Math 1A Practice Midterm 2 – Solutions

1. $h'(\theta) = -\csc\theta\cot\theta + e^{\theta}\frac{d}{d\theta}\cot\theta + \cot\theta\frac{d}{d\theta}e^{\theta}$ product rule

$= -\csc\theta\cot\theta + e^{\theta}(-\csc^2\theta) + \cot\theta\,(e^{\theta})$

$= \boxed{-\csc\theta\cot\theta - e^{\theta}\csc^2\theta + e^{\theta}\cot\theta}.$

2. $\frac{d}{dx}\sin(\tan(\sqrt{\sin x}))$

$= \cos(\tan(\sqrt{\sin x}))\,\frac{d}{dx}\tan(\sqrt{\sin x})$ chain rule; $\frac{d}{du}\sin u = \cos u$

$= \cos(\tan(\sqrt{\sin x}))\sec^2(\sqrt{\sin x})\,\frac{d}{dx}\sqrt{\sin x}$ chain rule; $\frac{d}{du}\tan u = \sec^2 u$

$= \cos(\tan(\sqrt{\sin x}))\sec^2(\sqrt{\sin x})\left(\frac{1}{2\sqrt{\sin x}}\right)\frac{d}{dx}\sin x$ chain rule; $\frac{d}{du}\sqrt{u} = \frac{1}{2\sqrt{u}}$

$= \boxed{\dfrac{\cos(\tan(\sqrt{\sin x}))\sec^2(\sqrt{\sin x})\cos x}{2\sqrt{\sin x}}}.$

3. $3x^2 + (x^2y' + y(2x)) + 8yy' = 0$

$3x^2 + y'(x^2+8y) + 2xy = 0$

$y'(x^2+8y) = -3x^2 - 2xy$

$y' = \boxed{\dfrac{-3x^2-2xy}{x^2+8y}}.$

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4. (Note: $f(x) = \frac{1}{3x^3}$).

Derivative	
0	$\frac{1}{3}x^{-3}$
1	$-x^{-4}$
2	$4x^{-5}$
3	$-4\cdot 5x^{-6}$
4	$4\cdot 5\cdot 6\, x^{-7}$
5	$-4\cdot 5\cdot 6\cdot 7\, x^{-8}$
$\vdots$	
n	$(-1)^n\, 4\cdot 5\cdot 6\cdot \ldots \cdot (n+1)(n+2)x^{-(n+3)}$

odd derivatives have a minus sign in front

2 and 3 are missing, so divide by 6

$$= \boxed{\frac{(-1)^n (n+2)!}{6} x^{-(n+3)}}.$$

5. $y = \log_2(1-3x) = \dfrac{\ln(1-3x)}{\ln 2}$

chain rule; $\frac{d}{du}\ln u = \frac{1}{u}$

$$y' = \left(\frac{1}{\ln 2}\right)\left(\frac{1}{1-3x}\right)\frac{d}{dx}(1-3x)$$

$$= \frac{1}{(\ln 2)(1-3x)}(-3)$$

$$= \boxed{\frac{-3}{(\ln 2)(1-3x)}}.$$

6. $$y' = \frac{1}{1-(\sqrt{x})^2}\frac{d}{dx}\sqrt{x} \quad \left(\text{chain rule; } \frac{d}{du}\tanh^{-1}(u) = \frac{1}{1-u^2}\right)$$

$$= \frac{1}{1-x}\left(\frac{1}{2\sqrt{x}}\right) = \boxed{\frac{1}{2(1-x)\sqrt{x}}.}$$

7. The linearization of $f(x)$ at $x=a$ is

$$f(a) + f'(a)(x-a).$$

$$f(1) = 1^3 = 1 \qquad f'(x) = 3x^2$$

$$f'(1) = 3(1^2) = 3$$

We get

$$1 + 3(x-1) = \boxed{3x-2.}$$

8. (Note: $y = x\ln(x)$.)

$$\frac{dy}{dx} = x\frac{d}{dx}\ln(x) + \ln(x)\frac{d}{dx}x \qquad \text{(product rule)}$$

$$\frac{dy}{dx} = x\left(\frac{1}{x}\right) + \ln(x)(1)$$

$$\frac{dy}{dx} = 1 + \ln(x)$$

$$\boxed{dy = (1+\ln(x))\,dx.}$$

9. $f'(x) = 3x^2 + 6x - 24$

$$= 3(x^2 + 2x - 8)$$

$$= 3(x+4)(x-2).$$

$f'(x)$ exists for all x, so the critical numbers of $f(x)$ are when $f(x) = 0$, ie. $\boxed{x = -4, 2.}$

10. Let $f(x) = x^3 - 15x + c$. We need to show $f(x) = 0$ has at most one solution.

Solution 1

$f'(x) = 3x^2 - 15.$

For x in the interval $[-2,2]$, we have $x^2 \leq 4$. So,

$$f'(x) \leq 3(4) - 15$$

$$f'(x) \leq 12 - 15$$

$$f'(x) \leq -3$$

Because $f'(x) < 0$, $f(x)$ is decreasing on $[-2,2]$, so $f(x) = 0$ has at most one solution.

Solution 2

Suppose $f(x) = 0$ had two solutions on the interval $[-2,2]$, say $x = a$ and $x = b$. Because $f(x)$ is differentiable for all x, $f(x)$ satisfies the hypotheses of Rolle's theorem on the closed interval from a to b. This gives a number c between a and b (so c is in the interval $[-2,2]$) such that

$$f'(c) = 0.$$

But solving $f'(x) = 0$, we have

$$f'(x) = 0$$

$$3x^2 - 15 = 0$$

$$3x^2 = 15$$

$$x^2 = 5$$

$$x = \pm\sqrt{5}$$

So $c = \pm\sqrt{5}$. But neither $\sqrt{5}$ nor $-\sqrt{5}$ is in the interval $[-2,2]$. This is a contradiction.

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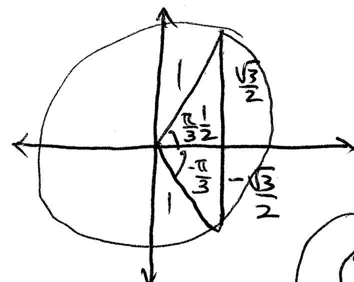
11. $f'(x) = 1 - 2\cos x$.

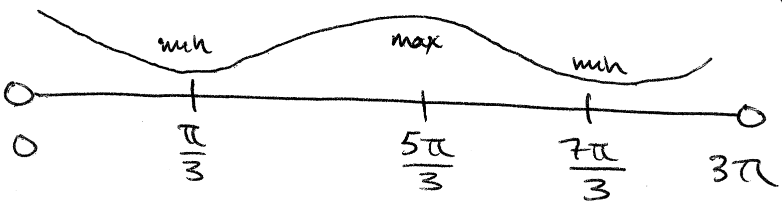
$f'(x)$ exists for all x, so the critical values occur when $f'(x) = 0$:

$$1 - 2\cos x = 0$$
$$1 = 2\cos x$$
$$\cos x = \frac{1}{2}$$



$$x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}.$$

general solution is $x = \pm\frac{\pi}{3} + 2\pi n$ (n an integer); these are the values in the interval $(0, 3\pi)$



$f'(x)$	$-$	$+$	$-$	$+$
$f(x)$	decreasing	increasing	decreasing	increasing

So, $f(x)$ is increasing on $\left(\frac{\pi}{3}, \frac{5\pi}{3}\right)$ and $\left(\frac{7\pi}{3}, 3\pi\right)$, and decreasing on $\left(0, \frac{\pi}{3}\right)$ and $\left(\frac{5\pi}{3}, \frac{7\pi}{3}\right)$. $f(x)$ has a local maximum at $x = \frac{5\pi}{3}$ with value $f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} + \sqrt{3}$, and local minima at $x = \frac{\pi}{3}$ and $x = \frac{7\pi}{3}$ with values $f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - \sqrt{3}$, $f\left(\frac{7\pi}{3}\right) = \frac{7\pi}{3} - \sqrt{3}$.

12.

Solution 1

$$\lim_{x\to 0} \frac{\tan(px)}{\tan(qx)}$$

indeterminate of the form $\frac{0}{0}$, so we can apply L'Hôpital's rule

$$= \lim_{x\to 0} \frac{p\sec^2(px)}{q\sec^2(qx)}$$

$$= \frac{p\sec^2(0)}{q\sec^2(0)} = \boxed{\frac{p}{q}}.$$

Solution 2

$$\lim_{x\to 0} \frac{\tan(px)}{\tan(qx)} = \lim_{x\to 0} \frac{\frac{\sin(px)}{\cos(px)}}{\frac{\sin(qx)}{\cos(qx)}}$$

$$= \lim_{x\to 0} \left(\frac{p}{q}\right)\left(\frac{\frac{\sin(px)}{px}}{\frac{\sin(qx)}{qx}}\right)\left(\frac{\frac{1}{\cos(px)}}{\frac{1}{\cos(qx)}}\right)$$

$$= \left(\frac{p}{q}\right)\left(\frac{1}{1}\right)\left(\frac{\frac{1}{1}}{\frac{1}{1}}\right) = \boxed{\frac{p}{q}.}$$

using $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$.

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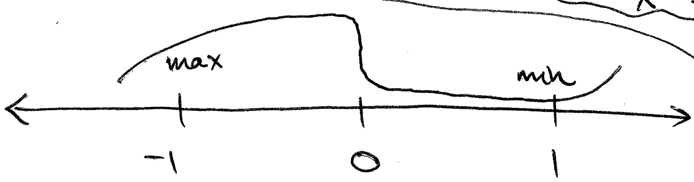
13. $\lim\limits_{x\to 0} \dfrac{x+\sin x}{x+\cos x} = \dfrac{0+\sin 0}{0+\cos 0} = \dfrac{0+0}{0+1} = \boxed{0.}$

14. $y = x - 3x^{\frac{1}{3}} = x^{\frac{1}{3}}\left(x^{\frac{2}{3}}-3\right).$ zeros: $x=0,\ x=\pm 3^{\frac{3}{2}} = \pm\sqrt{27}$

$y' = 1 - x^{-\frac{2}{3}}$ critical values: $x=0$ (y' not defined), $x=\pm 1$ ($y'=0$)

y is defined at $x=0$ and $\lim\limits_{x\to 0} 1-x^{-\frac{2}{3}} = \infty$, so there is a vertical tangent line at $x=0$

$\lim\limits_{x\to\pm\infty} 1-x^{-\frac{2}{3}} = 1$, so as $x\to\pm\infty$ the slope of $y=f(x)$ approaches 1



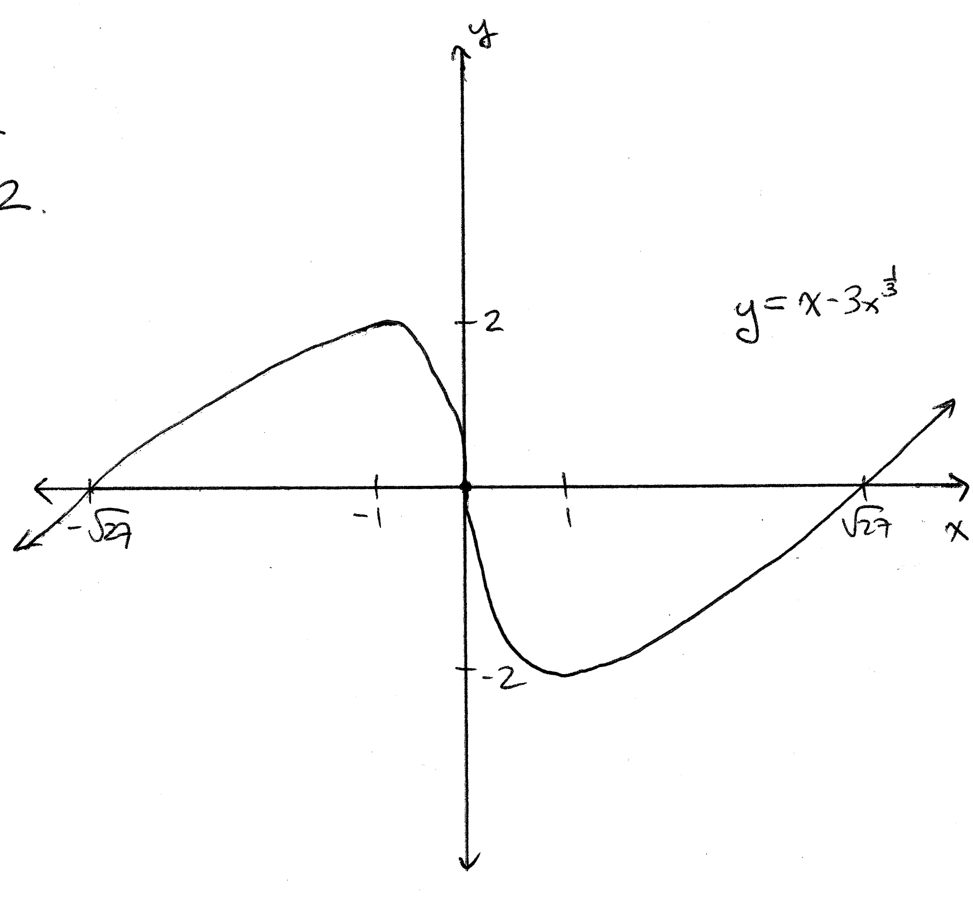
y'	+	−	−	+
y	increasing	decreasing	decreasing	increasing

$y'' = \frac{2}{3}x^{-\frac{5}{3}}$ inflection point at $x=0$

plug in points:

$y(1) = 1 - 3\sqrt[3]{1} = -2$

$y(-1) = -1 - 3\sqrt[3]{-1} = 2.$



15. The domain of $f(x)$ is $(0,\infty)$. {no zeros}

$y = x^{\frac{1}{x}}$

$\ln y = \ln\left(x^{\frac{1}{x}}\right)$

$\ln y = \frac{1}{x}\ln(x)$

$\frac{y'}{y} = \frac{1}{x}\frac{d}{dx}\ln(x) + \ln(x)\frac{d}{dx}\frac{1}{x}$ (product rule)

$\frac{y'}{y} = \frac{1}{x}\left(\frac{1}{x}\right) + \ln(x)\left(-\frac{1}{x^2}\right) = \frac{1}{x^2}(1-\ln x)$

$y' = \frac{x^{\frac{1}{x}}}{x^2}(1-\ln x)$ {y' exists for all $x>0$, so the critical points occur when $y'=0$, ie, $x=e$}

max

0 ——— e ———>

y' (+) (−)

y increasing decreasing

y'' = too complicated

Behavior as $x \to \infty$:

{indeterminate of the form $\frac{\infty}{\infty}$, so we can apply L'Hôpital's rule}

$$\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty}\left(e^{\ln(x)}\right)^{\frac{1}{x}} = e^{\lim_{x\to\infty}\frac{\ln(x)}{x}} = e^{\lim_{x\to\infty}\frac{\frac{1}{x}}{1}} = e^0 = 1.$$

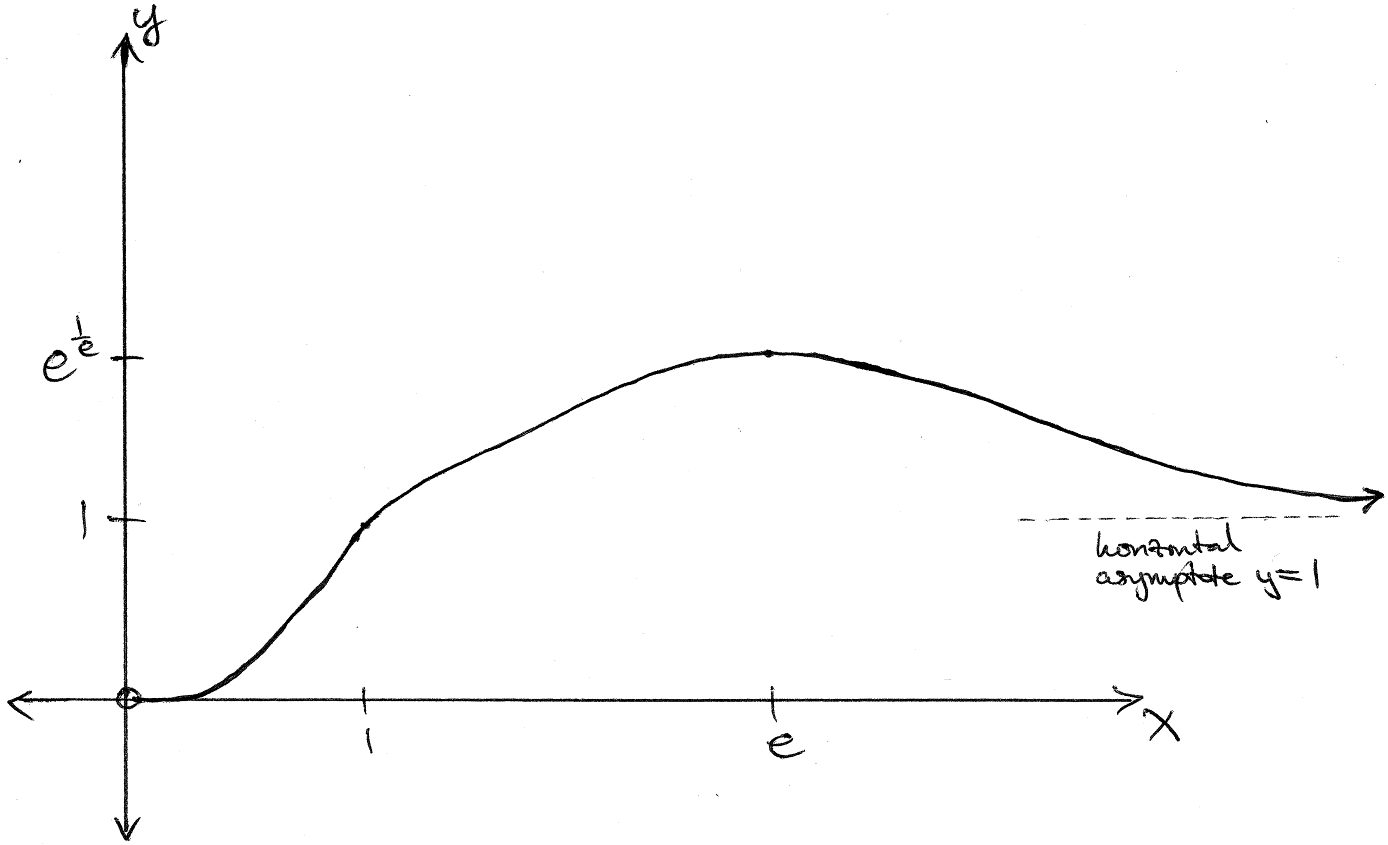
{horizontal asymptote at $y=1$ as $x\to\infty$}

Behavior as $x \to 0+$: plug in points

$f(1) = 1$

$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

$f\left(\frac{1}{4}\right) = \left(\frac{1}{4}\right)^4 = \frac{1}{256}$ {$f(x)$ decreases rapidly to 0 as $x \to 0+$ $\Rightarrow$ horizontal tangent line at $x=0$}



y
$e^{\frac{1}{e}}$
1
horizontal asymptote y = 1
1
e
x