MATH 1A: 2004 Final Solutions

1. $x = \left(\frac{y-2}{3}\right)^{1/3}$.

2. By Binomial Theorem $\lim_{x\to 0} \dfrac{(2+x)^3 - 8}{x} = 12$.

3. Set $f(x) = (2+x) - e^x$. Note $f(x)$ continuous. Note $f(0) = 2 - e^0 > 0$ and $f(-2) = -e^{-2} < 0$. Therefore I.V.T. implies that $f(x)$ has a root.

4. By Quotient Rule $f'(x) = \dfrac{ad-bc}{(cx+d)^2}$.

5. By Chain Rule $\frac{d}{dx}[y] = 6\tan^2(2x)\sec^2(2x)$.

6. Implicit differentiation of y with respect to x gives $\frac{d}{dx}[y] = \tan x \tan y$.

7. Note $f(x)$ is differentiable on all real numbers. Have $f'(x) = e^{2x} + 2xe^{2x} = 0$ if and only if $x = -1/2$. Therefore $x = -1/2$ critical point of $f(x)$.

8. Set $f(x) = x^4 + 4x + c$. By Rolle's Theorem there exists a root of $f'(x) = 4x^3 + 4$ between two consecutive roots of $f(x)$. Since $4x^3 + 4 = 0$ if and only if $x = -1$, this implies that $f(x)$ has at most two roots.

9. By L'Hospital's Rule have $\lim_{x\to 0} \dfrac{\cos x - 1}{x^2} = \lim_{x\to 0} \dfrac{-\sin x}{2x} = \lim_{x\to 0} \dfrac{-\cos x}{2} = -1/2$.

10. Consider ellipse $x^2 + (y/2)^2 = 1$. Distance between (x, y) on ellipse and $(1, 0)$ is $\sqrt{(x-1)^2 + 4 - 4x^2}$. Set $f(x) = (x-1)^2 + 4 - 4x^2$. Note $f'(x) = -6x - 2 = 0$ if and only if $x = -1/3$. Have $f(-1) = 4$, $f(1) = 0$, and $f(-1/3) = 5\frac{1}{3}$. Therefore $f(x)$ maximized on the interval $[-1, 1]$ at $x = -1/3$. Conclude distance maximized on ellipse at $(-1/3, \pm 4\sqrt{2}/3)$.

11. Tangent line is horizontal at $x_1 = 1$.

12. $x_2 = 2 - \frac{32-36}{5\cdot 16} = 2.05$.

13. $f(x) = \frac{1}{4}x^4 + \frac{3}{4}$. Note $f(x)$ tangent to $x + y = 0$ at $(-1, 1)$.

14. Set $f(x) = cx + d - \ln x$. Note $0 = f(1) = c + d$ implies $c = -d$; note $0 = f(2) = c - \ln 2$ implies $c = -d = \ln 2$. Therefore $f(x) = \ln(2)x - \ln(2) - \ln x$.

15. $1 \cdot f(2) + 1 \cdot f(3) + 1 \cdot f(4) = 1/2 + 1/3 + 1/4 = 13/12$.

16. $\int_1^2 f(x)dx + \int_2^5 f(x)dx = \int_1^5 f(x)dx$ implies $\int_1^2 f(x)dx + 14 = 12$. Therefore $\int_1^2 f(x)dx = -2$.

17. Integral corresponds to area of region between $y = 0$ and $x^2 + y^2 = 4$ for $y > 0$. Since $x^2 + y^2 = 4$ equation of a circle of radius 2, have $\int_{-2}^{2} \sqrt{4 - x^2}dx = 2\pi$.

18. Since $1 \leq \sqrt{1 + x^2} \leq \sqrt{2}$ for x in $[-1, 1]$, have
$2 = \int_{-1}^{1} 1dx \leq \int_{-1}^{1} \sqrt{1 + x^2}dx \leq \int_{-1}^{1} \sqrt{2}dx = 2\sqrt{2}$.

19. By F.T.C. 1 have $g'(x) = \ln x$.

20. By F.T.C. 1 and Chain Rule have $\frac{d}{dx}[y] = \cos x - \cos(x)\cos(\sin x)$.

21. By F.T.C. 2 have $\int_{-1}^{0} 2x - e^x dx = x^2 - e^x|_{x=-1}^{x=0} = e^{-1} - 2$.

22. By F.T.C. 2 have $\int_0^{\pi/4} \frac{1 + \cos^2 x}{\cos^2 x} dx = \tan x + x|_{x=0}^{x=\pi/4} = 1 + \pi/4$.

23. $\int x^3\sqrt{2x^4 - 1}dx = \frac{1}{12}\left(2y^4 - 1\right)^{3/2} + C$.

24. $\int \tan x dx = \ln|\sec x| + C$.

25. By F.T.C. 2 have $\int_0^4 (x - 2)^7 dx = \frac{1}{8}(x - 2)^8|_{x=0}^{x=4} = 0$.

26. Note $\int_1^n 1/x dx = \ln n$. Take left endpoint approximation with $n - 1$ rectangles of width 1. Since respective heights are $1, \ldots, \frac{1}{n-1}$, have $1 + \frac{1}{2} + \cdots + \frac{1}{n-1} > \ln n$.

27. Area enclosed by curves is $\int_{-6}^{2} (3 - \frac{1}{4}y^2) - y dy = 24 - \frac{56}{3} + 16 = \frac{64}{3}$.

28. Volume of solid is $\int_0^1 \pi e^{2x} dx = \frac{\pi}{2}(e^2 - 1)$.

29. Volume of sphere of radius r is $2\int_0^r 2\pi r\sqrt{r^2 - x^2}dx = 4\pi\left[\frac{-1}{3}(r^2 - x^2)^{3/2}\right]_{x=0}^{x=r} = 4\pi r^3/3$.

30. $\frac{1}{4}\int_0^4 \sqrt{x}dx = 4/3.$