# Gateset design for contemporary systems

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# Cartan decomp$^{ns}$

- For $K$ a nice subgroup of $G$ a nice group, and $A$ a maximal torus perpendicular to $K$ $\Rightarrow$ $G$ decomposes as $G = K \cdot A \cdot K$.

  (For most $g = k_L \cdot a \cdot k_R$ in $G$, the three factors can be made unique.)

- **Linear algebra** ("the spectral theorem")**:**
  If $U$ is **symmetric unitary**, then $U = O^{-1} \cdot D \cdot O$ for $O$ orthogonal, $D$ diagonal

- **Cartan:** If $U$ is just **unitary**, then $U = O_L \cdot D \cdot O_R$
  $\rightsquigarrow$ "imbalanced diagonalization"

# Cartan decomp$^{ns}$ and quantum info$^{n}$

- Euler decomposition (1Q operators)

$$U = O' \cdot D \cdot O = Y_\psi \cdot Z_\theta \cdot Y_\lambda$$ math$^{ician}$ basis

$$= Z_{\psi'} \cdot X_{\theta'} \cdot Z_{\lambda'}$$ quantum info$^{n}$ basis

- Canonical decomposition (2Q operators)

$$U = O' \cdot D \cdot O$$ math$^{ician}$ basis

$$= (L_1 \otimes L_2) \cdot \mathrm{CAN}(\alpha, \beta, \delta) \cdot (R_1 \otimes R_2)$$ quantum info$^{n}$ basis

  - Generally unique
  - $L$, $R$ are local gates
  - $\mathrm{CAN}(\alpha, \beta, \delta) = \exp(-i(\alpha\, XX + \beta\, YY + \delta\, ZZ))$ the "nonlocal signature"

# Cartan decomp$^{ns}$: Canonical coord$^{s}$

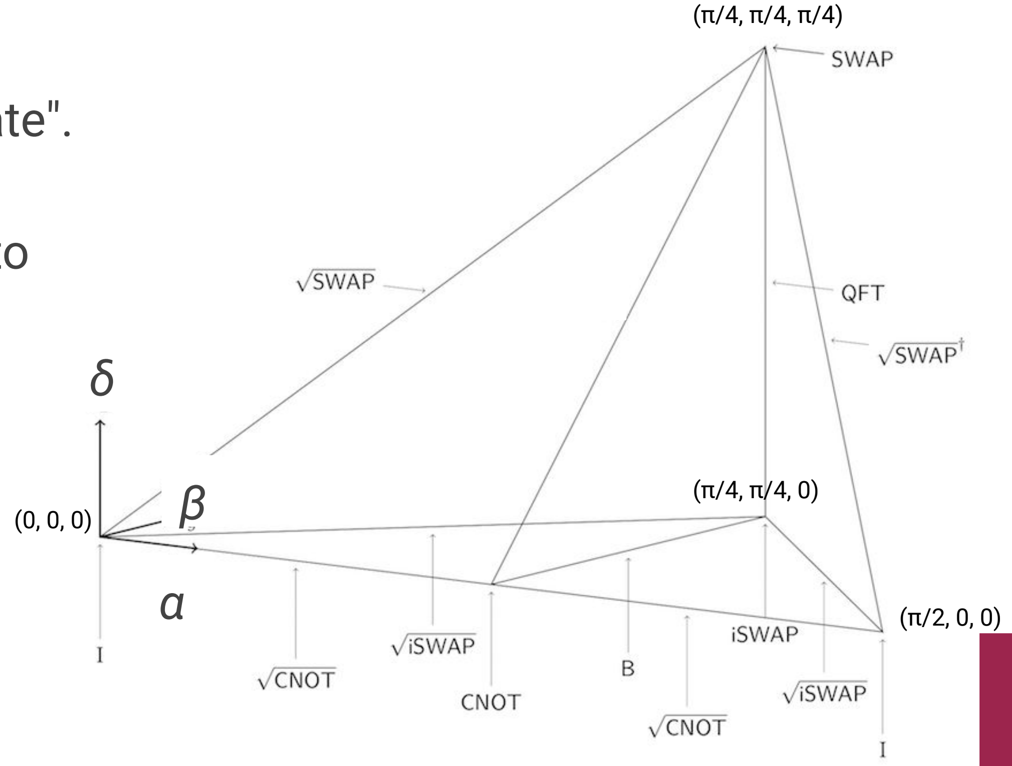
- $(\alpha, \beta, \delta)$ is the "canonical coordinate".

- By convention, these are subject to

$$\alpha \geq \beta \geq \delta \geq 0\,,$$
$$\pi/2 \geq \alpha + \beta\,.$$

- Gates on the bottom plane, $\delta = 0$, have two avatars:

$$(\alpha, \beta, 0) \sim (\pi/2 - \alpha, \beta, 0).$$

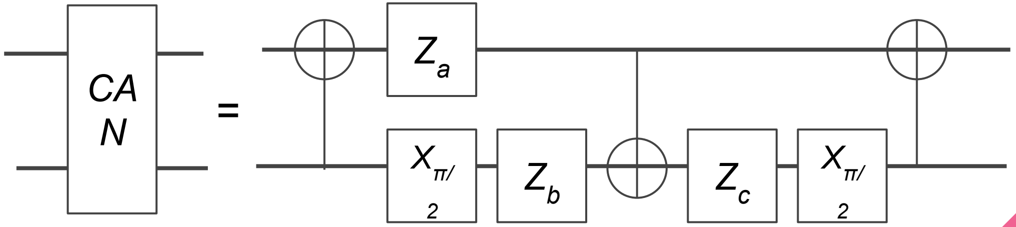


# Cartan decomp$^{ns}$ and quantum circuits

- Euler decomposition

$$\begin{aligned} U &= Z_{\psi'} \cdot X_{\theta'} \cdot Z_{\lambda'} \\ &= Z_{\psi'+\pi} \cdot X_{\pi/2} \cdot Z_{\pi+\theta'} \cdot X_{\pi/2} \cdot Z_{\lambda'} \end{aligned}$$

- Canonical decomposition



# Research program

**Goal:** Optimize device performance by modifying the gate set.

- **Inputs / constraints:**
  - Device Hamiltonian
  - Error model
  - Budget for calibration effort
  - (Target application)
  - (...)
- **Outputs:**
  - Set of gates to calibrate
  - Tailored decomposition routines

# Gate set design: single qubits

- **Assume:** $Z_\theta$ can be done perfectly, $X_\theta$ costs ($m\ \theta + b$) infid[ty] + engineer time.

- **Euler Part 1:** Given $X_{\pi/2}$, every 1Q operator $U$ costs ($m\ \pi + 2\ b$).
- **Euler Part 2:** In $U = Z_{\lambda'} \cdot X_\psi \cdot Z_\lambda$, the "signature" is $\psi$.
- **Exact:** If $X_\psi$ were native, then $U = Z_{\lambda'} \cdot X_\psi \cdot Z_\lambda$ would cost ($m\ \psi + b$).

- **Q:** Which $X_\theta$ can be built out of $X_{\psi 1}$, ..., $X_{\psi n}$? How? What's the cheapest combo?

- **Q:** Fixing $n$, what $\psi_1$, ..., $\psi_n$ give
  the best average-case performance?

# Gate set design: single qubits

**Q:** Which $X_\theta$ are simulable using $X_\psi$ and $X_{\psi'}$ ?

$$Z_{\iota'} \cdot X_\psi \cdot Z_\iota \cdot X_{\psi'} \cdot Z_{\iota''} = \qquad X_\theta$$

# Gate set design: single qubits

**Q:** Which $X_\theta$ are simulable using $X_\psi$ and $X_{\psi'}$ ?

$$X_\psi \cdot Z_\iota \cdot X_{\psi'} \qquad = Z_{\iota'} \cdot X_\theta \cdot Z_{\iota''}$$

# Gate set design: single qubits

**Q:** Which $X_\theta$ are simulable using $X_\psi$ and $X_{\psi'}$ ?

Easier to understand in math[ns] basis:

commingling e[vals] $\longrightarrow$ $Z_\psi \cdot Y_\iota \cdot Z_{\psi'} = Y_{\iota'} \cdot Z_\theta \cdot Y_{\iota''}$

# Gate set design: single qubits

**Q:** Which $X_\theta$ are simulable using $X_\psi$ and $X_{\psi'}$ ?

Easier to understand in math$^{ns}$ basis:

commingling $e^{vals}$ $\longrightarrow$ $Z_\psi \cdot Y_\iota \cdot Z_{\psi'} \quad = Y_{\iota'} \cdot Z_\theta \cdot Y_{\iota''}$

Write out the characteristic polynomial...

$$\cos\theta = \cos\psi\cos\psi' - \cos\iota\sin\psi\sin\psi'$$

destructive interference $\longrightarrow$ $|\psi - \psi'| \leq \theta \leq \pi - |\psi + \psi' - \pi|$ $\longleftarrow$ constructive interference

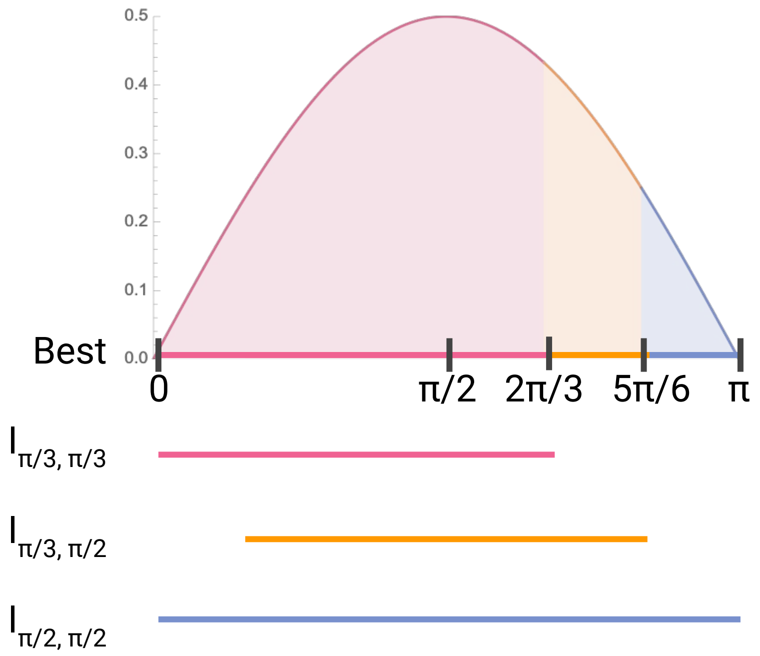
$\psi + \psi'$,

but accounting for wraparound

# Gate set design: single qubits

- Analyze coverage regions!

e.g., $\{ X_{\pi/3}, X_{\pi/2} \}$ →




- Haar-random $\theta$: $p(\theta) = 1/2 \sin(\theta/2)$

- **A:** Best single gate to use is $X_{\pi/2}$ .
  **A:** Best gate to use in tandem with $X_{\pi/2}$ is $X_{\pi/4}$ .
  (If $m / b \approx 5$, **average cost drops to ≈ 79%** .)

# Gate set design: multiple qubits

Same deal, but more param$^s$ / e$^{vals}$.

$$\mathrm{CAN}(a', b', c') \cdot (L \otimes R) \cdot \mathrm{CAN}(a'', b'', c'') = (L' \otimes R') \cdot \mathrm{CAN}(a, b, c) \cdot (L'' \otimes R'')$$

four-way e$^{val}$ interference ⟶ $D' \cdot O \cdot D'' = O' \cdot D \cdot O''$

# Gate set design: multiple qubits

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four-way e$^{val}$ interference ⟶ $D' \cdot O \cdot D'' = O' \cdot D \cdot O''$

**Theorem** (P.–Crooks–Smith)**:** If *CAN*' and *CAN*'' range over polyhedra *P*', *P*'', then *CAN* ranges over a polyhedron *P* which is determined explicitly by *P*', *P*''.

*Proof:* Nonabelian Yang-Mills, Riemann surfaces, symplectic reduction, moduli of $\mathfrak{g}$-valued connections, monodromy, moment map, moduli of algebraic curves, parabolic bundle, semistability, Grassmannian, Schubert classes, quantum cohomology ring, intersection form, Gromov-Witten invariants, ... .

**Important warning:** No control over *O*, *O*', *O*''.

# monodromy: A polytope CAS

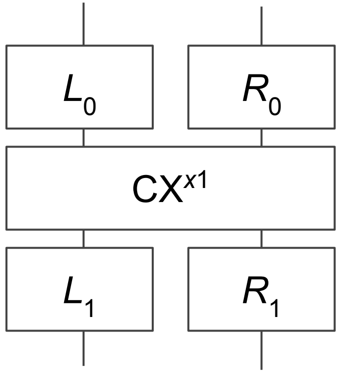
- **Data structures:**
  House non/convex polytopes.
  Precompute static data from the P.–Crooks–Smith theorem.

- **Manipulation:**
  Whatever it takes to isolate CAN from CAN' and CAN".
  Namely: intersection, union, containment, equality, projection, ... .

- **Simplification:**
  Small polytope descriptions for fast computations.

# monodromy example: fractional CX

```
Working on an XX interaction sequence of length 1...
Took 0.06217901199999998 + 0.04937912900000008 + 0.04701946599999984 seconds.
[
    0 +     0 alpha +     0 beta +     1 delta +     0 x1 == 0
    0 +     0 alpha +     1 beta +     0 delta +     0 x1 == 0
    0 +     1 alpha +     0 beta +     0 delta +    -1 x1 == 0
    1 +     0 alpha +     0 beta +     0 delta +    -2 x1 >= 0
    0 +     0 alpha +     0 beta +     0 delta +     1 x1 >= 0
,
    0 +     0 alpha +     0 beta +     1 delta +     0 x1 == 0
    0 +     0 alpha +     1 beta +     0 delta +     0 x1 == 0
    1 +     0 alpha +     0 beta +     0 delta +    -2 x1 >= 0
    0 +     0 alpha +     0 beta +     0 delta +     1 x1 >= 0
   -1 +     2 alpha +     0 beta +     0 delta +    -2 x1 == 0
]
```




$\alpha, \beta, \delta$ normalized so that SWAP is at ½, ½, ½

$x_n$ normalized so that CX is at $x_n$ = ½

# monodromy example: fractional CX

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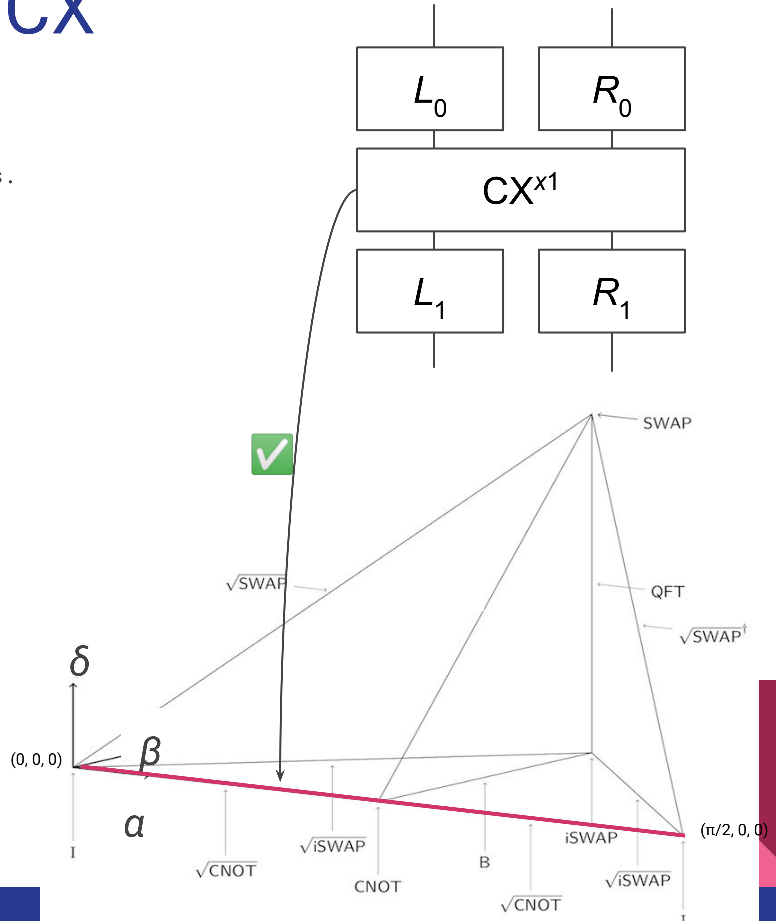
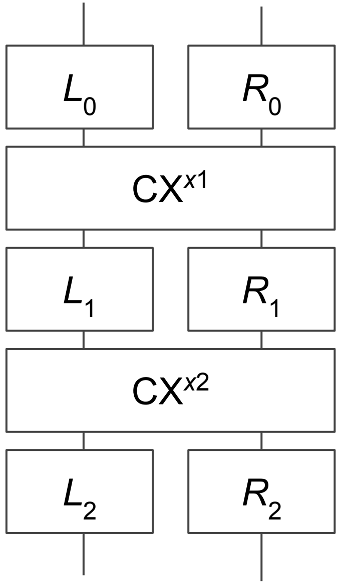
Took 0.06217901199999998 + 0.04937912900000008 + 0.04701946599999984 seconds.
[
     0 +      0 alpha +      0 beta +      1 delta +      0 x1 == 0
     0 +      0 alpha +      1 beta +      0 delta +      0 x1 == 0
     0 +      1 alpha +      0 beta +      0 delta +     -1 x1 == 0
     1 +      0 alpha +      0 beta +      0 delta +     -2 x1 >= 0
     0 +      0 alpha +      0 beta +      0 delta +      1 x1 >= 0
,
     0 +      0 alpha +      0 beta +      1 delta +      0 x1 == 0
     0 +      0 alpha +      1 beta +      0 delta +      0 x1 == 0
     1 +      0 alpha +      0 beta +      0 delta +     -2 x1 >= 0
     0 +      0 alpha +      0 beta +      0 delta +      1 x1 >= 0
    -1 +      2 alpha +      0 beta +      0 delta +     -2 x1 == 0
]
```



# monodromy example: fractional CX

```
Working on an XX interaction sequence of length 2...
Took 0.2472996639999998 + 0.11630750199999995 + 0.257707522 seconds.
[
     1 +      0 alpha +      0 beta +      0 delta +     -2 x1 +      0 x2 >= 0
     0 +      0 alpha +      0 beta +      1 delta +      0 x1 +      0 x2 >= 0
     1 +      0 alpha +      0 beta +      0 delta +      0 x1 +     -2 x2 >= 0
     0 +      0 alpha +      1 beta +      0 delta +      0 x1 +      0 x2 >= 0
     0 +      0 alpha +      0 beta +     -1 delta +      0 x1 +      0 x2 >= 0
     0 +      1 alpha +     -1 beta +      0 delta +     -1 x1 +      1 x2 >= 0
     0 +      1 alpha +     -1 beta +      0 delta +      1 x1 +     -1 x2 >= 0
     0 +     -1 alpha +     -1 beta +      0 delta +      1 x1 +      1 x2 >= 0
,
     1 +      0 alpha +      0 beta +      0 delta +     -2 x1 +      0 x2 >= 0
     0 +      0 alpha +      0 beta +      1 delta +      0 x1 +      0 x2 >= 0
     1 +      0 alpha +      0 beta +      0 delta +      0 x1 +     -2 x2 >= 0
     0 +      0 alpha +      1 beta +      0 delta +      0 x1 +      0 x2 >= 0
     0 +      0 alpha +      0 beta +     -1 delta +      0 x1 +      0 x2 >= 0
     1 +     -1 alpha +     -1 beta +      0 delta +     -1 x1 +      1 x2 >= 0
     1 +     -1 alpha +     -1 beta +      0 delta +      1 x1 +     -1 x2 >= 0
    -1 +      1 alpha +     -1 beta +      0 delta +      1 x1 +      1 x2 >= 0
]
```

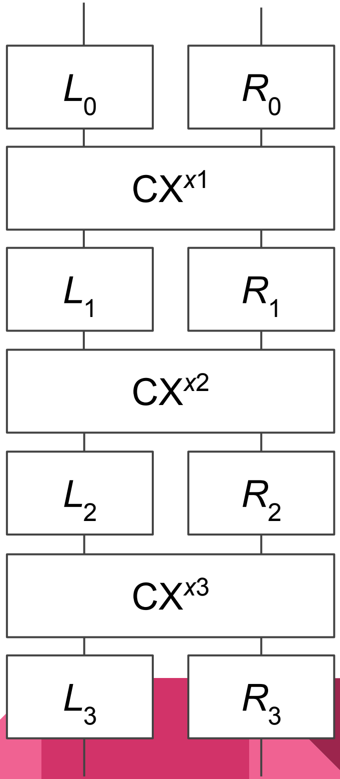


# monodromy example: fractional CX

```
Working on an XX interaction sequence of length 3...
Took 0.4972898670000001 + 0.40358383799999986 + 0.39905767999999986 seconds.
[
    1 +     0 alpha +     0 beta +     0 delta +    -2 x1 +     0 x2 +     0 x3 >= 0
    1 +     0 alpha +     0 beta +     0 delta +     0 x1 +    -2 x2 +     0 x3 >= 0
    1 +    -1 alpha +    -1 beta +     0 delta +     0 x1 +     0 x2 +     0 x3 >= 0
    0 +     0 alpha +     0 beta +     1 delta +     0 x1 +     0 x2 +     0 x3 >= 0
    0 +     0 alpha +     1 beta +    -1 delta +     0 x1 +     0 x2 +     0 x3 >= 0
    0 +     1 alpha +    -1 beta +     0 delta +     0 x1 +     0 x2 +     0 x3 >= 0
    1 +     0 alpha +     0 beta +     0 delta +     0 x1 +     0 x2 +    -2 x3 >= 0
    0 +     1 alpha +    -1 beta +    -1 delta +     1 x1 +     1 x2 +    -1 x3 >= 0
    0 +     1 alpha +    -1 beta +    -1 delta +    -1 x1 +     1 x2 +     1 x3 >= 0
    0 +     1 alpha +    -1 beta +    -1 delta +     1 x1 +    -1 x2 +     1 x3 >= 0
    0 +    -1 alpha +    -1 beta +    -1 delta +     1 x1 +     1 x2 +     1 x3 >= 0
    0 +     0 alpha +     0 beta +    -1 delta +     0 x1 +     1 x2 +     0 x3 >= 0
    0 +     0 alpha +     0 beta +    -1 delta +     1 x1 +     0 x2 +     0 x3 >= 0
    0 +     0 alpha +     0 beta +    -1 delta +     0 x1 +     0 x2 +     1 x3 >= 0
,

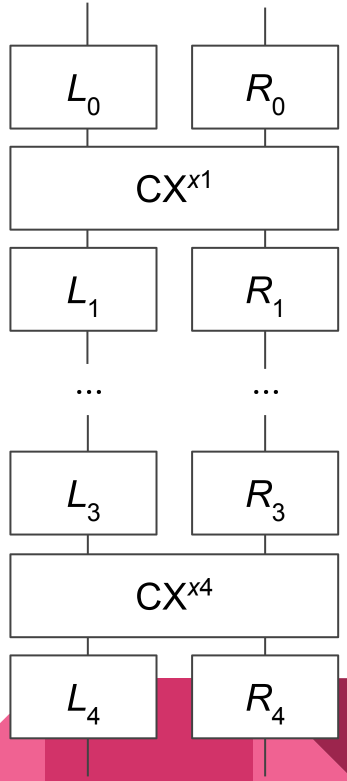
    ...

]
```



# monodromy example: fractional CX

```
Working on an XX interaction sequence of length 4...
Took 9.483033161999998 + 3.4342919700000003 + 1.014309728999999 seconds.
[
    1 +    0 alpha +    0 beta +    0 delta +   -2 x1 +    0 x2 +    0 x3 +    0 x4 >= 0
    1 +    0 alpha +    0 beta +    0 delta +    0 x1 +   -2 x2 +    0 x3 +    0 x4 >= 0
    1 +    0 alpha +    0 beta +    0 delta +    0 x1 +    0 x2 +   -2 x3 +    0 x4 >= 0
    1 +   -1 alpha +   -1 beta +    0 delta +    0 x1 +    0 x2 +    0 x3 +    0 x4 >= 0
    0 +    0 alpha +    0 beta +    1 delta +    0 x1 +    0 x2 +    0 x3 +    0 x4 >= 0
    0 +    0 alpha +    1 beta +   -1 delta +    0 x1 +    0 x2 +    0 x3 +    0 x4 >= 0
    0 +    1 alpha +   -1 beta +    0 delta +    0 x1 +    0 x2 +    0 x3 +    0 x4 >= 0
    1 +    0 alpha +    0 beta +    0 delta +    0 x1 +    0 x2 +    0 x3 +   -2 x4 >= 0
    0 +    0 alpha +    0 beta +    0 delta +    0 x1 +    1 x2 +    0 x3 +    0 x4 >= 0
    0 +    0 alpha +    0 beta +    0 delta +    1 x1 +    0 x2 +    0 x3 +    0 x4 >= 0
    0 +    0 alpha +    0 beta +    0 delta +    0 x1 +    0 x2 +    1 x3 +    0 x4 >= 0
    0 +    1 alpha +   -1 beta +   -1 delta +    1 x1 +    1 x2 +   -1 x3 +    1 x4 >= 0
    0 +    1 alpha +   -1 beta +   -1 delta +   -1 x1 +    1 x2 +    1 x3 +    1 x4 >= 0
    0 +    1 alpha +   -1 beta +   -1 delta +    1 x1 +   -1 x2 +    1 x3 +    1 x4 >= 0
    0 +   -1 alpha +   -1 beta +   -1 delta +    1 x1 +    1 x2 +    1 x3 +    1 x4 >= 0
    0 +    1 alpha +   -1 beta +   -1 delta +    1 x1 +    1 x2 +    1 x3 +   -1 x4 >= 0
    0 +    0 alpha +    0 beta +   -1 delta +    1 x1 +    1 x2 +    0 x3 +    0 x4 >= 0
    0 +    0 alpha +    0 beta +   -1 delta +    0 x1 +    1 x2 +    1 x3 +    0 x4 >= 0
    0 +    0 alpha +    0 beta +   -1 delta +    1 x1 +    0 x2 +    1 x3 +    0 x4 >= 0
    0 +    0 alpha +    0 beta +   -1 delta +    0 x1 +    1 x2 +    0 x3 +    1 x4 >= 0
    0 +    0 alpha +    0 beta +   -1 delta +    1 x1 +    0 x2 +    0 x3 +    1 x4 >= 0
    0 +    0 alpha +    0 beta +   -1 delta +    0 x1 +    0 x2 +    1 x3 +    1 x4 >= 0
    0 +    0 alpha +    0 beta +    0 delta +    0 x1 +    0 x2 +    0 x3 +    1 x4 >= 0
,
    ...
]
```



# monodromy example: fractional CX

```
[
    1 +   0 alpha +   0 beta +   0 delta +  -2 x1 +   0 x2 +   0 x3 +   0 x4 >= 0
    1 +   0 alpha +   0 beta +   0 delta +   0 x1 +  -2 x2 +   0 x3 +   0 x4 >= 0
    1 +   0 alpha +   0 beta +   0 delta +   0 x1 +   0 x2 +  -2 x3 +   0 x4 >= 0
    1 +   0 alpha +   0 beta +   0 delta +   0 x1 +   0 x2 +   0 x3 +  -2 x4 >= 0
    0 +   0 alpha +   0 beta +   0 delta +   1 x1 +   0 x2 +   0 x3 +   0 x4 >= 0
    0 +   0 alpha +   0 beta +   0 delta +   0 x1 +   1 x2 +   0 x3 +   0 x4 >= 0
    0 +   0 alpha +   0 beta +   0 delta +   0 x1 +   0 x2 +   1 x3 +   0 x4 >= 0
    0 +   0 alpha +   0 beta +   0 delta +   0 x1 +   0 x2 +   0 x3 +   1 x4 >= 0

    1 +  -1 alpha +  -1 beta +   0 delta +   0 x1 +   0 x2 +   0 x3 +   0 x4 >= 0
    0 +   0 alpha +   0 beta +   1 delta +   0 x1 +   0 x2 +   0 x3 +   0 x4 >= 0
    0 +   0 alpha +   1 beta +  -1 delta +   0 x1 +   0 x2 +   0 x3 +   0 x4 >= 0
    0 +   1 alpha +  -1 beta +   0 delta +   0 x1 +   0 x2 +   0 x3 +   0 x4 >= 0

    0 +  -1 alpha +  -1 beta +  -1 delta +   1 x1 +   1 x2 +   1 x3 +   1 x4 >= 0

    0 +   1 alpha +  -1 beta +  -1 delta +  -1 x1 +   1 x2 +   1 x3 +   1 x4 >= 0
    0 +   1 alpha +  -1 beta +  -1 delta +   1 x1 +  -1 x2 +   1 x3 +   1 x4 >= 0
    0 +   1 alpha +  -1 beta +  -1 delta +   1 x1 +   1 x2 +  -1 x3 +   1 x4 >= 0
    0 +   1 alpha +  -1 beta +  -1 delta +   1 x1 +   1 x2 +   1 x3 +  -1 x4 >= 0

    0 +   0 alpha +   0 beta +  -1 delta +   1 x1 +   1 x2 +   0 x3 +   0 x4 >= 0
    0 +   0 alpha +   0 beta +  -1 delta +   1 x1 +   0 x2 +   1 x3 +   0 x4 >= 0
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    0 +   0 alpha +   0 beta +  -1 delta +   0 x1 +   1 x2 +   1 x3 +   0 x4 >= 0
    0 +   0 alpha +   0 beta +  -1 delta +   0 x1 +   1 x2 +   0 x3 +   1 x4 >= 0
    0 +   0 alpha +   0 beta +  -1 delta +   0 x1 +   0 x2 +   1 x3 +   1 x4 >= 0
]
```

**interaction ineq$^s$**

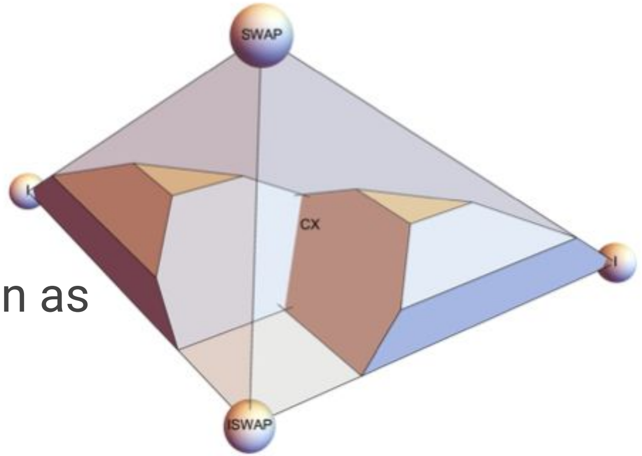
**CAN ineq$^s$**

**"total strength inequality"**

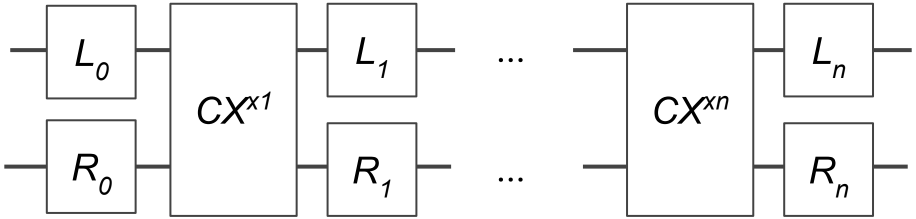
**"slant inequality"**

**"height inequality"**

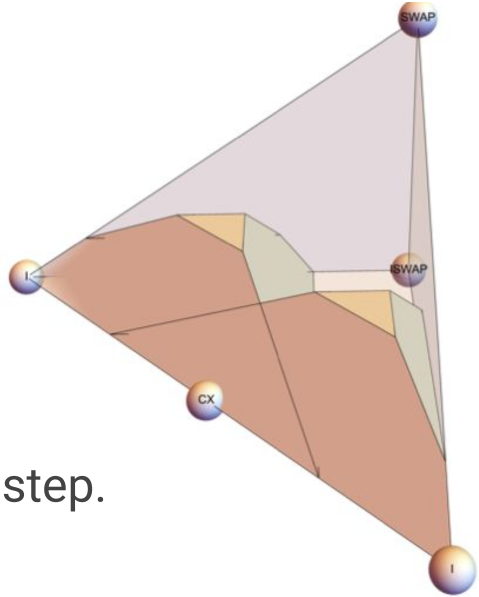
# Gate set design: fractional CX




**Theorem** (P.–Bishop–Javadi): CAN($\alpha, \beta, \delta$) can be written as



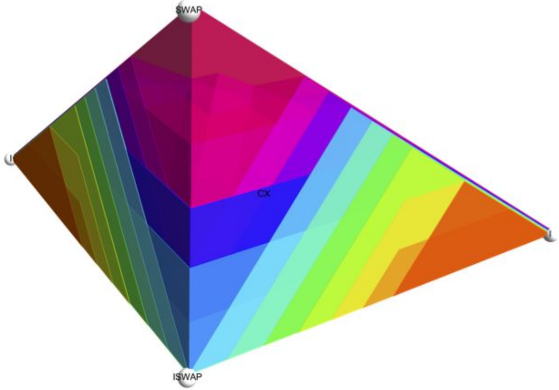

iff $$\alpha + \beta + \delta \leq \textstyle\sum_j x_j\,,$$
$$-\alpha + \beta + \delta \leq \textstyle\sum_j x_j - \max_k \{ 2x_k \}\,,$$
$$\delta \leq \textstyle\sum_j x_j - \max_{k \neq m} \{ x_k + x_m \}$$
are met (or are met after a certain reflection).

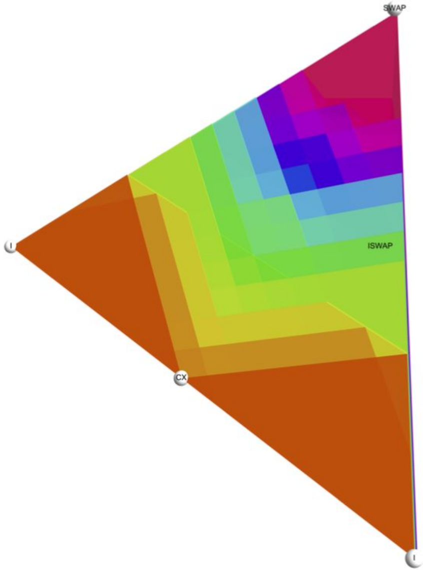



$P^f$: Only 6 free variables. Use the CAS to check inductive step.

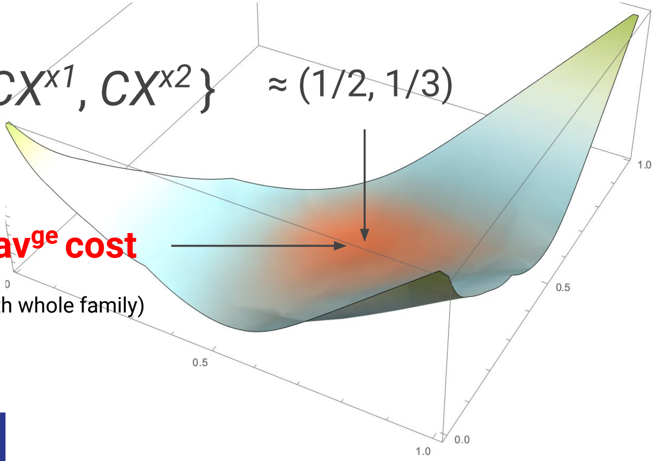
# Gate set design: fractional CX

- Circuit regions,
  e.g., for { $CX$, $CX^{1/2}$, $CX^{1/3}$ }




- Expected fidelity for { $CX$, $CX^{x1}$, $CX^{x2}$ } ≈ (1/2, 1/3)
  and $m$ / $b$ ≈ 5

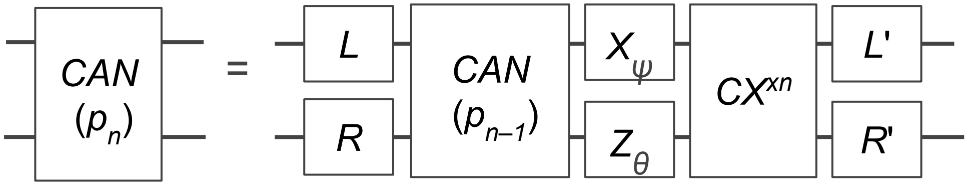


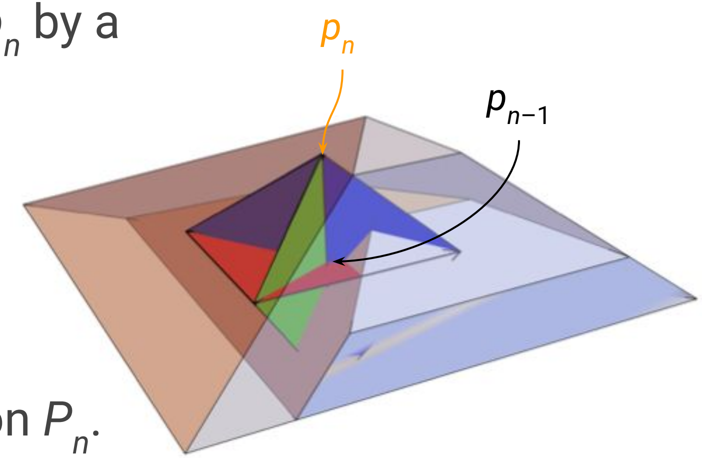

**≈66% av$^{ge}$ cost**

(≈63.1% with whole family)

# Compiling to fractional CX

**Theorem** (P.–Bishop–Javadi): If $p_n$ meets the condition for $x_1, \ldots, x_n$, then there is a $p_{n-1}$ for $x_1, \ldots, x_{n-1}$ related to $p_n$ by a local circuit w/ known parameters, as in:




P[f]: Say $p_{n-1}$ belongs to little region $P_{n-1}$, $p_n$ to big region $P_n$.
Fixing $p_{n-1}$ and letting $\psi$ and $\theta$ range gives the star.
Set $P'_n = \{ p_n \text{ in } P_n \mid p_n \text{ reachable by the star from some } p_{n-1} \}$.
CAS concludes $P_n' = P_n$. (Then isolate $p_{n-1}$ and solve!)

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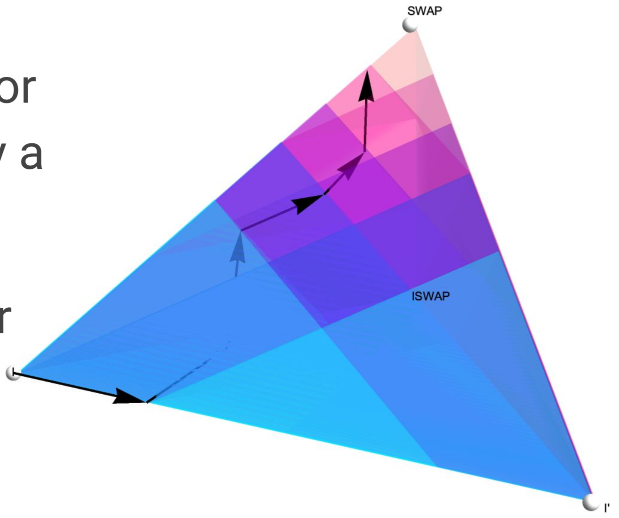
**Corollary:** Apply repeatedly to get optimal compilation for fractional *CX*.

**Observ^ns^:**

Fast! ≈ 2× slower than standard *CX*.

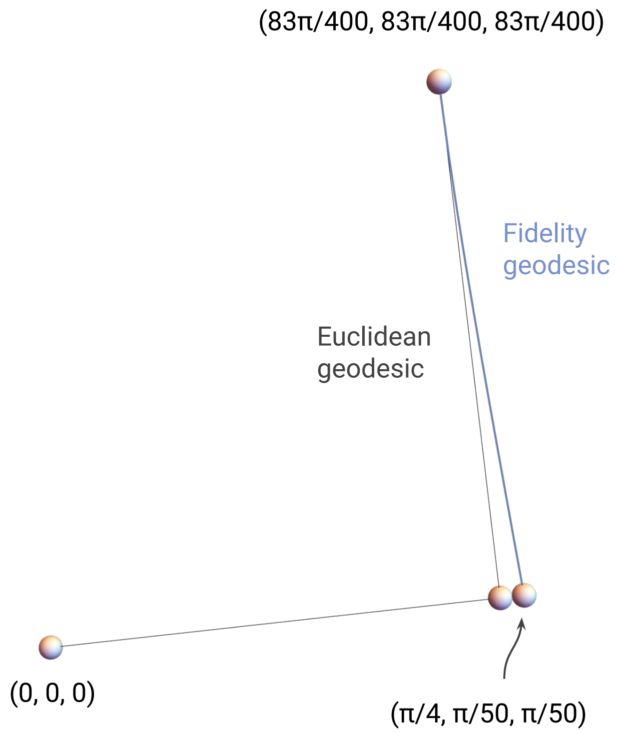
No precomputation, tables, ... .

Can outperform brute search in *quality*.



# Fractional CX and approximation

**Fact:** If local pre-/post-corrections are allowed, can calculate best approximation in av$^g$ gate fidelity using canonical components.

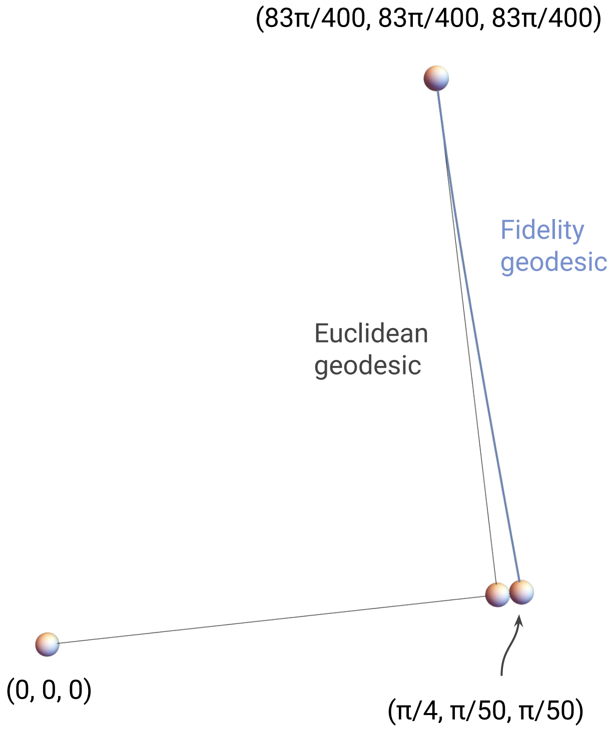


# Fractional CX and approximation

**Fact:** If local pre-/post-corrections are allowed,
can calculate best approximation in av$^g$ gate fidelity
using canonical components.

**Theorem** (P.–Bishop–Javadi):
When $P$ is a fractional CX polytope,
Euclidean-nearest agrees with fidelity-nearest.

This means it's *fast* to calculate best approximations.
Super mysterious why this is true!



# Thank you!

https://chromotopy.org

http://github.com/ecpeterson

Paper on arXiv, code on GH

# Appendix: Fractional *ISWAP*

```
Working on an XY interaction sequence of length 2...
Took 0.8788942500000303 + 0.1939395639999475 + 0.3812988629999836 seconds.
[
    0 +     1 alpha +    -1 beta +    -1 delta +     0 x1 +     0 x2 >= 0
    0 +     0 alpha +     1 beta +    -1 delta +     0 x1 +     0 x2 >= 0
    1 +    -1 alpha +    -1 beta +     0 delta +     0 x1 +     0 x2 >= 0
    0 +     0 alpha +     0 beta +     1 delta +     0 x1 +     0 x2 >= 0

    1 +     0 alpha +     0 beta +     0 delta +     0 x1 +    -2 x2 >= 0
    1 +     0 alpha +     0 beta +     0 delta +    -2 x1 +     0 x2 >= 0

    0 +     0 alpha +     1 beta +     0 delta +     1 x1 +    -1 x2 >= 0
    0 +     0 alpha +     1 beta +     0 delta +    -1 x1 +     1 x2 >= 0

    2 +    -1 alpha +    -1 beta +    -1 delta +    -2 x1 +     0 x2 >= 0
    2 +    -1 alpha +    -1 beta +    -1 delta +     0 x1 +    -2 x2 >= 0

    0 +    -1 alpha +     1 beta +    -1 delta +     0 x1 +     2 x2 >= 0
    0 +    -1 alpha +     1 beta +    -1 delta +     2 x1 +     0 x2 >= 0

    1 +     0 alpha +     0 beta +    -1 delta +    -1 x1 +    -1 x2 >= 0
    0 +    -1 alpha +     0 beta +     0 delta +     1 x1 +     1 x2 >= 0
,
    ...
]
```

alcove inequalities

interaction inequalities

$\beta \geq |x_1 - x_2|$

$2 - \alpha_+ \geq \max \{ x_1, x_2 \}$

$2 \min \{ x_1, x_2 \} \geq \alpha - \beta + \delta$

$1 - \delta \geq x_1 + x_2 \geq \alpha$

# Appendix: Fractional *ISWAP*

```
Working on an XY interaction sequence of length 3...
Took 115.42721978700001 + 99.31264188099999 + 6.086576625000134 seconds.
[
    # (suppressed alcove and interaction inequalities)

    0 +     0 alpha +     0 beta +    -1 delta +     0 x1 +     1 x2 +     1 x3 >= 0
    0 +     0 alpha +     0 beta +    -1 delta +     1 x1 +     0 x2 +     1 x3 >= 0
    0 +     0 alpha +     0 beta +    -1 delta +     1 x1 +     1 x2 +     0 x3 >= 0

    1 +     0 alpha +     0 beta +    -1 delta +     1 x1 +    -1 x2 +    -1 x3 >= 0
    1 +     0 alpha +     0 beta +    -1 delta +    -1 x1 +     1 x2 +    -1 x3 >= 0
    1 +     0 alpha +     0 beta +    -1 delta +    -1 x1 +    -1 x2 +     1 x3 >= 0

    0 +     0 alpha +     1 beta +     0 delta +    -1 x1 +     1 x2 +     1 x3 >= 0
    0 +     0 alpha +     1 beta +     0 delta +     1 x1 +    -1 x2 +     1 x3 >= 0
    0 +     0 alpha +     1 beta +     0 delta +     1 x1 +     1 x2 +    -1 x3 >= 0

    0 +    -1 alpha +     0 beta +     0 delta +     1 x1 +     1 x2 +     1 x3 >= 0

    0 +     1 alpha +     1 beta +    -1 delta +    -2 x1 +     2 x2 +     2 x3 >= 0
    0 +     1 alpha +     1 beta +    -1 delta +     2 x1 +    -2 x2 +     2 x3 >= 0
    0 +     1 alpha +     1 beta +    -1 delta +     2 x1 +     2 x2 +    -2 x3 >= 0

    0 +    -1 alpha +     1 beta +    -1 delta +     0 x1 +     2 x2 +     2 x3 >= 0
    0 +    -1 alpha +     1 beta +    -1 delta +     2 x1 +     0 x2 +     2 x3 >= 0
    0 +    -1 alpha +     1 beta +    -1 delta +     2 x1 +     2 x2 +     0 x3 >= 0

    ... more ...
```

# Appendix: Fractional *ISWAP*

```
Working on an XY interaction sequence of length 3...
Took 115.42721978700001 + 99.31264188099999 + 6.086576625000134 seconds.
[
    ... more ...

    2 +    -1 alpha +    -1 beta +    -1 delta +    -2 x1 +     2 x2 +     0 x3 >= 0
    2 +    -1 alpha +    -1 beta +    -1 delta +    -2 x1 +     0 x2 +     2 x3 >= 0
    2 +    -1 alpha +    -1 beta +    -1 delta +     2 x1 +    -2 x2 +     0 x3 >= 0
    2 +    -1 alpha +    -1 beta +    -1 delta +     0 x1 +    -2 x2 +     2 x3 >= 0
    2 +    -1 alpha +    -1 beta +    -1 delta +     2 x1 +     0 x2 +    -2 x3 >= 0
    2 +    -1 alpha +    -1 beta +    -1 delta +     0 x1 +     2 x2 +    -2 x3 >= 0

    0 +     1 alpha +    -1 beta +    -1 delta +     2 x1 +     0 x2 +     0 x3 >= 0
    0 +     1 alpha +    -1 beta +    -1 delta +     0 x1 +     2 x2 +     0 x3 >= 0
    0 +     1 alpha +    -1 beta +    -1 delta +     0 x1 +     0 x2 +     2 x3 >= 0

    0 +    -1 alpha +    -1 beta +    -1 delta +     2 x1 +     2 x2 +     2 x3 >= 0

    1 +    -1 alpha +     0 beta +    -1 delta +     1 x1 +     1 x2 +    -1 x3 >= 0

,

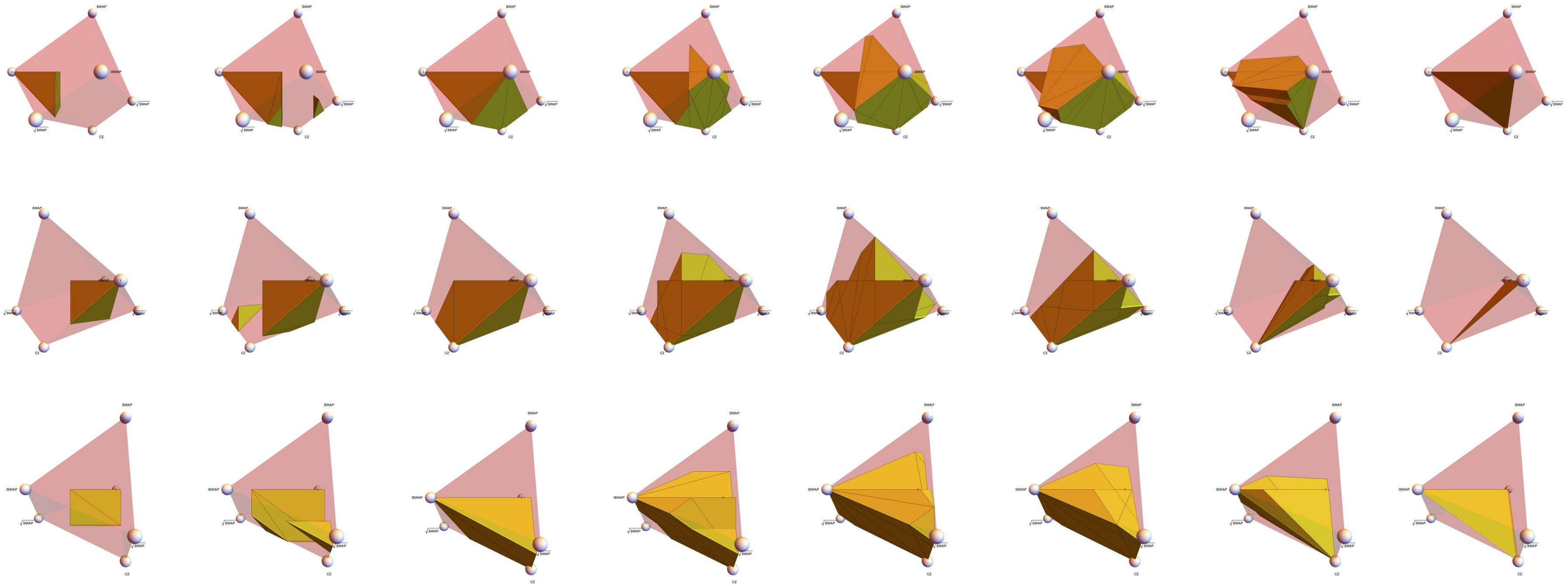
    ... 3 other convex summands ...

]
```

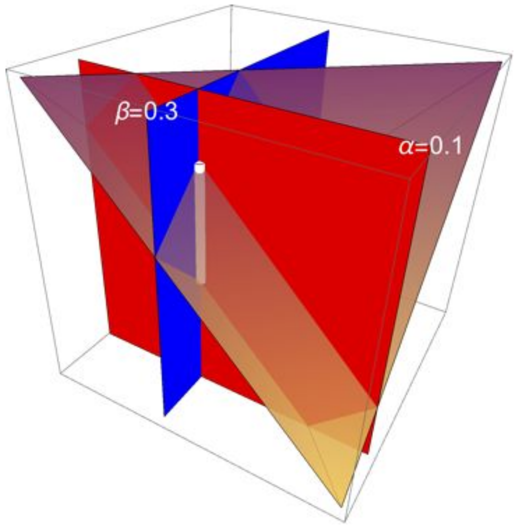
Asymmetric!

# Appendix: Symmetric fractional *ISWAP*

3/10 4/10 5/10 6/10 7/10 8/10 9/10 *ISWAP*



# Appendix: The monodromy polytope




- Yang–Mills for $SU(4)$ on $\Sigma = \mathbb{P}^1 \setminus \{0, 1, \infty\}$
  "Energy functional": $\Phi^\flat : \mathcal{M}^\flat(\Sigma; SU(4)) \to \mathfrak{t}^{\times 3}$.
  **(Atiyah et al.:)** The image of any energy functional
  is* a convex polytope.
  **(Falbel–Wentworth:)** Restrict mixers $SO(n) \leq SU(n)$.

- Bundles on Σ (w/ monodromy) ⇜⇝ Moduli of curves on Gr($n$, $k$) (w/ markings).
  "Schubert cells": geometric regions tracking ways $e^{vals}$ can interact.
  $Log^{mic}$ $e^{vals}$ can't wrap around "too much", where "too much"
  is given by intersecting the conditions that
  $e^{vals}$ interact at all.