Lecture notes

December 9, 2009

1 Administrative notes

I finally managed to tally all the student informal feedback responses sent to me; the major weaknesses (though they were by no means unanimous) were classroom pacing, handwriting, and going off on long-winded tangents. The first two at least I can try to fix; they're sort of related complaints, since my handwriting is only really terrible when I feel rushed and everything turns into a bumpy script. One student noted that the pacing problem probably isn't my fault either, and that the class (both lecture and discussion) ought to meet three times a week; I completely agree, but we'll see what we can do anyway.

Tangents, on the other hand, I don't intend on fixing (though maybe I could somehow mark them as "you don't necessarily need to completely understand this"). Last class was excellent in illustrating the point that introducing seemingly irrelevant material or doing things slightly out of order isn't always a bad thing. In the graphing section, Stewart requests that we compute slant asymptotes of the function, which we covered in the section on limits of a function at infinity, to demonstrate that there's more than we can say about the limiting behavior of a function than just what single value it takes. We also used linear approximation to show why L'Hôpital's rule works, at least in a very particular case. Math is something that only makes more sense as you see more of the underlying, unifying theory; being able to make intuitive statements about how functions and so forth "should" behave is just as important in making sure you're not doing something wildly incorrect on exams as actually being able to make the computation. If you don't need to make the logarithmic argument that $\lim_{x\to\infty}\left(\frac{2}{3}\right)^x = 0$ and you can see "why" it must be the case, then that saves you work. Both abilities are important.

Again, I'll do what I can about pacing and handwriting.

2 More on L'Hôpital's Rule

Last time we talked about L'Hôpital's rule, which gave us a tool to compute limits of particular quotients: if $f(x)$ and $g(x)$ are differentiable at a and satisfy

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0,$$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

This also holds if $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$.

2.1 Indeterminate forms

We also discussed the existence of other "indeterminate forms," which were expressions involving 0, 1, and ∞ that didn't immediately allow evaluation of a limit but could be worked into a form where we could instead use L'Hôpital. Namely, consider the following expressions:

- $0/0$: Suppose we have f, g such that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$. Then L'Hôpital's rule applies to the quotient directly, and we have $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$.
- ∞/∞: Suppose we have f, g such that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$. Then L'Hôpital's rule applies to the quotient directly, and we have $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$.
- $\infty - \infty$: Suppose we have f, g such that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$. Then we can manipulate the indeterminate form $\lim_{x\to a}(f(x) - g(x))$ as follows:

$$\lim_{x\to a}(f(x) - g(x)) = \lim_{x\to a}\left(f(x)\cdot\left(1 - \frac{g(x)}{f(x)}\right)\right),$$

which reduces to the case ∞/∞.

- $\infty \cdot 0$: Suppose we have f, g such that $\lim_{x\to a} f(x) = \infty$ while $\lim_{x\to a} g(x) = 0$. Then the limit of their product can be rewritten as

$$\lim_{x\to a}(f(x)\cdot g(x)) = \lim_{x\to a}\frac{f(x)}{\frac{1}{g(x)}} = \lim_{x\to a}\frac{g(x)}{\frac{1}{f(x)}},$$

which reduces to the cases $0/0$ and ∞/∞ where L'Hôpital's rule applies.

- 0^0: Let f, g be such that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$. Then,

$$\begin{aligned} L &= \lim_{x\to a} f(x)^{g(x)} \\ \ln L &= \lim_{x\to a}\left(\ln\left(f(x)^{g(x)}\right)\right) \\ &= \lim_{x\to a}\left(g(x)\ln f(x)\right), \end{aligned}$$

which is indeterminate of the form $0\cdot\infty$, so we reduce to the case above.

- 1^∞: Suppose we have f, g such that $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \infty$. Then setting $\lim_{x\to a} f(x)^{g(x)} = L$, we have

$$\begin{aligned} L &= \lim_{x\to a} f(x)^{g(x)} \\ \ln L &= \lim_{x\to a}\ln\left(f(x)^{g(x)}\right) \\ &= \lim_{x\to a}\left(g(x)\cdot\ln f(x)\right) \end{aligned}$$

which is indeterminate of the form $\infty\cdot 0$, so we reduce to the case above.

- ∞^0: Finally, suppose we have f, g such that $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$. Then,

$$\begin{aligned} L &= \lim_{x\to a} f(x)^{g(x)} \\ \ln L &= \lim_{x\to a}\ln\left(f(x)^{g(x)}\right) \\ &= \lim_{x\to a}\left(g(x)\ln f(x)\right), \end{aligned}$$

which is indeterminate of the form $0\cdot\infty$, so we reduce to the case above.

2.2 Sample problems

Using these tools, we can solve a variety of problems involving L'Hôpital. The plan in class was to ask which ones you had questions about from the homework; I'll just sample a few from WebAssign to do here.

Problem 1: Suppose we're asked to compute

$$\lim_{x\to 0} \frac{7\sin x - 7x}{x^3}.$$

Since $\sin 0 = 0$, this limit is of the indeterminate form $0/0$, and so we can apply L'Hôpital directly to get

$$\lim_{x\to 0} \frac{7\sin x - 7x}{x^3} \overset{H}{=} \lim_{x\to 0} \frac{7\cos x - 7}{3x^2}.$$

Again, attempting to substitute 0 in for x results in the expression $0/0$, so we apply L'Hôpital's rule a second time:

$$\lim_{x\to 0} \frac{7\cos x - 7}{3x^2} \overset{H}{=} \lim_{x\to 0} \frac{-7\sin x}{6x}.$$

This limit is still of the form $0/0$ (but note that since each time we differentiate the denominator loses a power of x, so we remain hopeful that eventually we'll end up with a quotient of a function by a constant), so we apply L'Hôpital's rule one last time to get

$$\lim_{x\to 0} \frac{-7\sin x}{6x} \overset{H}{=} \lim_{x\to 0} \frac{-7\cos x}{6} = \frac{-7}{6}.$$

You're encouraged to verify this graphically by visiting Wolfram-alpha, entering the expression for the function, and looking at the behavior of the graph near zero.

Problem 2: Now suppose we're asked to compute a more complicated indeterminate form, for example, $\lim_{x\to 0}(\csc x - \cot x)$. Naïvely evaluated, this expression takes the form $\infty - \infty$, and so we algebraically manipulate it toward something we can apply L'Hôpital to:

$$\begin{aligned}
\lim_{x\to 0}(\csc x - \cot x) &= \lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right) \\
&= \lim_{x\to 0} \frac{1-\cos x}{\sin x} \\
&\overset{H}{=} \lim_{x\to 0} \frac{\sin x}{\cos x} = 0.
\end{aligned}$$

3 Newton's Method

Newton's method is best explained with a picture, which is awkward to include a hand-written PDF; the idea is that, rather than solving directly for the roots of some function $f(x)$, we solve for the roots of various linear approximations of it, which collectively (under nice conditions) yield a sequence of values converging to a limit of the original function. As far as actually solving problems goes, there is exactly one kind of problem: take a function $f(x)$ and an initial value x_0. Then, recursively define the following sequence of values:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

This next value comes from solving for the root of the linear approximation to $f(x)$ at x_n. Recall that that approximation is given by $L_f(t) = f(x_n) + f'(x_n)(t - x_n)$. Setting this equal to zero and solving for t

gives

$$\begin{aligned} f(x_n) + f'(x_n)(t - x_n) &= 0 \\ f'(x_n)(t - x_n) &= -f(x_n) \\ t - x_n &= -\frac{f(x_n)}{f'(x_n)} \\ t &= x_n - \frac{f(x_n)}{f'(x_n)}. \end{aligned}$$

Because Newton's method is designed to be expressly algorithmic, there's not much else to say; any problems about it will involve computing various terms in this sequence, which you can get at by just plugging the numbers in.

4 More Related Rates Problems

I've also tacked on a couple of related rates and optimization problems; students are encouraged to read the setup of the problem and attempt a solution on their own before reading the solution I've supplied. Had we held section today, I'd have made you work on the problems alone for 5-10 minutes before discussing them as a group.

4.1 A bead on a wire

We managed to set this problem up for BD2 but didn't get to it in BD1. The description of the problem is as follows: a bead travels along a wire described by the curve $5xy^3 = 8(1 + y^2)$. At time t_0 it is position $(1, 2)$ with a horizontal velocity of 6 units per second. Find the total speed.

To start, the total speed is given by the length of the velocity vector, corresponding to $\sqrt{s_x^2 + s_y^2}$, where s_x is the horizontal component of the velocity and s_y is the vertical component. We're given the horizontal component, but we have to find the vertical one.

It makes sense that in order for the bead to stay on the wire, it must have vertical velocity so that its velocity vector matches the slope of the curve – if the velocity vector is too highly sloped, then the bead will fly off to the north, and if it's not sloped enough, the bead will fly off to the east. So, thinking of both x and y as functions of a time parameter t, we can implicitly differentiate the curve to get

$$\begin{aligned} \frac{d}{dt}(5xy^3) &= \frac{d}{dt}(8 + 8y^2) \\ 5x'y^3 + 15xy^2y' &= 16yy' \\ \frac{5x'y^3 + 15xy^2y'}{16y} &= y'. \end{aligned}$$

Evaluating y' at t_0 using the constants supplied above yields $s_y = y'(t_0)$, which we can then put into the equation for total speed.

4.2 Sectors of circles

Find the radius r and inner angle θ of a sector[1] of a circle of minimal perimeter P, such that the sector has fixed area A.

The area of a sector depends upon two things: the radius of a circle and the inner angle swept out. Namely, we have $A(r, \theta) = \frac{1}{2}\theta r^2$. The perimeter of a sector is the sum of the outer arc plus the two radial segments, so $P(r, \theta) = r\theta + 2r$ – recall the formula $s = r\theta$ for arclength.

[1] Recall that a "sector" of a circle is the geometrically-inclined name for a slice of pizza.

Because this is an optimization problem, we need to produce a function of one variable so that we can differentiate and find critical points. Since A is fixed, the relation $A = \frac{\theta r^2}{2}$ gives us the relation $\theta = \frac{2A}{r^2}$ between θ and r. Substituting this into the second expression gives

$$P(r) = \frac{2A}{r^2} \cdot r + 2r,$$

which is a function of one variable that we can minimize. Its derivative is given by

$$P'(r) = \frac{-2A}{r^2} + 2,$$

and solving for the zeroes of this derivative gives

$$\begin{aligned} \frac{-2A}{r_0^2} + 2 &= 0 \\ \frac{2A}{r_0^2} &= 2 \\ \frac{r_0^2}{2A} &= \frac{1}{2} \\ r_0 &= \sqrt{A}. \end{aligned}$$

One quickly checks that this is a local minimum and that the "endpoints" of the definition of P do not make sense as local minima. Solving for θ_0 using the formula derived above gives

$$\begin{aligned} \theta_0 &= \frac{2A}{r_0^2} \\ &= \frac{2A}{A} \\ &= 2, \end{aligned}$$

which solves the problem.

5 Quiz

We should have had a quiz today; expect it Tuesday!