2006 Final – Solutions

1) $$\lim_{h\to 0} \frac{(3+h)^{-1}-3^{-1}}{h} = \lim_{h\to 0} \frac{\frac{1}{3+h}-\frac{1}{3}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{3-(3+h)}{(3+h)3}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{-h}{(3+h)3}}{h}$$

$$= \lim_{h\to 0} \frac{-1}{(3+h)3}$$

$$= \frac{-1}{(3+0)3} = \boxed{-\frac{1}{9}}.$$

2) $$\frac{d}{dx}\frac{x}{1+x^2} = \frac{(1+x^2)\frac{d}{dx}x - x\frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$ (quotient rule)

$$= \frac{(1+x^2)(1)-x(2x)}{(1+x^2)^2}$$

$$= \frac{1+x^2-2x^2}{(1+x^2)^2}$$

$$= \boxed{\frac{1-x^2}{(1+x^2)^2} = \frac{1-x^2}{1+2x^2+x^4}}.$$

3) $$\frac{d}{dx}\sin(\cos(\sqrt{x})) = \cos(\cos(\sqrt{x}))\frac{d}{dx}\cos(\sqrt{x})$$ (chain rule)

$$= \cos(\cos(\sqrt{x}))(-\sin(\sqrt{x}))\frac{d}{dx}\sqrt{x}$$ (chain rule again)

$$= \cos(\cos(\sqrt{x}))(-\sin(\sqrt{x}))\left(\frac{1}{2\sqrt{x}}\right)$$

$$= \boxed{\frac{-\cos(\cos(\sqrt{x}))\sin(\sqrt{x})}{2\sqrt{x}}}$$

4) Taking the derivative with respect to x,

$$3x^2 + \left(x^2\frac{dy}{dx} + y\frac{d}{dx}x^2\right) + \frac{d}{dx}y^2 = 0$$

$$3x^2 + x^2\frac{dy}{dx} + y(2x) + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(x^2+2y) = -3x^2 - 2xy$$

$$\boxed{\frac{dy}{dx} = \frac{-3x^2-2xy}{x^2+2y}}$$

5)

derivative	function
0	e^{3x}
1	$3e^{3x}$
2	3^2e^{3x}
3	3^3e^{3x}
4	3^4e^{3x}
$\vdots$	$\vdots$
57	$\boxed{3^{57}e^{3x}}$

6) $\lim\limits_{x\to 0+} x^{x^2} = \lim\limits_{x\to 0+} \left(e^{\ln x}\right)^{x^2}$

$= e^{\lim\limits_{x\to 0+} (\ln x)(x^2)}$ ← turn into a fraction so that we can use L'Hôpital's rule

$= e^{\lim\limits_{x\to 0+} \frac{\ln x}{x^{-2}}}$ ← $\frac{\infty}{\infty}$, so we can apply L'Hôpital's rule

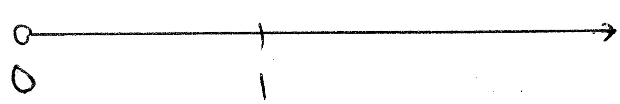
$= e^{\lim\limits_{x\to 0+} \frac{\frac{1}{x}}{-2x^{-3}}}$

$= e^{\lim\limits_{x\to 0+} -\frac{1}{2}x^2} = e^0 = \boxed{1.}$

7) Let $f(x) = x + \frac{1}{x}$ for $x > 0$. We want to minimize $f(x)$ for $x > 0$. The derivative of $f(x)$ is

$$f'(x) = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2} = \frac{(x-1)(x+1)}{x^2}.$$

So, $f(x)$ has exactly one critical point, at $x=1$:



$f'(x)$	$-$	$+$
$f(x)$	decreasing	increasing

So, $f(x)$ is minimized at $\boxed{x=1.}$

8) Let $f(x) = x^2 - 26$, which has $\sqrt{26}$ as a root. We apply one iteration of Newton's method with $x_1 = 5$:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$f(5) = 5^2 - 26 = -1$

$f'(x) = 2x$
$f'(5) = 2(5) = 10$

$$= 5 - \frac{-1}{10}$$

$$= 5 + \frac{1}{10} = \boxed{5.1}$$

9) $$\int \frac{\sin\theta}{\cos^2\theta}\, d\theta$$

substitute $u = \cos\theta$:
$\frac{du}{d\theta} = -\sin\theta \qquad -du = \sin\theta\, d\theta$

(a good choice because the derivative of $\cos\theta$ is $-\sin\theta$, which appears in the integrand

$$= \int \frac{(-du)}{u^2}$$

$$= \frac{1}{u} + C = \frac{1}{\cos\theta} + C = \sec\theta + C$$

Alternatively, we could have noticed right away that $\frac{\sin\theta}{\cos^2\theta} = \sec\theta\tan\theta = \frac{d}{d\theta}\sec\theta$.

Now, the domain of $\frac{\sin\theta}{\cos^2\theta}$ is when $\cos\theta \neq 0$, i.e. $\theta \neq \frac{\pi}{2} + \pi n$, where n is an integer. We may write the domain as a union of intervals:

$$\cdots \cup \left(-\tfrac{5\pi}{2}, -\tfrac{3}{2}\pi\right) \cup \left(-\tfrac{3}{2}\pi, -\tfrac{1}{2}\pi\right) \cup \left(-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi\right) \cup \left(\tfrac{1}{2}\pi, \tfrac{3}{2}\pi\right) \cup \cdots$$

We may pick a different constant C on each interval. Indexing the intervals as $\left(\pi n - \frac{\pi}{2}, \pi n + \frac{\pi}{2}\right)$ for integers n, the most general antiderivative of $\frac{\sin\theta}{\cos^2\theta}$ is

$$\sec\theta + C_n \text{ for } \theta \text{ in the interval } \left(\pi n - \tfrac{\pi}{2}, \pi n + \tfrac{\pi}{2}\right),$$
n an integer,

where C_n are constants defined for every integer n.

10) (We assume that the domain of $f(x)$ is $(0, \infty)$.)

Since $f'(x)$ is an antiderivative of $f''(x)$,

$$f'(x) = -\frac{1}{x} + C. \qquad (C \text{ a constant})$$

Since $f(x)$ is an antiderivative of $f'(x)$,

$$f(x) = -\ln(x) + Cx + D. \qquad (D \text{ a constant})$$

We plug in values to find C and D:

$f(1) = 1$

$-\ln(1) + C(1) + D = 1$

$C + D = 1$

$f(2) = 0$

$-\ln(2) + C(2) + D = 0$

$2C + D = \ln(2)$

This gives the system

$$\begin{cases} C + D = 1 \\ 2C + D = \ln(2). \end{cases}$$

Subtracting the first equation from the second gives

$$C = \ln(2) - 1,$$

and from the first equation,

$$D = 1 - C = 1 - (\ln(2) - 1) = 2 - \ln(2).$$

So $\boxed{f(x) = -\ln(x) + (\ln(2) - 1)x + 2 - \ln(2).}$

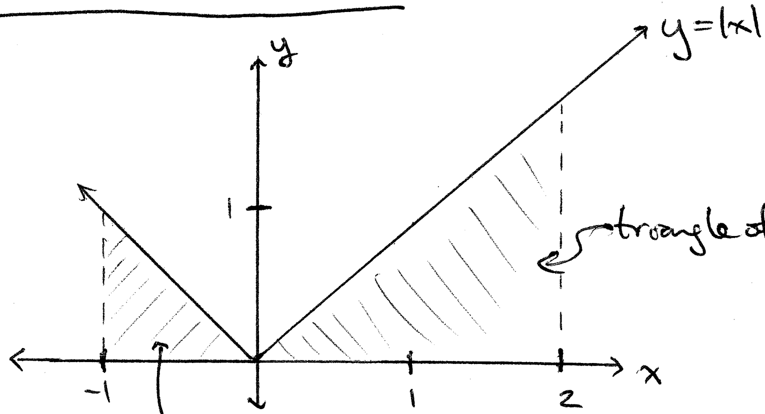
11) $\int_1^5 2f(x) - 3g(x)\,dx = 2\int_1^5 f(x)\,dx - 3\int_1^5 g(x)\,dx$

$= 2(1) - 3(2)$

$= 2 - 6$

$= \boxed{-4.}$

12)



triangle of area $\frac{1}{2}$(base)(height) $= \frac{1}{2}(2)(2) = 2$

triangle of area $\frac{1}{2}$(base)(height) $= \frac{1}{2}(1)(1) = \frac{1}{2}$

By computing areas of triangles,

$$\int_{-1}^{2} |x|\,dx = \frac{1}{2} + 2 = \boxed{\frac{5}{2}}.$$

13) By the fundamental theorem of calculus, $\boxed{g'(x) = x^2 \ln(x)}$

14) Write $y = \int_{\cos(x)}^{x} \cos(t^2)\,dt = \int_0^x \cos(t^2)\,dt - \int_0^{\cos(x)} \cos(t^2)\,dt.$

By the fundamental theorem of calculus,

$$y' = \cos(x^2) - \cos(\cos^2(x))\,\frac{d}{dx}\cos(x) \qquad \text{(chain rule)}$$

$$= \cos(x^2) - \cos(\cos^2(x))(-\sin(x))$$

$$= \boxed{\cos(x^2) + \cos(\cos^2(x))\sin(x).}$$

15) $$\int_1^{64} \frac{1+x^{\frac{1}{3}}}{\sqrt{x}}\,dx = \int_1^{64} \frac{1+x^{\frac{1}{3}}}{x^{\frac{1}{2}}}\,dx = \int_1^{64} x^{-\frac{1}{2}} + x^{-\frac{1}{6}}\,dx$$

$$= \left[2x^{\frac{1}{2}} + \frac{6}{5}x^{\frac{5}{6}}\right]_1^{64}$$

$64^{\frac{1}{2}} = \sqrt{64} = 8$

$64^{\frac{1}{6}} = 2$ since $2^6 = 64$

$$= \left(2(64^{\frac{1}{2}}) + \frac{6}{5}(64^{\frac{5}{6}})\right) - \left(2(1^{\frac{1}{2}}) + \frac{6}{5}(1^{\frac{5}{6}})\right)$$

$$= \left(2(8) + \frac{6}{5}(2^5)\right) - \left(2 + \frac{6}{5}\right)$$

$$= 16 + \frac{6}{5}(32) - 2 - \frac{6}{5} = \frac{80+192-10-6}{5} = \boxed{\frac{256}{5}}$$

16) $\int_0^{\frac{\pi}{4}} \frac{1+\cos^2\theta}{\cos^2\theta}\,d\theta = \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2\theta}+1\,d\theta = \int_0^{\frac{\pi}{4}} \sec^2\theta+1\,d\theta$

$= \left[\tan\theta+\theta\right]_0^{\frac{\pi}{4}}$

 $\tan\left(\frac{\pi}{4}\right)=1$

$= \left(\tan\left(\frac{\pi}{4}\right)+\frac{\pi}{4}\right)-\left(\tan(0)+0\right)$

$= \left(1+\frac{\pi}{4}\right)-(0+0) = \boxed{1+\frac{\pi}{4}.}$

(a good choice since the derivative of $2y^4-1$ is $8y^3$, which appears in the integrand)

17) $\int y^3\sqrt{2y^4-1}\,dy$ ← substitute $u=2y^4-1$: $\frac{du}{dy}=8y^3$ $\quad \frac{du}{8}=y^3dy$

$= \int \sqrt{u}\left(\frac{du}{8}\right)$

$= \int \frac{1}{8}u^{\frac{1}{2}}du$

$= \frac{1}{8}\left(\frac{2}{3}u^{\frac{3}{2}}\right)+C$ (C a constant)

$= \frac{1}{12}(2y^4-1)^{\frac{3}{2}}+C.$

(We may pick different constants on each interval $\left(-\infty,-\frac{1}{\sqrt[4]{2}}\right)$ and $\left(\frac{1}{\sqrt[4]{2}},\infty\right)$ of the domain.)

(a good choice since the derivative of $\cos(x)$ is $-\sin(x)$, which appears in the integrand)

18) $\int \tan(x)\ln(\cos(x))\,dx = \int \frac{\sin(x)}{\cos(x)}\ln(\cos(x))\,dx$ ← substitute $u=\cos(x)$: $\frac{du}{dx}=-\sin(x)$ $\quad -du=\sin(x)\,dx$

$= \int \frac{\ln(u)}{u}(-du) = \int -\frac{\ln(u)}{u}du$ ← substitute $t=\ln(u)$: $\frac{dt}{du}=\frac{1}{u}$ $\quad dt=\frac{du}{u}$

$= \int -t\,dt$

$= \frac{-t^2}{2}+C$ (C a constant)

(a good choice since the derivative of $\ln(u)$ is $\frac{1}{u}$, which appears in the integrand)

$= -\frac{(\ln(u))^2}{2}+C = \boxed{-\frac{(\ln(\cos(x)))^2}{2}+C.}$

(If you see right away that $\frac{d}{dx}\ln(\cos(x))=\tan(x)$, then substitute $u=\ln(\cos(x))$.)

19) $\int_1^e \frac{\ln(x)^3}{x}\,dx$ ← substitute $u=\ln(x)$: $\frac{du}{dx}=\frac{1}{x}$ $\quad du=\frac{1}{x}\,dx$

(a good choice because the derivative of $\ln(x)$ is $\frac{1}{x}$, which appears in the integrand)

$= \int_{\ln(1)}^{\ln(e)} u^3\,du$

$= \int_0^1 u^3\,du$

$= \left[\frac{u^4}{4}\right]_0^1$

$= \frac{1^4}{4} - \frac{0^4}{4} = \boxed{1/4.}$

20) $\int \frac{\cos(x)}{\sqrt{1+\sin(x)}}\,dx$ ← substitute $u=1+\sin(x)$: $\frac{du}{dx}=\cos(x)$ $\quad du=\cos(x)\,dx$

(a good choice because the derivative of $1+\sin(x)$ is $\cos(x)$, which appears in the integrand)

$= \int \frac{du}{\sqrt{u}}$

$= \int u^{-\frac{1}{2}}\,du$

$= 2u^{\frac{1}{2}} + C$ $\quad$ (C a constant)

$= 2\sqrt{u} + C$

$= 2\sqrt{1+\sin(x)} + C.$

(A note on the constant C: the domain of $\frac{\cos(x)}{\sqrt{1+\sin(x)}}$ is when $\sin(x) \neq -1$, ie, $x \neq -\frac{\pi}{2} + 2\pi n$ (n an integer). We can write this as a union of intervals

$$\cdots \cup \left(-\tfrac{\pi}{2}-2\pi, -\tfrac{\pi}{2}\right) \cup \left(-\tfrac{\pi}{2}, -\tfrac{\pi}{2}+2\pi\right) \cup \left(-\tfrac{\pi}{2}+2\pi, -\tfrac{\pi}{2}+4\pi\right) \cup \cdots$$

We may choose a different constant C on every interval of the domain $\left(-\frac{\pi}{2}+2\pi n, -\frac{\pi}{2}+2\pi(n+1)\right)$ (n an integer).)

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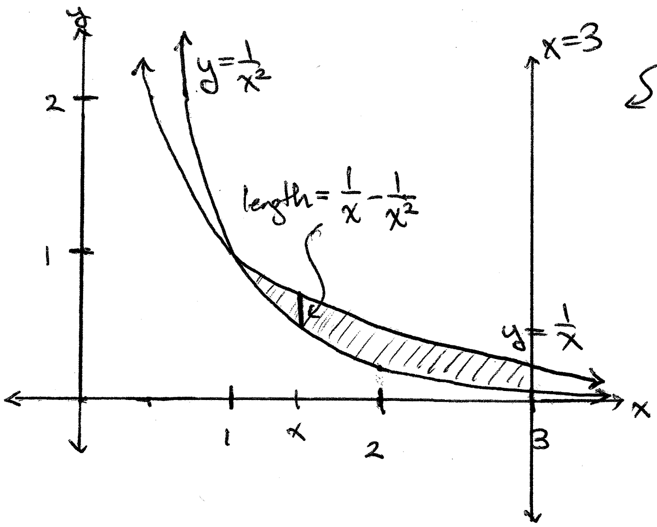
21)



plot the curves by plugging in points

$$\text{area} = \int_1^3 \text{length}\, dx$$

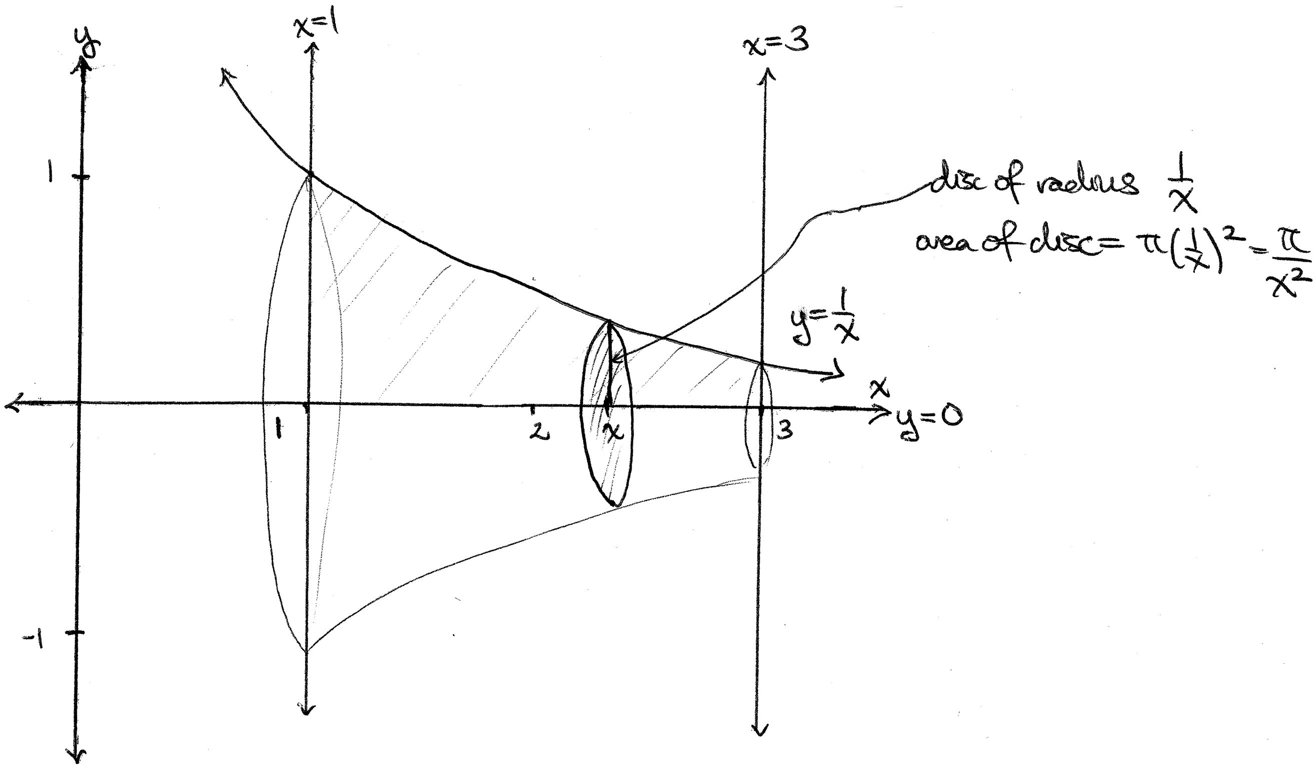
$$= \int_1^3 \frac{1}{x} - \frac{1}{x^2}\, dx$$

$$= \left[\ln(x) + \frac{1}{x}\right]_1^3$$

$$= \left(\ln(3) + \frac{1}{3}\right) - \left(\ln(1) + \frac{1}{1}\right)$$

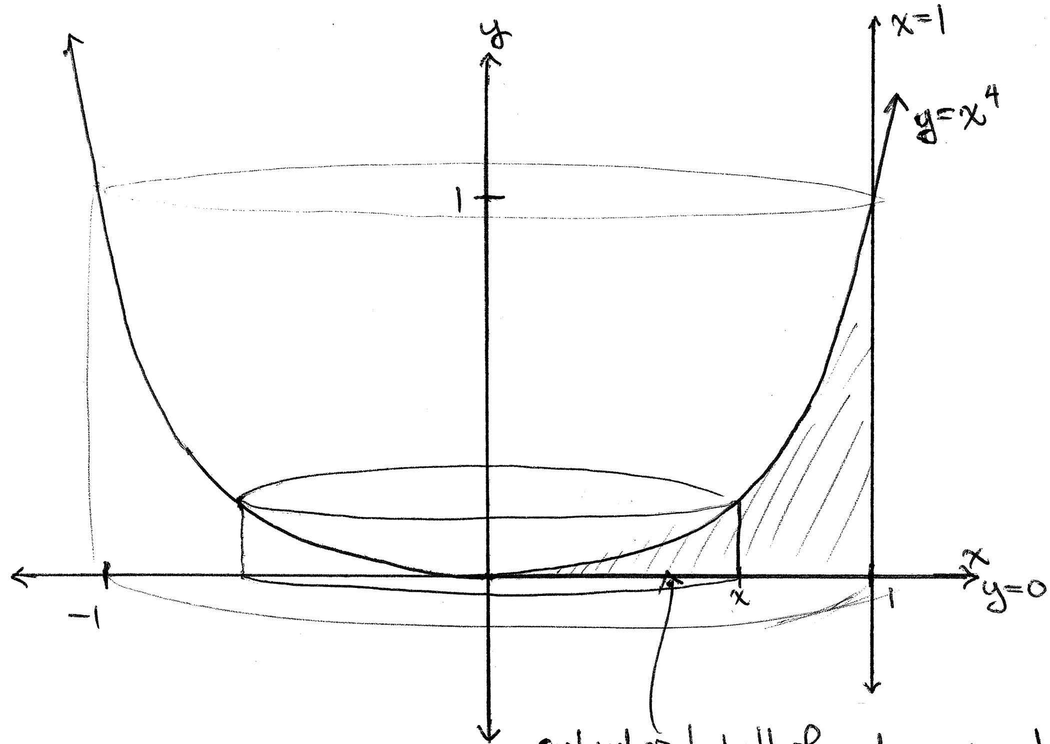
$$= \ln(3) + \frac{1}{3} - 0 - 1 = \boxed{\ln(3) - \frac{2}{3}}$$

22)



$$\text{volume} = \int_1^3 \text{area}\, dx = \int_1^3 \frac{\pi}{x^2}\, dx = \left[-\frac{\pi}{x}\right]_1^3 = -\frac{\pi}{3} + \frac{\pi}{1} = \boxed{\frac{2\pi}{3}}$$

23)



cylindrical shell of radius x and height x^4

area of cylindrical shell $= (2\pi x)(x^4) = 2\pi x^5$

$$\text{volume} = \int_0^1 \text{area}\, dx = \int_0^1 2\pi x^5\, dx = \left[\frac{\pi x^6}{3}\right]_0^1 = \frac{\pi(1^6)}{3} - \frac{\pi(0^6)}{3} = \boxed{\frac{\pi}{3}.}$$

24) $$\text{average value} = \frac{\int}{\text{length of interval}}$$

a good choice because the derivative of $\sin(x)$ is $\cos(x)$, which appears in the integrand

substitute $u = \sin(x)$:

$\frac{du}{dx} = \cos(x)$ $\qquad du = \cos(x)\, dx$

$$= \frac{\int_0^{\pi} \cos(x)\sin^4(x)\, dx}{\pi}$$

$$= \frac{1}{\pi}\int_{\sin(0)}^{\sin(\pi)} u^4\, du$$

$$= \frac{1}{\pi}\int_0^0 u^4\, du$$

$$= \boxed{0.}$$