Math 1A
2009 Final

1. Evaluate the limit $\lim_{x\to-4}\frac{\sqrt{x^2+9}-5}{x+4}$.

   Solution. For $x \neq -4$,

$$\begin{aligned}\frac{\sqrt{x^2+9}-5}{x+4} &= \frac{\sqrt{x^2+9}-5}{x+4}\,\frac{\sqrt{x^2+9}+5}{\sqrt{x^2+9}+5}\\ &= \frac{(x^2+9)-25}{x+4(\sqrt{x^2+9}+5)}\\ &= \frac{x^2-16}{(x+4)(\sqrt{x^2+9}+5)}\\ &= \frac{(x+4)(x-4)}{(x+4)(\sqrt{x^2+9}+5)}\\ &= \frac{x-4}{\sqrt{x^2+9}+5}\end{aligned}$$

   Thus

$$\begin{aligned}\lim_{x\to-4}\frac{\sqrt{x^2+9}-5}{x+4} &= \lim_{x\to-4}\frac{x-4}{\sqrt{x^2+9}+5}\\ &= \frac{(-4)-4}{\sqrt{(-4)^2+9}+5}\\ &= \frac{-8}{10}\\ &= -\frac{4}{5}\end{aligned}$$

□

2. Differentiate $e^x/\sin(x)$.

   Solution.

$$\begin{aligned}\frac{d}{dx}\left(\frac{e^x}{\sin(x)}\right) &= \frac{\sin(x)e^x - e^x\cos(x)}{\sin^2(x)}\\ &= e^x\csc(x)(1-\cot(x))\end{aligned}$$

□

3. Find the derivative of the function $y = \ln(\ln(\ln(x)))$.

Solution. If we let $f(x) = \ln(x)$ then $f'(x) = \frac{1}{x}$ and so

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(f(f(f(x)))) \\ &= f'(f(f(x)))f'(f(x))f'(x) \text{ by the chain rule} \\ &= \frac{1}{\ln(\ln(x))}\frac{1}{\ln(x)}\frac{1}{x} \\ &= \frac{1}{x\ln(x)\ln(\ln(x))}\end{aligned}$$

□

4. Find dy/dx if $x\sin(y^2) = y\sin(x^2)$.

Solution.

$$\begin{aligned}x\sin(y^2) = y\sin(x^2) &\implies \sin(y^2) + x\cos(y^2)2y\frac{dy}{dx} = \frac{dy}{dx}\sin(x^2) + y\cos(x^2)2x \\ &\iff (\sin(y^2) - 2xy\cos(x^2)) = \frac{dy}{dx}(\sin(x^2) - 2xy\cos(y^2)) \\ &\iff \frac{dy}{dx} = \frac{\sin(y^2) - 2xy\cos(x^2)}{\sin(x^2) - 2xy\cos(y^2)}\end{aligned}$$

□

5. A bacteria culture initially contains 1000 cells, and grows at a rate proportional to its size. After 1 hour the population is 1100 cells. Find an expression for the approximate number of cells after t hours.

Solution. Let $C(t)$ be the number of cells after t hours. Then

$$\frac{dC}{dt} = kC \text{ for some constant } k.$$

Thus $C(t) = C_0e^{kt}$ for some constant C_0. Since $C(0) = C_0$ and $C(0) = 1000$ we have that $C_0 = 1000$. We are also told that $C(1) = 1100$ but $C(1) = 1000e^k$ and so

$$\begin{aligned}1100 = 1000e^k &\iff 1.1 = e^k \\ &\iff k = \ln(1.1)\end{aligned}$$

Thus

$$C(t) = 1000e^{\ln(1.1)t}$$

□

6. Find $\lim_{x\to\infty} xe^{1/x} - x$.

Solution.

$$xe^{1/x} - x = x(e^{1/x} - 1) = \frac{e^{1/x} - 1}{1/x}$$

where

$$\lim_{x\to\infty} e^{1/x} - 1 = 0, \qquad \lim_{x\to\infty} 1/x = 0$$

Thus

$$\begin{aligned}\lim_{x\to\infty} xe^{1/x} - x &= \lim_{x\to\infty} \frac{e^{1/x} - 1}{1/x}\\ &= \lim_{x\to\infty} \frac{-\frac{1}{x^2}e^{1/x}}{-1/x^2} \text{ by l'Hôpital's rule}\\ &= \lim_{x\to\infty} e^{1/x}\\ &= 1\end{aligned}$$

□

7. Sketch the graph of $y = \sin(x^2)$.

Solution. Click here to see the graph. □

8. Explain why Newton's method does not converge to the root -1 of $2x^3 - x^2 + 3 = 0$ if the initial approximation is chosen to be $x_1 = 1$.

Solution. Let $f(x) = 2x^3 - x^2 + 3$. Then $f'(x) = 6x^2 - 2x$. Thus

$$\begin{aligned}x_2 &= x_1 - \frac{f(x_1}{f'(x_1)}\\ &= 1 - \frac{f(1)}{f'(1)}\\ &= 0\end{aligned}$$

But then $f'(x_2) = 0$ and we cannot use Newton's method to obtain a third approximation. □

9. Find the most general anti-derivative of $(2 + x^2)/(1 + x^2)$.

Solution.

$$\begin{aligned}\int \frac{2 + x^2}{1 + x^2}\,dx &= \int \frac{1 + (1 + x^2)}{1 + x^2}\,dx\\ &= \int \frac{1}{1 + x^2} + 1\,dx\\ &= \arctan(x) + x + C\end{aligned}$$

□

10. A stone is dropped off a cliff and hits the ground after 2 seconds. What is the height of the cliff, assuming that acceleration due to gravity is 10 m/s^{-2}.

Solution. Let $a(t)$ be the acceleration of the stone as a function of time. Let $v(t)$ be the velocity of the stone as a function of time. Let $s(t)$ be the height of the stone as a function of time. Then

$$\begin{aligned} a(t) = -10 &\implies v(t) = -10t + v_0 \\ &\implies v(t) = -10t \text{ since v(0)} = 0 \\ &\implies s(t) = -5t^2 + s_0 \\ &\implies s_0 = 5(2)^2 \text{ since } s(2) = 0 \\ &\implies s_0 = 20 \end{aligned}$$

Thus the height of cliff is 20 m. □

11. Use the midpoint rule for 3 points to approximate the integral $\int_0^6 x^2\, dx$.

Solution.

$$\begin{aligned} \int_0^6 x^2\, dx &\approx (1)^2(2) + (3)^2(2) + (5)^2(2) \\ &= 2 + 18 + 50 \\ &= 70 \end{aligned}$$

□

12. Evaluate the integral $\int_{-3}^3 \sqrt{9 - x^2}\, dx$ by interpreting it as an area.

Solution. The integral represents the half the area of a circle of radius 3 (see picture) and so

$$\int_{-3}^3 \sqrt{9 - x^2}\, dx = \frac{1}{2}\pi(3)^2 = \frac{9\pi}{2}$$

□

13. Find the derivative of the function $g(x) = \int_x^1 e^{-t^2}\, dt$.

Solution.

$$\begin{aligned} \frac{d}{dx}\left(\int_x^1 e^{-t^2}\, dt\right) &= \frac{d}{dx}\left(-\int_1^x e^{-t^2}\, dt\right) \\ &= -\frac{d}{dx}\left(\int_1^x e^{-t^2}\, dt\right) \\ &= -e^{-x^2} \end{aligned}$$

□

14. The Fresnel integral is given by $S(x) = \int_0^x \sin(\pi t^2/2)\ dt$. Sketch its graph, and find the values of x at which it has local maximum values.

Solution.

$$S'(x) = \sin(\pi x^2/2)$$

and so

$$\begin{aligned} S'(x) = 0 &\iff \pi x^2/2 = \pi n \text{ for some } n \in \mathbb{Z} \\ &\iff x^2 = 2n \text{ for some } n \in \mathbb{Z} \\ &\iff x = \pm\sqrt{2n} \end{aligned}$$

$S'(x)$ changes sign, from positive to negative, at $x = \sqrt{4n-2}$ and $x = -2\sqrt{n}$ for $n \geq 1$. Thus the local maximum occur at $x = \sqrt{4n-2}$ and $x = -2\sqrt{n}$ for $n \geq 1$. See page 390 of the textbook for the graph (Figure 8). □

15. Evaluate the integral $\int_0^{3\pi/2} |\sin(x)|\ dx$.

Solution.

$$\sin(x) \geq 0 \text{ for } x \in [0, \pi] \text{ and } \sin(x) \leq 0 \text{ for } x \in [\pi, \frac{3\pi}{2}]$$

so

$$\begin{aligned} \int_0^{3\pi/2} |\sin(x)|\ dx &= \int_0^{\pi} \sin(x)\ dx + \int_{\pi}^{3\pi/2} -\sin(x)\ dx \\ &= (-\cos(x)\Big|_0^{\pi}) + (\cos(x)\Big|_{\pi}^{3\pi/2}) \\ &= (1 - (-1)) + (0 - (-1)) \\ &= 3 \end{aligned}$$

□

16. Find the area of the finite region bounded by the lines $x = 0, y = 1, y = x^{1/4}$.

Solution. Click here to see the region. Thus the area is

$$\begin{aligned} \int_0^1 1 - x^{1/4}\ dx &= x - \frac{4}{5}x^{5/4}\Big|_0^1 \\ &= 1 - \frac{4}{5} \\ &= \frac{1}{5} \end{aligned}$$

□

17. Prove that if n is a positive integer then $1 + 1/2 + 1/3 + \cdots + 1/n > \ln(n)$.

Solution.

$$\ln(n) = \ln(n) - \ln(1) = \int_1^n 1/x \, dx$$

while

$$1 + 1/2 + 1/3 + \cdots + 1/n$$

corresponds to a left Riemann sum for $1/x$ from 1 to $n + 1$. Since $1/x$ is a decreasing function, a left Riemann sum produces an upper sum which overapproximates the area under $1/x$ from 1 to $n + 1$. Click here to see a picture. The sum

$$1 + 1/2 + \cdots + 1/n$$

corresponds to the area of the rectangles in the picture while

$$\ln(n)$$

corresponds to the area under the curve $f(x) = 1/x$ from 1 to n in the picture. □

18. Evaluate the indefinite integral $\int \frac{\cos(\ln(x))}{x} \, dx$.

Solution.

$$\begin{aligned}\int \frac{\cos(\ln(x))}{x} \, dx &= \int \cos(u) \, du \text{ where } u = \ln(x) \\ &= \sin(u) + C \\ &= \sin(\ln(x)) + C\end{aligned}$$

□

19. Evaluate the definite integral $\int_0^2 (x - 1)^{10} \, dx$.

Solution.

$$\begin{aligned}\int_0^2 (x - 1)^{10} \, dx &= \frac{1}{11}(x - 1)^{11}\Big|_0^2 \\ &= \frac{1}{11} - (-\frac{1}{11}) \\ &= \frac{2}{11}\end{aligned}$$

□

20. Evaluate the indefinite integral $\int \tan(x)\ dx$.

Solution.

$$\begin{aligned}\int \tan(x)\ dx &= \int \frac{\sin(x)}{\cos(x)}\ dx \\ &= \int -\frac{1}{u}\ du \text{ where } u = \cos(x) \\ &= -\ln(|u|) + C \\ &= -\ln(|\cos(x)|) + C \\ &= \ln(|\sec(x)|) + C\end{aligned}$$

□

21. Find the volume of the region obtained by rotating the region bounded by the curves $y = x^3, y = 0, x = 1$, about the x-axis.

Solution.

$$\begin{aligned}\int_0^1 \pi(x^3)^2\ dx &= \pi \int_0^1 x^6\ dx \\ &= \frac{\pi}{7}x^7\Big|_0^1 \\ &= \frac{\pi}{7}\end{aligned}$$

□