Computations with spectral sequences

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0.1 Foreword

Mathematics is a field where computations lead theory, and this is especially evident in the subfield of algebraic topology, which is positively rife with computations. These often take the form of spectral sequences, which are notorious among students of any field that makes use of homological algebra for being pathologically cryptic and complex. Nevertheless, their utility is immense, and students, often with much groaning, at least learn to stomach the sight of them, if not fully embrace the idea of computing with one.

There are many reasons spectral sequences are viewed as impossibly complex, large parts of which are due to the following two reasons. First, spectral sequences are often triply-indexed — and each index is often infinite, or bi-infinite, or indexed over a group more complicated than the integers! This means that an enormous amount of information is available in a spectral sequence, which begets the second point: effective computation with a spectral sequence appears to require that one keep an outlandish number of things in mind while working, along with an array of subtle tricks and facts from elsewhere in topology, not presently visible on the page. In turn, these have lead to a derth of textbooks covering the art of computing with spectral sequences; if they're so difficult to think about, then the situation is even worse when trying to linearize them into writing and then typeset the whole mess. For this reason, knowing how to compute with spectral sequences is often referred to as an "oral tradition," passed down in ritual form from advisor to student, behind closed doors and with endless scratch paper.

The purpose of this text is to fill this gap. In conversation with an expert, time plays the role of linearizer, as one watches the spectral sequence play out on a page in real time. Our goal is to turn these conversations into text, where the linearization instead takes place across pages, in the form of an elementary school student's "flip book." On each page the reader can find a single step of the larger computation highlighted and dissected, then turn to the next to find the diagram slightly modified, as in real-time. This should dramatically ease the learning curve for students who are interested in spectral sequences but who don't enjoy ready access to lunches with Doug Ravenel and crew.

This book does not have exercises; instead, it is written more like a solutions manual for a text that does not exist. However, the methods described are extremely general, and the reader looking to try them out for himself should be able to pick a favorite space and plug it into these machines, following roughly the same process to compute its associated invariants. For this reason, the examples worked here have been selected with illustration kept in mind rather than exhaustiveness.

An important thing to remind the reader of is that spectral sequences, as massive mathematical machines, are designed to take their users' minds off the details of a problem. Some of these details will be addressed and discussed lightly in the text surrounding the computations, but the uninterested, bored, or befuddled reader should not hesitate to skip over these parts of the text for now. In the same way, schoolchildren are taught arithmetic algorithms long before they investigate what makes the algorithms tick, and in this intervening period the utility of knowing how to perform long division is not diminished.

We should also immediately mention other textbooks on this subject. McCleary's book *A User's Guide to Spectral Sequences* is excellent and contains all of the details we omit here and then some. Mosher & Tangora's *Cohomology Operations and Applications in Homotopy Theory* centers around the interactions of the Steenrod algebra with spectral sequences, and is rife with the computations that spurred the development of this field. Every homological algebra textbook in existence (Weibel's *Homological algebra*, Cartan and Eilenberg's *Homological algebra*, ...) contains a section on the construction and maintenance of spectral sequences, where technical details can be found. Hatcher has made available an unfinished book project on spectral sequences at `http://www.math.cornell.edu/~hatcher/SSAT/SSATpage.html`. Miller has published course notes that use in a central way the EHP spectral sequence, available in full at `http://www-math.mit.edu/~hrm/papers/` and in the process of being converted to LaTeX. Ravenel's *Complex cobordism and stable homotopy groups of spheres* remains the standard reference for the analysis of the beginning of the Adams spectral sequence for the sphere. And, of course, there are many others.

Finally, this is a draft version of this textbook, compiled on August 6, 2012. I'm sure that it's rife with errors, inconsistencies, omissions, and generally confused language, and I would greatly appreciate any or all of corrections, remarks, and expansions. I can easily be reached at `ericp@math.berkeley.edu`. This project progresses slowly, as I tend to work on it only when I'm stuck on and tired of my other mathematical projects, but I hope that it grows into something genuinely useful as it goes.

Drafts of this document are available at `http://math.berkeley.edu/~ericp/ss-book/main.pdf`, and the software used to generate it is available in the directory `http://math.berkeley.edu/~ericp/ss-book/`.

0.2 Acknowledgements

People who directly taught me: Matthew Ando

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Chapter 1

Spectral sequences in general

BEFORE we start in on computations with spectral sequences, we should take a moment to outline what they are and where they come from. Once we've pinned these down, we will also mention some of the most common complications and useful structures.

1.1 Homology theories

Spectral sequences arise naturally in homological algebra, which is the study in the abstract of where homology functors come from. Since this book is geared toward algebraic topologists, we will not be so abstract; instead, a (reduced) homology functor for us is a sequence of functors $(\tilde{H}_n)_{n\in\mathbb{Z}} : Ho(\mathsf{PointedSpaces}) \to \mathsf{AbelianGroups}$ from the category of pointed homotopy types to abelian groups which collectively satisfy the following two axioms:

- Wedge sum: For any collection of spaces[1] $\{X_\alpha\}_{\alpha\in A}$, we have a natural isomorphism

$$\tilde{H}_n\left(\bigvee_\alpha X_\alpha\right) \cong \bigoplus_\alpha \tilde{H}_n X_\alpha.$$

- Triangulation: For A a subspace[2] of X, the "short exact sequence"

$$A \xrightarrow{i} X \xrightarrow{p} X/A$$

  of spaces begets a long exact sequence

[1] Throughout this book, we will suppress the basepoint we carry along with our spaces. It's an important technicality, but it's not worth dwelling on constantly by bringing into the notation.

[2] We require $i : A \to X$ to be quite reasonable, namely a cofibration. For example, the inclusion of a subcomplex counts.

$$\begin{array}{l} \cdots \longrightarrow \tilde{H}_{n+1}(X/A) \\ \hookrightarrow \tilde{H}_n A \longrightarrow \tilde{H}_n X \longrightarrow \tilde{H}_n(X/A) \\ \hookrightarrow H_{n-1}A \longrightarrow \cdots . \end{array}$$

The middle maps are specified by functoriality, but the maps labeled ∂ are new data.

TODO: Mention unreduced.

These axioms alone can be used to compute a small handful of things. For instance, the first axiom tells us that the homology of a point must vanish, since $\mathrm{pt} \vee \mathrm{pt} \simeq \mathrm{pt}$. To see the utility of the second axiom, let X be a $(d+1)$-dimensional hemisphere, and let A be the inclusion of the equatorial band, itself a d-dimensional sphere. The space X is homotopy equivalent to a point, so has vanishing homology, whereas the quotient X/A is homeomorphic to a $(d+1)$-dimensional sphere. The long exact sequence in homology reads

$$\cdots \to \tilde{H}_{n+1} S^{d+1} \xrightarrow{\partial} \tilde{H}_n S^d \xrightarrow{\tilde{H}_n i} \tilde{H}_n \,\mathrm{pt} \xrightarrow{\tilde{H}_n p} \tilde{H}_n S^{d+1} \xrightarrow{\partial} \tilde{H}_{n-1} S^d \to \cdots .$$

Hence, the homology of the $(d+1)$-sphere is exactly the homology of the d-sphere, shifted up by one degree.

TODO: Mention cohomology. **TODO**: A useful fact is $H_* \operatorname{colim} F = \operatorname{colim} H_* F$.

1.2 Filtrations and spectral sequences

This is all well and good, and one can compute a great many things manually by specifying $H_* S^0$ and working with these two axioms from there. For more complex situations, manual computations become tedious, and this is where spectral sequences enter the picture. To perform the homology computation of a complex space X, we must first break it down into a sequence of simple spaces X_q, each including into the next. Not only should all of them include into X, but we should have $X = \operatorname{colim}_q X_q$. On the other end, we require $X_{-1} = \mathrm{pt}$. Here is a diagram of the situation:

$$\cdots \rightarrowtail X_{q-1} \xrightarrow{i_{q-1}} X_q \xrightarrow{i_q} X_{q+1} \rightarrowtail \cdots \rightarrowtail X.$$

We're seeking to relate the homology of X to the homologies of these pieces X_q. Looking back at our axioms for a homology theory, we do see that inclusions play a special role in the triangulation axiom, but to apply the triangulation axiom we must also consider various quotients. We extend our diagram to match:

$$\begin{array}{ccccccccc} \cdots \rightarrowtail & X_{q-1} & \overset{i_{q-1}}{\rightarrowtail} & X_q & \overset{i_q}{\rightarrowtail} & X_{q+1} & \rightarrowtail \cdots \rightarrowtail & X. \\ & \big\downarrow p_{q-1} & & \big\downarrow p_q & & \big\downarrow p_{q+1} & & \\ & F_{q-1} & & F_q & & F_{q+1} & & \end{array}$$

Now we apply homology $\tilde{H}^*$ to our diagram, and in doing so we also apply the triangulation axiom to each of these angled arms:

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \tilde{H}_*X_{q-1} & \xrightarrow{\tilde{H}_* i_{q-1}} & \tilde{H}_*X_q & \xrightarrow{\tilde{H}_* i_q} & \tilde{H}_*X_{q+1} & \longrightarrow \cdots \longrightarrow & \tilde{H}_*X. \\ & \big\downarrow \tilde{H}_* p_{q-1} & \overset{\partial}{\nwarrow} & \big\downarrow \tilde{H}_* p_q & \overset{\partial}{\nwarrow} & \big\downarrow \tilde{H}_* p_{q+1} & & \\ & \tilde{H}_*F_{q-1} & & \tilde{H}_*F_q & & \tilde{H}_*F_{q+1} & & \end{array}$$

Note that each of these maps ∂ is *not* degree-preserving[3] but shifts the degree down by 1.

Now that we have this picture, we are tasked with tying this discussion up and saying something meaningful about $\tilde{H}_*X$. Before formalizing the process, we will describe its goal. Suppose that we pick some homology class $\alpha \in \tilde{H}_*F_q$; the question we then pose is whether α is in some way visible in H_*X. The only map we have in front of us by which we can push back up into the Xes is ∂, so we produce an element $\partial\alpha \in H_{*-1}X_{q-1}$. Naïvely, we'd want to then push forward into H_*X by tracking the maps to the right, but because the triangles in our diagram are exact we immediately know that $(\tilde{H}_* i_q) \circ \partial\alpha = 0$. We must be more creative.

Another thing we could try to do is to find an element $\beta \in \tilde{H}_*X_q$ for which $(\tilde{H}_* p_q)\beta = \alpha$. Again employing exactness of the triangle, such a β exists exactly when $\partial\alpha = 0$. However, at the moment we have no way of telling whether this is the case, since the rules of the game are that we only understand the groups $\tilde{H}_*F_*$. So, to get back into the land of things we understand, we follow the vertical map down to produce $(\tilde{H}_* p_q) \circ \partial\alpha$.

At this point there are two options. First, $(\tilde{H}_* p_q) \circ \partial\alpha$ could be nonzero, in which case $\partial\alpha$ itself must have been nonzero, and there is no hope for producing β. In this case, we should discard α as an unfortunate artifact of the filtering process, without contribution to the total homology. On the other hand, if $(\tilde{H}_* p_q) \circ \partial\alpha = 0$, it's possible that either $\partial\alpha = 0$ or merely that $\partial\alpha \in \ker \tilde{H}_* p_q$. But, in either case, we can employ the exactness of the next triangle in the sequence to preimage the element $\partial\alpha$ through the map $\tilde{H}_* i_{q-1}$ to produce an element $(\tilde{H}_* i_{q-1})^{-1}\partial\alpha$, with which we can play the same game.

[3]A key to successfully doing homological algebra successfully is to suppress as many indices as possible, so we don't draw this in the diagram.

Eventually, however, we will hit the bottom of our filtration. If we can play this game all the way back to then, then we have produced an element $\gamma = (\tilde{H}_* i_*)^{\circ(-q)} \partial \alpha$ for which $(\tilde{H}_* i_*)^{\circ q} \gamma = \alpha$. However, because $X_{-1} = \mathrm{pt}$, we know that $\gamma = 0$, and hence $\partial \alpha = (\tilde{H}_* i_*)^{\circ q}(0) = 0$, and we win — β exists!

This process is formalized by packaging up these composites. We write $E^1_{*,q} = \tilde{H}_* F_q$, and the map $(\tilde{H}_* p_*) \circ \partial$ is called $d^1 : E^1_{*,q} \to E^1_{*-1,q-1}$. One quickly checks that d^1 is a differential, as the two maps in the middle of $d^1 \circ d^1 = (\tilde{H}_* p_*) \circ \partial \circ (\tilde{H}_* p_*) \circ \partial$ belong to the same exact triangle, and hence compose to zero. We are interested only in keeping classes in the kernel of the outgoing d^1 while deleting all the classes in the kernel of the incoming d^1, and so advancing to the next stage in the game corresponds exactly to taking cohomology against the differentials d^1. This cohomology group we label $E^2_{*,q}$. By a small miracle, it turns out that this same quotient is what is required to eliminate the indeterminacy in picking the preimage $(\tilde{H}_* p_{q-1}) \circ (\tilde{H}_* i_{q-1})^{-1} \circ \partial \alpha$, and this composite we label $d^2 : E^2_{*,q} \to E^2_{*-1,q-2}$. This pattern in producing differentials and computing their cohomology continues, and in general we have groups $E^r_{*,q}$, which are sub-quotients of $E^{r-1}_{*,q}$, and differentials $d^r : E^r_{*,q} \to E^r_{*-1,q-r}$. The index r is called the "page" or "sheet," and altogether this data forms a "spectral sequence."

1.3 Convergence and the endgame

One can produce spectral sequences for cohomology as well, using an identical setup. The only difference is in the endgame: in homology, we kept lowering filtration degree, so we eventually hit the bottom and deduced something about our element α. In cohomology, we will instead *raise* filtration degree, and so we will never hit bottom and be able to conclude something solid. We will, however, continuously march toward $\tilde{H}^* X$ with which filtration degree we climb up, and so our spectral sequence will compute something about $\lim_q H^* X_q$, the limit of the cohomology groups. Whether this compares well with $H^* X$ is one of the things we discuss now.

The general theory of spectral sequences is quite wild, and it is possible to construct spectral sequences not arising naturally from a filtration in the way we've described. However, almost all of the examples witnessed in the wild (and certainly those with which one should learn to compute) do come from this construction, and assuming we're in this situation simplifies the theory of convergence considerably.

Label the groups $H_* X_q$ of the above construction by $F_{*,q}$, and label $H_* X$ by G_*. The spectral sequence is said to be …

- … *weakly convergent* (to G_*) if $\operatorname{colim}_q F_{*,q} = G_*$ and $E^\infty_q = F_{*,q}/F_{*,q-1}$.
- … *convergent* if it is weakly convergent and furthermore $\lim_q F_{*,q} = 0$.
- … *strongly convergent* if it is convergent and furthermore $\lim^1_q F_{*,q} = 0$.

- …*conditionally convergent* if $\lim F_{*,q} = 0$.[4]

The first three conditions neatly summarize what extra steps we will need to take in the end to compare the "result" of our spectral sequence with the target of convergence. In the case of strong convergence, we need only to deal with extension problems. The individual homology groups G_p are, by construction, sliced up and scattered through the homology groups $\{E^\infty_{p,q}\}_q$ as q ranges. To recover G_p from this sequence, we are faced with a nest of extension problems: there is some intermediate group extending $E^\infty_{p,0}$ by $E^\infty_{p,1}$, which in turn has an intermediate group extending it by $E^\infty_{p,2}$, and so forth. In the strongly convergent case, the limit of this process yields G_p.

In the convergent case, we are faced exactly with the issue presented by the cohomological spectral sequence above. We can attempt to solve the extension problem, just as before, but the resulting groups G'_p sit in a short exact sequence $0 \to G'_p \to G_p \to \lim^1_q F_{*,q} \to 0$ obstructing honest equality, so we must also address this.

In the weakly convergent case, we are faced with the above two issues, but we additionally ... uh actually I'm not sure what we have to do. Throw in the intersection as a summand?

The conditionally convergent case is differently flavored from the rest. Conditional convergence on its own is worse than weak convergence, but it appears frequently, and there are various extra mild assumptions, easily verified in practice, that turn conditional convergence into strong convergence. For example, if $F_{*,q}$ stabilizes for $q \ll 0$, the spectral sequence converges conditionally, and if $\lim^1_q E^\infty_{*,q} = 0$, the convergence is strong.

There are two more vocabulary words worth knowing: in the case $X_{-1} = \mathrm{pt}$, the spectral sequence lives entirely on one half of the full doubly-integer-indexed plane, and so is called a half-plane spectral sequence. In the homological case, where the differentials eventually land in the unoccupied half-plane, the associated spectral sequence is said to have *exiting differentials*. In the cohomological case, where all differentials eventually land in the occupied half-plane, the associated spectral sequence is said to have *entering differentials*.

In the exiting case, we have few convergence issues to worry about: provided the filtration is Hausdorff, we have strong convergence. If the differentials are entering, however, we need conditional convergence together with the vanishing $\lim^1$-term to get strong convergence.

1.4 Grading conventions and multiplicative structures

Pairings. The Leibniz rule.

[4]In this situation, the filtration is said to be *Hausdorff*.

Chapter 2

Atiyah-Hirzebruch-Serre

2.1 The Atiyah-Hirzebruch spectral sequence

The essential building blocks of the spaces in non-pathological topology (including algebraic topology) are the unit balls D^n of dimension n, and their surface spheres S^{d-1} of dimension $(d-1)$. A topological space X is said to be a CW-complex when it can be decomposed into a sequence of spaces $X^{(d)}$, called d-skeleta[1], such that $X^{(-1)} =$ pt a single point and $X^{(d+1)}$ is formed from $X^{(d)}$ by gluing in (unpointed) $(d+1)$-balls along their d-spherical surface shells, and such that X is given as the colimit of the $X^{(d)}$ as d grows large. These are somehow the most reasonable spaces on which we can "do homotopy theory," and from here on out all our spaces will be assumed to be CW-complexes[2].

This gives an ascending filtration of X by $X^{(d)}$ which is Hausdorff (the condition on $X^{(-1)}$) and exhaustive (the condition $X = \operatorname{colim}_d X^{(d)}$). Moreover, the filtration quotients are easy to compute: the cofiber of the map $X^{(d-1)} \hookrightarrow X^{(d)}$ collapses the $(d-1)$-skeleton to a point, to which all the d-cells get attached, resulting in a bouquet of spheres $X^{(d)}/X^{(d-1)} \simeq \bigvee_\alpha S^d_\alpha$ in filtration grading d. Selecting our favorite homology theory h_* and cohomology theory h^*, this gives a pair of spectral sequences with signatures

$$E^1_{s,t} = h_s\left(\bigvee_\alpha S^t_\alpha\right) = h_{s-t}(\mathrm{pt}) \Rightarrow h_s X, \qquad d^r : E^1_{s,t} \to E^1_{s-1,t-r},$$

$$E_1^{s,t} = h^s\left(\bigvee_\alpha S^t_\alpha\right) = h^{s-t}(\mathrm{pt}) \Rightarrow h^s X, \qquad d_r : E_1^{s,t} \to E_1^{s+1,t+r}.$$

In fact, we can do better: the differential on the first pages of these spectral sequences is exactly the differential that appears in the $(s-t)$th degree of the cellular chain complex for computing cohomology with $h^{s-t}(\mathrm{pt})$-coefficients. This spectral sequence

[1]Skeleta is the mathematician's plural of skeleton.

[2]The exact decomposition into the spaces $X^{(d)}$ isn't so important, just that there exists one.

also carries the structure of a $h^*(\mathrm{pt})$-module in the case that h takes its values in rings, though not with this grading. **TODO**: Straighten out this grading discussion. Putting all this together produces the more familiar form of these spectral sequences:

$$E^2_{p,q} = H^{cell}_p(X; h_q(\mathrm{pt})) \Rightarrow H_{p+q}X, \qquad d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1},$$
$$E_2^{p,q} = H_{cell}^p(X; h^q(\mathrm{pt})) \Rightarrow H^{p+q}X, \qquad d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}.$$

2.2 $H^*\mathbb{C}\mathrm{P}^\infty$

The motivic cell decomposition.

2.3 $H^*\mathbb{R}\mathrm{P}^\infty$

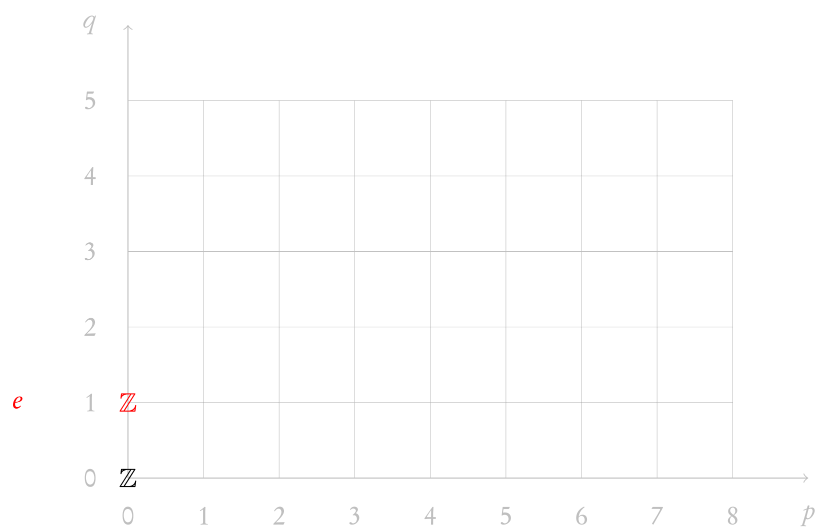
The motivic cell decomposition.

2.4 $KU^*B\mathbb{Z}/2$

Even-concentrated, but has extension problems. See Strickland's bestiary. This might be hard to do before the Gysin sequence description of $h^*\mathbb{R}\mathrm{P}^\infty$...

2.5 The Serre spectral sequence

The E_1-page is easy, but d_1 is hard. Multiplicative structure.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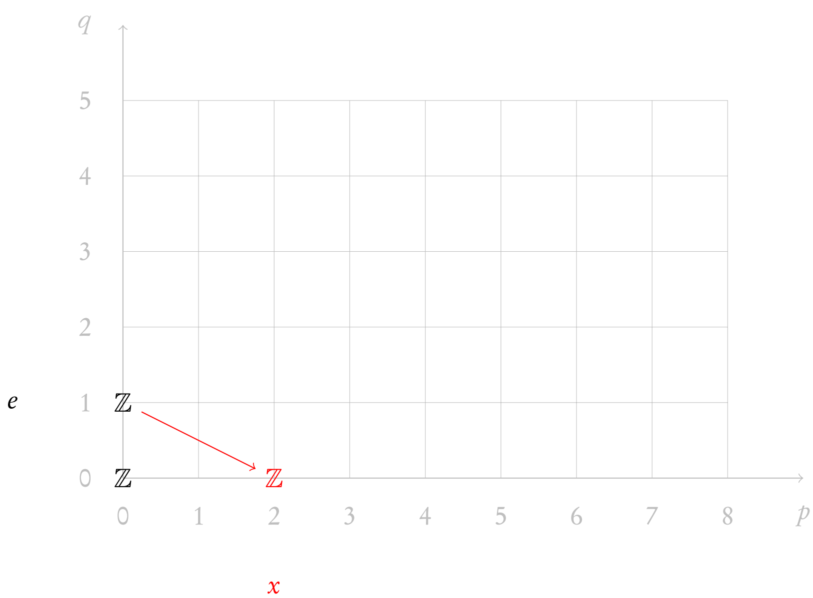
2.6 $H^*\mathbb{C}\mathrm{P}^\infty$ redux

Consider the spherical fibration

$$S^1 \to \mathbb{C}^\infty \setminus \{0\} \to \mathbb{C}\mathrm{P}^\infty.$$

The total space, $\mathbb{C}^\infty \setminus \{0\} \simeq S^\infty$, is contractible, hence has vanishing cohomology. The fiber S^1 has known cohomology groups, $H^*(S^1;\mathbb{Z}) = \Lambda[e]$. We know that, $\mathbb{CP}^\infty$ is connected, and hence we can compute, $H^0(\mathbb{CP}^\infty; H^*S^1)$ — it has two free, generators 1 and e in q-degrees 0 and 1.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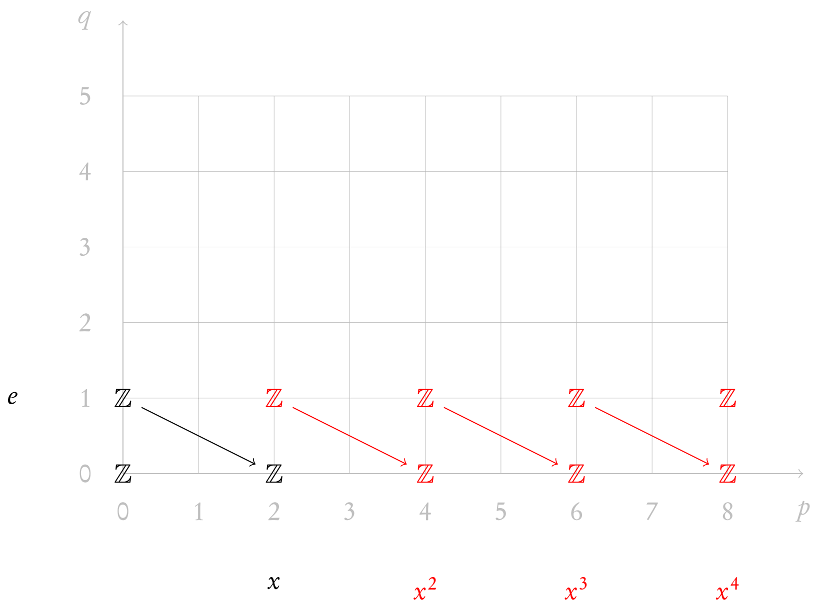
The Serre spectral sequence associated to a singular theory is a first-quadrant spectral sequence, and hence $E_2^{p,q} = 0$ whenever p or q is negative. The differentials have the type signature

$$d_r : E_r^{p,q} o E_r^{p+r,q-r+1}$$

and hence if the class e is to be killed by a differential — and it must, since $H^*(S^\infty, \mathbb{Z}) = \mathbb{Z}$ — it must happen on this page. Therefore, there must be a class x in $E_2^{2,0} = H^2(\mathbb{C}\mathrm{P}^\infty; H^0(S^1; \mathbb{Z}))$ with $d_2(e) = x$.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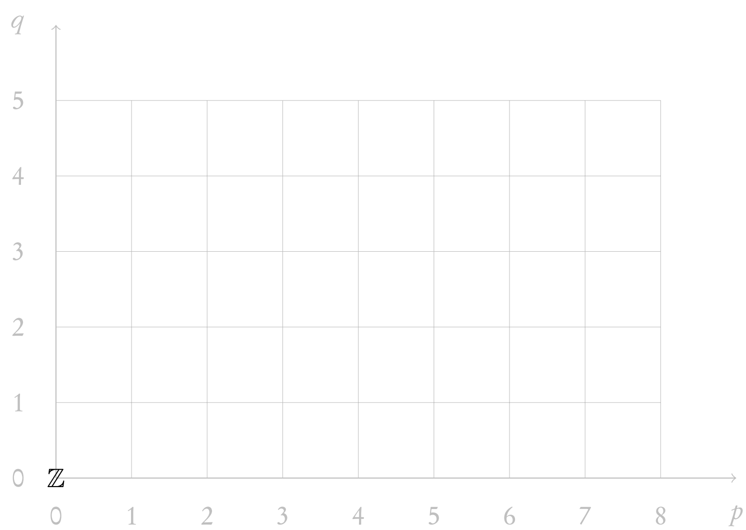
But, if $E_2^{2,0} = H^2(\mathbb{C}\mathrm{P}^\infty; H^0(S^1;\mathbb{Z}))$ is nonzero, then $E_2^{2,1} = H^2(\mathbb{C}\mathrm{P}^\infty; H^1(S^1;\mathbb{Z}))$ is also nonzero, since $H^0(S^1;\mathbb{Z}) \cong H^1(S^1;\mathbb{Z}) \cong \mathbb{Z}$. The Serre spectral sequence is multiplicative, and so we already have a name for this element: $e \cdot x$. Moreover, d_2 is a derivation, so

$$d_2(e \cdot x) = d_2(e) \cdot x + (-1)e \cdot d_2(x) \qquad = x^2 + 0 = x^2.$$

For degree reasons, $e \cdot x$ must also be killed on the E_2-page, and hence x^2 must exist in $E_2^{4,0}$. This pattern continues, as $d_2(e \cdot x^n) = x^{n+1} + (-1)e \cdot n x^{n-1} \cdot 0 = x^{n+1}$.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To build the E_3 page, we take cohomology with the d_2 differentials, and we find nothing left but 1 in the spectral sequence. Hence, $E_3 \cong E_\infty$, and the spectral sequence collapses at E_3.

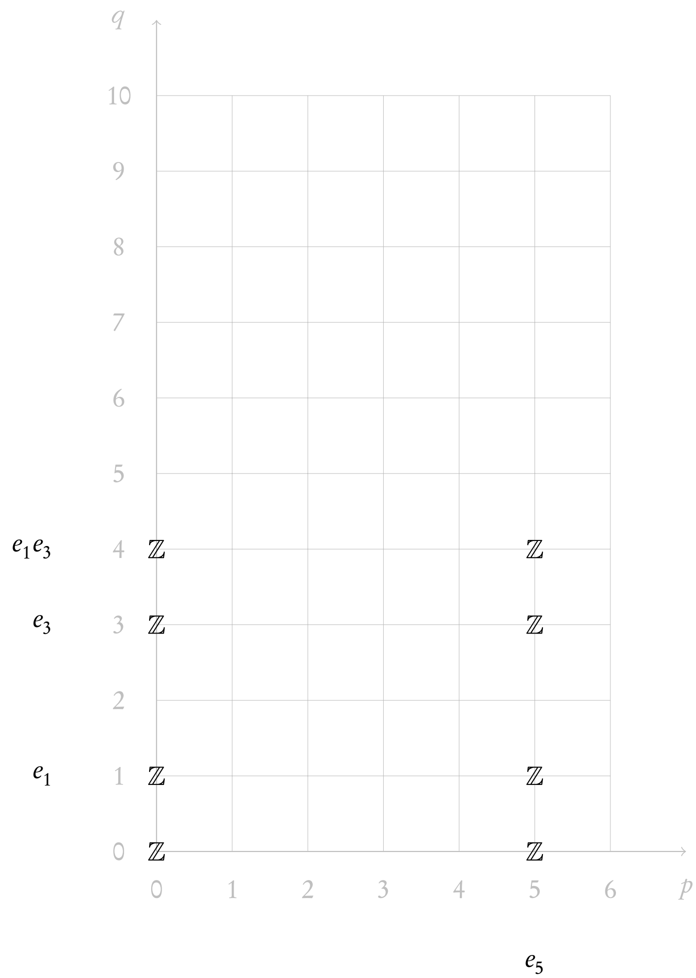
Recall that $E_2^{p,0} = H^p(\mathbb{C}\mathrm{P}^\infty; H^0(S^1;\mathbb{Z})) = H^p(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z})$. So, we can now read off the cohomology of $\mathbb{C}\mathrm{P}^\infty$, together with its ring structure:

$$H^*(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}) \cong \mathbb{Z}[x],$$

where $|x| = 2$.

2.7 $H^*\mathbb{R}\mathrm{P}^\infty$ and Gysin sequences

$K(n)^*B\mathbb{Z}/n$ too? Then, deducing differentials in the AHSS for $K(n)^*B\mathbb{Z}/n$?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2.8 Unitary groups

Now we will compute the cohomology H^*BSU by inductively analyzing related spaces. We begin by computing the cohomology rings $H^*U(n)$, where our primary tool is the fibration

$$U(n-1) \to U(n) \to \mathbb{C}^{2n} \setminus \{0\} \simeq S^{2n-1}.$$

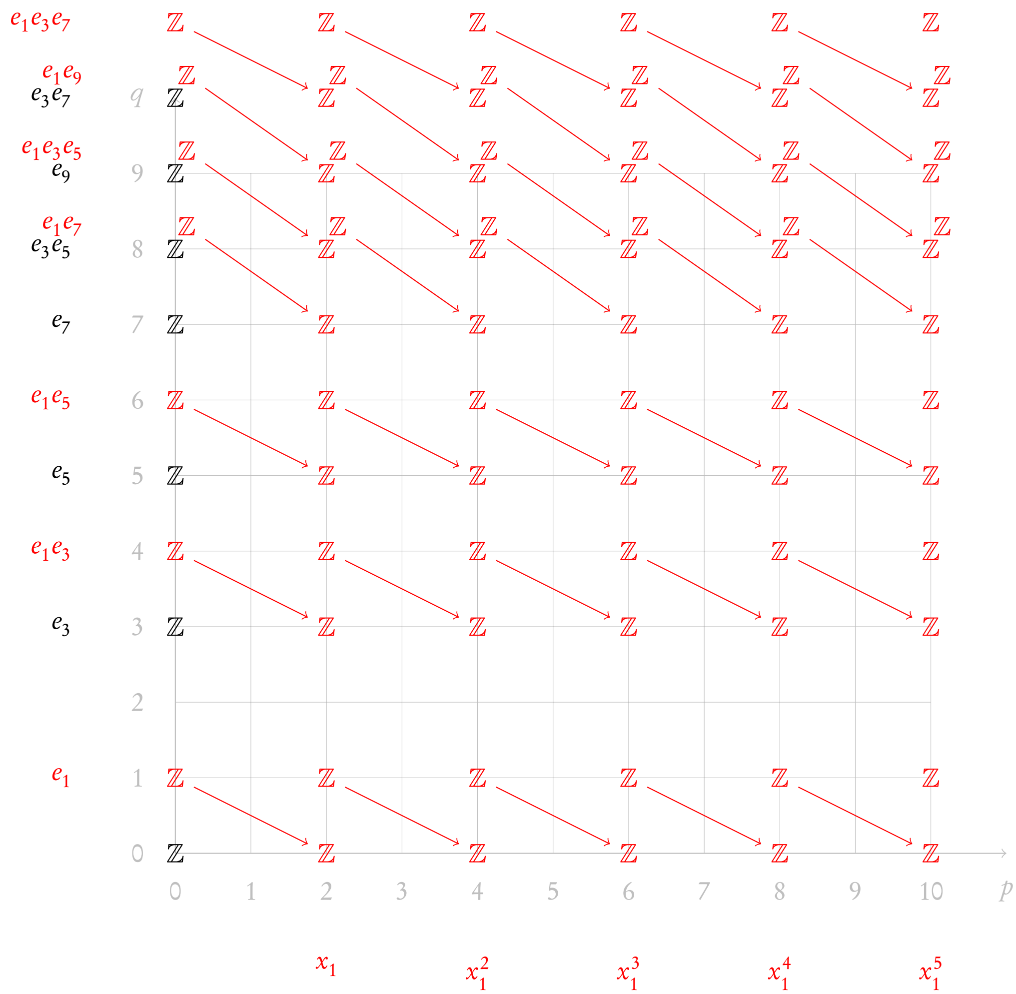
We identify $U(1) \simeq S^1$, which has cohomology $H^*U(1) = \Lambda[e_1]$ for $|e_1| = 1$. In general, we claim that $H^*U(n) = \bigotimes_{i \geq 1} \Lambda[e_{2i-1}]$. Let's consider the case $n = 3$, for example, whose spectral sequence is illustrated at left.

This spectral sequence collapses at this page, using an analysis in two parts. Firstly, consider the indecomposable elements in the fiber column: they are all of odd degree, of dimension bounded by $2n-3$. To support a differential, they must cross a large gap to reach the groups in the right-hand column, a distance of $2n-1$ across. This means that differentials can occur only on the E_{2n-1}-page, of signature $d_{2n-1} : E_{2n-1}^{0,q} \to E_{2n-1}^{2n-1,q-2n}$. The shift in vertical grading forces the differential to land below the p-axis, and so it cannot exist!

Secondly, for any decomposable element $\prod_{i \in I} e_i$, we can apply the Leibniz rule to get

$$d\left(\prod_{i \in I} e_I\right) = \sum_{i \in I} \pm d(e_i) \prod_{\substack{j \in I \\ j \neq i}} e_j.$$

We just showed that $d(e_i) = 0$ for any i, and so the sum collapses, determining all those differentials to be zero as well.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e3e7
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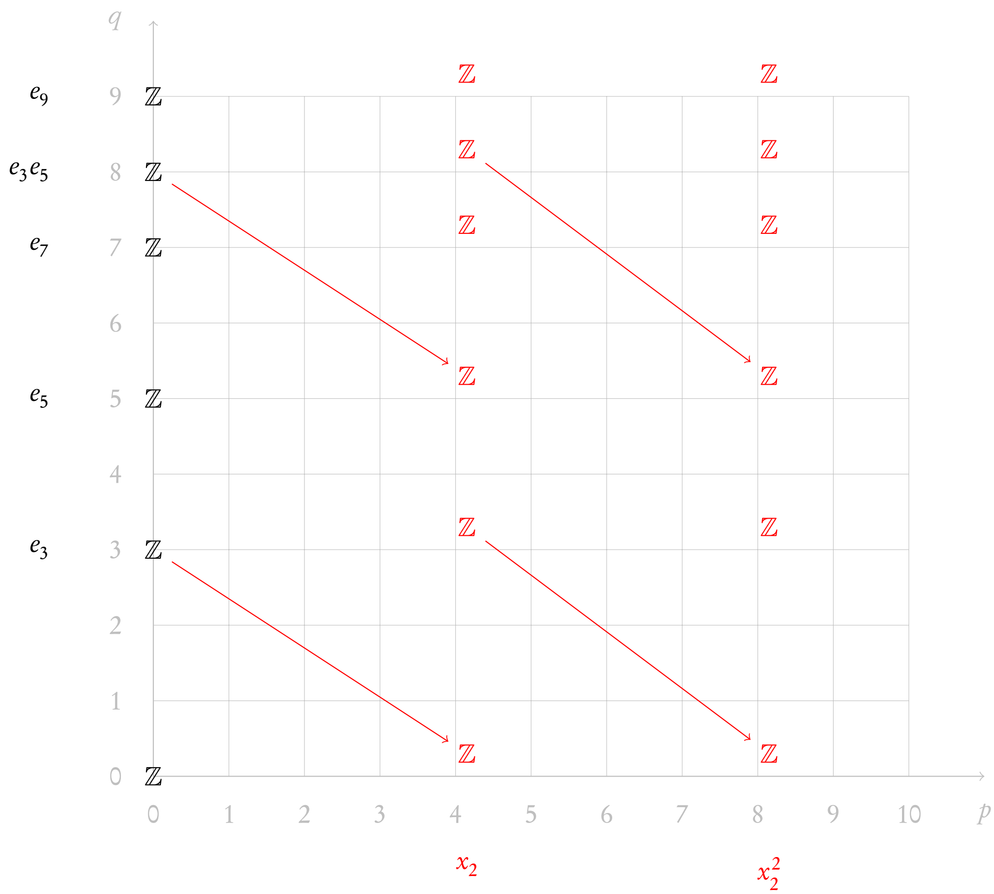
e9
e1e7
e3e5
e7
e1e5
e5
e1e3
e3
e1
q
p
x1
x1^2
x1^3
x1^4
x1^5

Next, we compute the cohomologies $H^*BU(n)$ using the fibration $U(n) \to EU(n) \to BU(n)$, where $EU(n) \simeq \mathrm{pt}$. This is very similar to the computation for $\mathbb{CP}^\infty$, since the fiber sequence $S^1 \to \mathbb{C}^\infty \setminus \{0\} \to \mathbb{CP}^\infty$ is equivalent to $U(1) \to EU(1) \to BU(1)$. Since the total space is contractible, the goal in this game is to clear the board by introducing classes in $H^*BU(n)$ to delete the classes already present coming from $H^*U(n)$.

At left, we consider the bottom of this spectral sequence for $n \geq 4$. We have one chance to delete the class e_1, by introducing a class $x_1 \in H^*BU(n)$ on page E_2, with differential $d(e_1) = x_1$. Application of the Leibniz rule yields a whole host of resulting differentials.


q
p
e_9
e_3e_5
e_7
e_5
e_3
x_2
x_2^2
$\mathbb{Z}$

blah.

$H^*U(n), H^*BU(n), H^*SU(n), H^*BSU(n), H^*BU, H^*BSU$

2.9 Loopspaces of spheres

$H^*\Omega S^{2n}$, $H^*\Omega S^{2n+1}$, $H^*\Omega^2 S^{2n+1}$. Edge homomorphisms.

2.10 The Steenrod algebra

Serre's $H^*(K(\mathbb{Z}/2,q);\mathbb{F}_2)$ and $H^*(K(\mathbb{Z},q);\mathbb{F}_2)$

2.11 $H^*(BU\langle 6\rangle;\mathbb{F}_2)$

Need Kudo transgression.

2.12 Unstable homotopy groups of S^3

π_3, π_4, $\pi_5 L_{(2)} S^3$

Chapter 3

Eilenberg-Moore

Filtration of a bicomplex. Take a homotopy pullback square $F \to E \to B$ and $F \to X \to B$. On cohomology, we don't get a pushout; instead, on the level of the derived category of chain complexes, we are taking the derived pushout, giving a spectral sequence from the tensor product of chain complexes to the chain complex of F. [[NOTE: How does this need to be graded for multiplicativity?]]

3.1 Computing Tor with Tate resolutions

In the previous section, it was mentioned that the Eilenberg-Moore spectral sequence is compatible with the multiplicative structure on Tor. If this is the input to the spectral sequence, then our next question should be: how do we compute this product structure? Or, even more basically, how do we compute Tor at all? In the specific case of R a Noetherian ring and the groups $\operatorname{Tor}^R_{*,*}(R/M, R/N)$, Tate has outlined an extremely useful and simple process for performing this computation, by constructing a DGA whose underlying chain complex is a free resolution of R/M.

Let's compute two examples to see Tate's method in action. First, let's select $R = \mathbb{Z}[x]$, $M = N = \langle x \rangle$, so that we're investigating $\operatorname{Tor}^{\mathbb{Z}[x]}(\mathbb{Z}, \mathbb{Z})$. Tate's resolution, like any resolution, begins with the left-hand argument $\mathbb{Z}$, depicted at left as a dot.

At the next stage in the resolution, we introduce a single copy of R, which surjects onto R/M by the quotient map $R \twoheadrightarrow R/M$. We haven't deviated from the usual process for building a free resolution yet, but Tate's big idea is that we should be giving these things names as algebra generators as we go. Since this copy of R lives in degree 0 of the resolution, and we expect an R-algebra in the end, we attach the name "1" to it, so that its various elements are of the form $r \cdot 1$ for $r \in R$.

To perform the next step, we investigate the kernel of the previous step, depicted beneath the resolution. The kernel here is the submodule of multiples of x, and so we introduce a shifted copy of R in resolution degree 1, mapping isomorphically into the kernel. Again, Tate suggests that we give this a name, so we make one up and call it "a". The differential connecting degree 1 to degree 0 is then described by $da = x$.

At this point, the resolution terminates, since the kernel at filtration degree 1 is empty. To compute $\mathrm{Tor}^{\mathbb{Z}[x]}(\mathbb{Z},\mathbb{Z})$, we drop the original $\mathbb{Z}$ from the resolution, tensor with $\mathbb{Z}$, and compute the cohomology of the resulting complex. Since all the differentials hit multiples of x, they vanish, and the differential structure evaporates. The algebra structure, however, does not disappear, and we compute $\mathrm{Tor}^{\mathbb{Z}[x]}(\mathbb{Z},\mathbb{Z}) = \mathbb{Z}[a]/\langle a^2\rangle$, referred to as an exterior algebra and denoted $\Lambda_{\mathbb{Z}}[x]$.

This example was too short to get interesting, so let's work through another: $\mathrm{Tor}^{\Lambda[x]}(\mathbb{Z},\mathbb{Z})$.

3.2 $H^*(\Omega S^{2n+1};\mathbb{F}_p)$

ΩS^{2n+1} using the square $\Omega S^{2n+1} \to \mathrm{pt} \to S^{2n+1}$ and $\Omega S^{2n+1} \to \mathrm{pt} \to S^{2n+1}$.

3.3 Complex projective spaces

Structure of the spectral sequence for $\mathrm{pt} \to \mathbb{C}\mathrm{P}^\infty$ pulled back to $S^{2n+1} \to \mathbb{C}\mathrm{P}^n$.

3.4 The James construction

The James construction and its filtration, comparison with the particular pullback square $\Omega\Sigma X \to \mathrm{pt} \to \Sigma X$.

3.5 $H^*BU\langle 6\rangle$ redux

Chapter 4

Co/simplicial objects

4.1 Mayer-Vietoris

4.2 The bar spectral sequence and $K(\mathbb{F}_p;*)$

Computations of $H_*(K(\mathbb{F}_p,*);\mathbb{F}_p)$ and $K(n)_*K(\mathbb{F}_p,*)$.

Comparison to the Rothenberg-Steenrod spectral sequence (i.e., the Eilenberg-Moore spectral sequence for the square $G \to EG \to BG$, $G \to \mathrm{pt} \to BG$).

4.3 The descent spectral sequence

Chapter 5

The Adams spectral sequence

Connection to simplicial objects.

Hill has homework assignments posted at http://people.virginia.edu/~mah7cd/Math885/Homework5.pdf and http://people.virginia.edu/~mah7cd/Math885/Homework6.pdf, which can probably be used for more examples.

5.1 The dual of the Steenrod algebra

5.2 Resolution by Hopf algebra quotients

5.3 $\pi_* ko_2^\wedge$

$\pi_* ko_2^\wedge$ from $\mathrm{Ext}_{\mathscr{A}(1)}(k,k)$. This uses $H^*(ko) = A//A(1)$, then a change-of-rings theorem to swap $\mathrm{Ext}_{\mathscr{A}}(\mathscr{A}//\mathscr{A}(1),k)$ for $\mathrm{Ext}_{\mathscr{A}(1)}(k,k)$. See the tail of http://www.math.ku.dk/~jg/students/masulli.msproject.2011.pdf, and also Hill's notes at http://people.virginia.edu/~mah7cd/Math885.html and specifically http://people.virginia.edu/~mah7cd/Math885/Lecture14.pdf. The only possible place for a differential is on the guy in $(1,1)$, called h_1, which has the potential to hit a guy in the tower to his left. This has a cute argument for nonexistence: calling the guy at the bottom of the tower h_0, we have $d(h_1) = kh_0^n$ for some k and n, so $d(h_0h_1) = 0 * h_1 + h_0 * kh_0^n$, but $h_0h_1 = 0$ so $d(h_0h_1) = 0$ so $k = 0$. Cuuuute!

5.4 $\pi_* ku_2^\wedge$

This is somehow the same story as $\pi_* ko$, but with an extra layer superimposed? I very barely remember this...

5.5 $ko^*M(2)$

The ASS over $\mathscr{A}(1)$ for the mod 2 Moore spectrum is computed at `http://people.virginia.edu/~mah7cd/Math885/ExtComps.pdf`; this gives $ko^*M(2)$

5.6 Massey products and secondary operations

5.7 Change of rings

5.8 π_*tmf

5.9 $\pi_{*\leq 16}S$

Hill has lecture notes covering the structure and differentials in the mod 2 Adams spectral sequence through dimension 16 (see Lecture 17). This is presumably accomplishable by considering just bits of the Steenrod algebra... but I haven't looked.

Chapter 6

Homotopy fixed points

Introduction: $H^p(G;\pi_q X) \Rightarrow \pi_{p-q} X^{hG}$. It comes from filtering EG_+ in $X^{hG} = \mathrm{Hom}(EG_+, X)$ using the cellular filtration; the E_1-page of the filtration quotients then looks like the cobar complex computing group cohomology for $\pi_q X$ considered as a G-module (so, remembering the G-action on the underlying spectrum!). Alternatively, X^{hG} can be written as a homotopy limit diagram over some category built from G, which should give an identical construction after piecing through a construction of 'homotopy limit' using a bar-type construction. Needs the Adams grading for the multiplicative structure.

6.1 Computing $H^*_{gp}(C_n, M)$

Computing the cohomology of cyclic groups with twisted coefficients is discussed in Weibel 6.2.1-6.2.2; there's a small, periodic resolution that is much better than the cobar construction.

6.2 $\pi_* KU^{hC_2}$

Needs $H^*(C_2; pi_* KU)$, which means knowing $H^*(\mathbb{RP}^\infty;\mathbb{Z}) = (\mathbb{Z}, 0, \mathbb{Z}/2, 0, \mathbb{Z}/2, 0, \ldots)$ in the untwisted case and $H^*(C_2;\mathbb{Z}) = (0, \mathbb{Z}/2, 0, \mathbb{Z}/2, 0, \ldots)$ in the twisted case. Has a 'multiplication-by-η' structure that's important for propagating differentials. The one generating differential is that the guy in degree $(4,0)$ hits the guy in degree $(3,3)$ (i.e., hits η^3), leaving behind the subgroup of 2-divisible elements. (How on earth is the existence of this differential proven?) See Lennart Meier's talk notes, or the photograph I took of Justin's blackboard.

6.3 $\pi_* ku^{hC_2}$

Essentially the same computation, but there's an extra diagonal vanishing line. This means some elements in negative degrees $-4n$ get missed, and so we don't get ko, which has no homotopy in negative degrees.

6.4 $\pi_* L_{K(1)} S^0_{(3)}$, $\pi_* L_{K(2)} S^0_{(5)}$

Hopkins-Miller says $E_n^{hS_n} = L_{K(n)} S^0$. This computation is accessible for $n = 1$ for sure, but may not involve much of a spectral sequence argument... It does involve the spectral sequence for composing fixed point functors, but it relies on degeneration.

The $K(2)$-local sphere is ridiculous (Shimomura-Wang, Behrens, ...), but maybe *something* can useful can be said about it without too much hassle. It may have to get downgraded to a picture.

Chapter 7

Some pictures of spectral sequences

Some spectral sequences are too hard to compute with, but now that readers know enough about spectral sequences to interpret diagrams, some completely unproven but important pictures might be appropriate to include in the tail of the book.

7.1 E_2 for the mod 2 Adams spectral sequence

7.2 E_2 for the MU-Adams-Novikov spectral sequence

7.3 Slice spectral sequence and the Kervaire invariant

7.4 The chromatic spectral sequence, stabilizer spectral sequences

The chromatic spectral sequence, using Wilson's BP sampler. Existence of the Morava stabilizer spectral sequence, maybe pictures of the stabilizer spectral sequence for the $K(1)$- and $K(2)$-local spheres (at $p \geq 5$?).

7.5 The EHP spectral sequence

Michael Donovan already has a start of a flipbook at `http://math.mit.edu/~mdono/_EHPSS.pdf`, which maybe can be cannibalized for this project. I pray that he hasn't automated this process and beat me to writing this book!

In fact, much of Haynes Miller's notes on the vector fields on spheres problem deals with and can be restated in terms of the EHP spectral sequence. Maybe this should be upgraded to its own section, so we can see some unstable phenomena?

7.6 Devinatz-Hopkins-Smith and the $X(n)$-Adams spectral sequence

Is this do-able? All they show are vanishing lines, but it seems like you should be able to draw a little bit...

7.7 $\mathscr{A}(2)$

Borrow Henriques-Hill?

— unsorted — The May / Bockstein spectral sequences. This seems hard to produce a good example of, since it mostly organizes computation rather than easing it in any particular way, but maybe it's worth it anyhow. The organization is pretty cool. The tangent spectral sequence: formal groups and tangent spaces, $H^*(F;TG) \Rightarrow H^*(F;G)$ Homotopy spectral sequences of spaces (details with π_0 and π_1, see Homotopy limits, completions, and localizations by Bousfield and Kan) Unbased spectral sequences (choose basepoints iteratively, kind of. See Bousfield's Homotopy spectral sequences and obstructions, which does this for unbased cosimplicial spaces.) Dan Dugger has a paper titled Multiplicative structures on homotopy spectral sequences. Could be worth looking at. `http://neil-strickland.staff.shef.ac.uk/courses/bestiary/ss.pdf` Pictures of $\mathscr{A}(1)$-resolutions at `http://math.wayne.edu/art/`