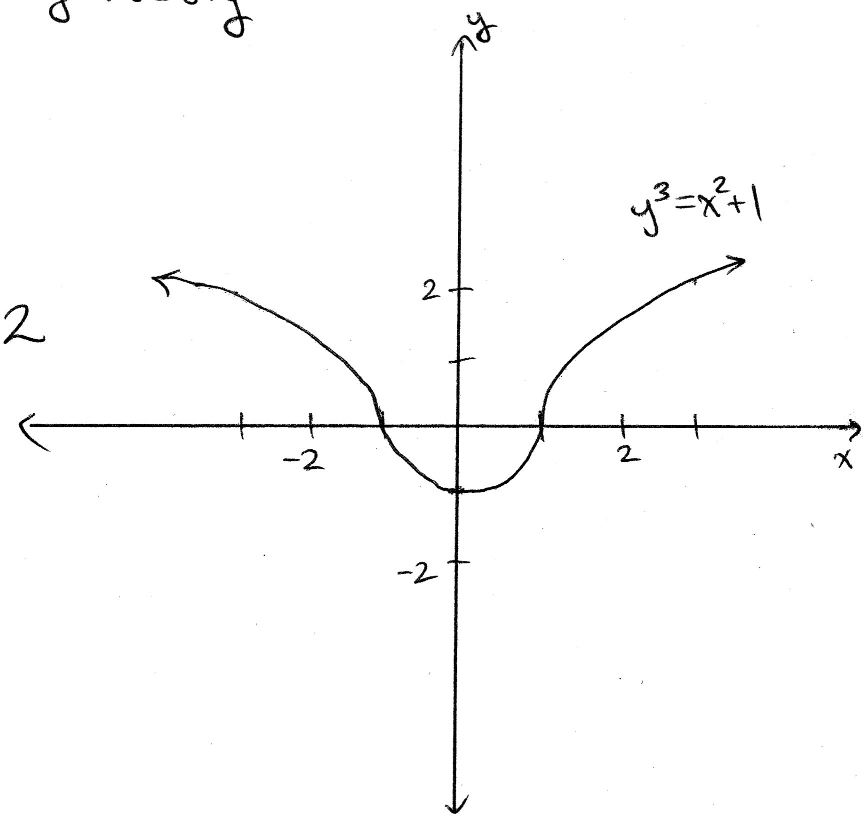
$y(0) = -1$

$y(\pm 1) = 0$

$y(\pm 3) = (9-1)^{\frac{1}{3}} = 2$

Behaviour as $x \to \pm\infty$:

$\lim_{x \to \pm\infty} (x^2-1)^{\frac{1}{3}} = \infty$

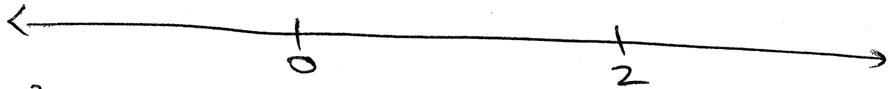


10. Sketch the curve $y = e^{x^3-3x^2}$. no zeros

$$y' = e^{x^3-3x^2}(3x^2-6x) = 3x\,e^{x^3-3x^2}(x-2)$$

critical values $x = 0, 2$

$y'' =$ too complicated



y'	$+$	$-$	$+$
y	increasing	decreasing	increasing

Plug in values:

$y(0) = e^0 = 1$

$y(1) = e^{1^3-3(1^2)} = e^{-2} = \frac{1}{e^2}$

$y(2) = e^{2^3-3(2^2)} = e^{8-12} = e^{-4} = \frac{1}{e^4}$

$y(3) = e^{3^3-3(3^2)} = e^0 = 1$

$y(-1) = e^{(-1^3)-3(-1)^2} = e^{-1-3} = e^{-4} = \frac{1}{e^4}$

Behavior as $x \to \pm\infty$:

the leading term x^3 dominates

$$\lim_{x\to\infty} e^{x^3-3x^2} = e^{\lim_{x\to\infty} x^3-3x^2} = e^{\infty} = \infty,$$

$$\lim_{x\to-\infty} e^{x^3-3x^2} = e^{\lim_{x\to-\infty} x^3-3x^2} = e^{-\infty} = 0$$

horizontal asymptote at $y=0$ as $x \to -\infty$