STEWART'S CALCULUS, EXERCISE 11.10.71

a) Fix an n, and set

$$g(x) = \sum_{k=0}^{\infty} \binom{n}{k} x^k.$$

Our eventual goal is to show $g(x) = (1+x)^n$, but more immediately we're to show

$$g'(x) = \frac{ng(x)}{(1+x)},$$

as this identity is satisfied by $(1+x)^n$. Clearing denominators, it is equivalent to show $(1+x)g'(x) = ng(x)$. We compute the derivative of g term-by-term as

$$g'(x) = \sum_{k=1}^{\infty} \binom{n}{k} kx^{k-1},$$

and we substitute our expressions into the left-hand side of the identity to get

$$\begin{aligned} g'(x) + xg'(x) &= \sum_{k=1}^{\infty} \binom{n}{k} kx^{k-1} + \sum_{k=1}^{\infty} \binom{n}{k} kx^k \\ &= \sum_{k=0}^{\infty} \binom{n}{k+1} (k+1)x^k + \sum_{k=1}^{\infty} \binom{n}{k} kx^k \\ &= n + \sum_{k=1}^{\infty} \left[\binom{n}{k+1} (k+1) + \binom{n}{k} k \right] x^k. \end{aligned}$$

Hence, by equating coefficients with the right-hand side, which is

$$ng(x) = n + \sum_{k=1}^{\infty} n \binom{n}{k} x^k,$$

we reduce to verifying the identity

$$\binom{n}{k+1}(k+1) + \binom{n}{k} k = n\binom{n}{k}.$$

The left-most summand can be written as

$$\begin{aligned} \binom{n}{k+1} &= \frac{n \cdot \cdots \cdot (n-k)}{(k+1)!} \cdot (k+1) \\ &= n \cdot \frac{(n-1) \cdot \cdots \cdot (n-k)}{k!} \\ &= n \cdot \binom{n-1}{k}, \end{aligned}$$

and the middle summand can similarly be written as

$$\begin{aligned} \binom{n}{k} k &= \frac{n \cdot \cdots \cdot (n-k+1)}{k!} \cdot k \\ &= n \cdot \frac{(n-1) \cdot \cdots \cdot (n-k+1)}{(k-1)!} \\ &= n \cdot \binom{n-1}{k-1}. \end{aligned}$$

Then, we use Pascal's identity, which we've discussed in class before. It states

$$\binom{a}{b} + \binom{a}{b+1} = \binom{a+1}{b+1}.$$

This means our expression above simplifies as

$$\begin{aligned}\binom{n}{k+1}(k+1) + \binom{n}{k}k &= n \cdot \binom{n-1}{k} + n \cdot \binom{n-1}{k-1} \\ &= n \cdot \binom{n}{k},\end{aligned}$$

as desired. So, the coefficients of the original two power series are equal, proving part a).

b) Now we're to show that $h'(x) = 0$ for the function $h(x) = (1+x)^{-n}g(x)$. This is not so bad:

$$\begin{aligned}\frac{d}{dx}h(x) &= \frac{d}{dx}\left((1+x)^{-n}\right) \cdot g(x) + (1+x)^{-n} \cdot \frac{d}{dx}g(x) \\ &= -k(1+x)^{-n-1}g(x) + (1+x)^{-n}g'(x) \\ &= (1+x)^{n}\left(-n(1+x)g(x) + g'(x)\right),\end{aligned}$$

which by part a) is zero.

c) Since $h'(x) = 0$, we know that it is a constant[1], hence we just need to figure out what constant it is! Evaluation at zero makes this easy: $h(0) = (1+0)^{-n}g(0) = 1 \cdot 1 = 1$. Hence,

$$1 = (1+x)^{-n}g(x),$$

or

$$g(x) = (1+x)^{n}.$$

[1]Or locally constant, anyway. Whatever.