RECURRENCE RELATIONS IN THOM SPECTRA

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(Throughout, H^* will default to mod-2 cohomology.)

1. A HIGHLY INTERESTING SPACE

We begin with a love letter to $\mathbb{RP}^\infty$. Its first appearance in the theory of algebraic topology comes about via the Steenrod operations: for any space X there is a natural map

$$\mathrm{Sq} : H^*X \to H^*X \otimes H^*\mathbb{RP}^\infty \cong H^*X \otimes \mathbb{F}_2[t].$$

The squaring operations themselves appear as coefficients in the resulting polynomial:

$$\mathrm{Sq}\, x = \sum_{n=0}^{\deg x} \mathrm{Sq}^n(x) t^n.$$

All of the standard properties but the Adem relations can be quickly deduced from a formal presentation. To get the Adem relations, one winds up investigating the Steenrod module structure of $H^*\mathbb{RP}^\infty$ itself:

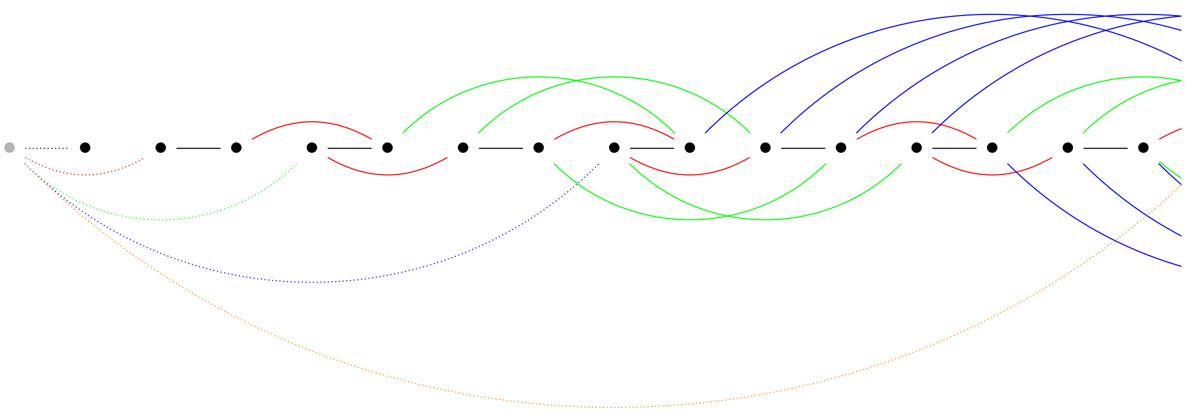

FIGURE 1. The Steenrod module structure of $H^*\mathbb{RP}^\infty_{-1}$ with Sq^1, Sq^2, Sq^4, Sq^8, and Sq^{16} drawn.

This module is visibly highly periodic. In order to explore this, we set $\mathbb{RP}^n_k$ to be the finite complex $\mathbb{RP}^n/\mathbb{RP}^{k-1}$ — i.e., the part of $\mathbb{RP}^\infty$ whose cells lie in dimensions k through n. These complexes come with cofiber sequences which interrelate them; for $0 \le k \le m < n$, there is the sequence

$$\mathbb{RP}^m_k \to \mathbb{RP}^n_k \to \mathbb{RP}^n_{m+1}.$$

If we consider the complexes $\mathbb{RP}^n_{n-2}$, then up to shift we have the following four Steenrod modules:

FIGURE 2. The Steenrod module structures for $\mathbb{RP}^{3+4n}_{1+4n}$, $\mathbb{RP}^{4+4n}_{2+4n}$, $\mathbb{RP}^{5+4n}_{3+4n}$, and $\mathbb{RP}^{6+4n}_{4+4n}$.

This sequence of four modules repeats itself as n varies, and this is one precise way that we can describe the periodicity seen. Another observation is that we can formally extend these patterns to the left — that is, there could be some fictional space with negative dimensional cells continuing the picture above. (For instance, we have drawn in the -1 cell at the extreme left of the diagram. It appears to be highly interesting, evidenced by the number of squaring operations it simultaneously supports.) Atiyah and James have made concrete both of these observations:

Theorem (Atiyah)**.** *There is a homeomorphism* $\mathrm{Thom}(\mathbb{RP}^n;\mathscr{L}) \cong \mathbb{RP}^{n+1}$*, where* $\mathscr{L}$ *denotes the tautological line bundle on* $\mathbb{RP}^n$*. More generally,*

$$\mathrm{Thom}(\mathbb{RP}^n; k\mathscr{L}) \cong \mathbb{RP}^{n+k}_k.$$

In particular, taking Thom spectra and setting k to be a negative integer gives complexes with negative dimensional cells.[1]

Theorem (James)**.** *There is a function* $f(n-k)$ *taking values in powers of* 2 *such that*

$$\mathbb{RP}^n_k \simeq \Sigma^{-f(n-k)} \mathbb{RP}^{n+f(n-k)}_{k+f(n-k)}.$$ [2]

James's theorem is actually much stronger than what has been described so far: not only is there a periodicity of Steenrod modules, but actually one of cell complexes. The *primary attaching map* of a cell in a $\mathbb{RP}^\infty$ is the value of

$$S^n \xrightarrow{i_n} \mathbb{RP}^n \longleftarrow \mathbb{RP}^{n-k} \longrightarrow S^{n-k}$$

for as large a value of k as the map i_n can be lifted across the inclusion $\mathbb{RP}^{n-k} \to \mathbb{RP}^n$. Here is a diagram of the primary attaching maps in $\mathbb{RP}^\infty$:

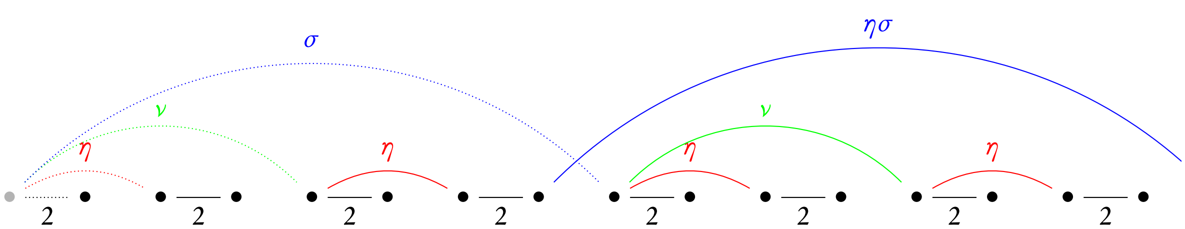

FIGURE 3. The primary attaching maps in $\mathbb{RP}^\infty_{-1}$.

It is an understatement to say that these maps are not trivial to calculate. Doing so was one of the first great feats of modern algebraic topology, as you can actually deduce Adams's Hopf invariant one theorem from knowledge of the primary attaching maps. Namely, the dimensions n in which S^n supports an H-space structure are exactly those n for which the top cell of $\mathbb{RP}^n_0$ is disconnected. In turn, this means that the Hopf invariant 1 classes are exactly the dashed lines extending from the (-1)-dimensional cell, named by Adams's celebrated theorem: 2, η, ν, and σ.

Our next observation is that our four named submodules have a horizontal duality among them: $\mathbb{RP}^4_2$ and $\mathbb{RP}^5_3$, for instance. There is another general statement explaining this:

Theorem (Atiyah)**.** *There is an equivalence* $D\mathbb{RP}^{n-1}_{-k} \simeq \Sigma\mathbb{RP}^{k-1}_{-n}$.

Using this equivalence, plus the fact that the 0-cell splits off of $\mathbb{RP}^n_0 \simeq \Sigma^\infty_+ \mathbb{RP}^n$, we can form the following system:

$$\begin{array}{ccccccccc}
\cdots & = & \mathbb{S}^0 & = & \mathbb{S}^0 & = & \mathbb{S}^0 & = & \mathbb{S}^0 & = & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & \mathbb{RP}^{n-2}_0 & \longrightarrow & \mathbb{RP}^{n-1}_0 & \longrightarrow & \mathbb{RP}^n_0 & \longrightarrow & \mathbb{RP}^{n+1}_0 & \longrightarrow & \cdots.
\end{array}$$

[1]This is proven by thinking about our excellent model for projective space: it is the projectivization of a vector space.

[2]This is proven by calculating the K-groups of $\mathbb{RP}^\infty$: they're all torsion and of predictable order, so $\mathscr{L}$ is of predictable order too.

An application of Atiyah's duality theorem turns this diagram into the following:

$$\begin{array}{ccccccccc}
\cdots & = & \mathbb{S}^{-1} & = & \mathbb{S}^{-1} & = & \mathbb{S}^{-1} & = & \mathbb{S}^{-1} & = & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longleftarrow & \mathbb{RP}^{-1}_{-n+1} & \longleftarrow & \mathbb{RP}^{-1}_{-n} & \longleftarrow & \mathbb{RP}^{-1}_{-n-1} & \longleftarrow & \mathbb{RP}^{-1}_{-n-2} & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longleftarrow & \mathbb{RP}^{\infty}_{-n+1} & \longleftarrow & \mathbb{RP}^{\infty}_{-n} & \longleftarrow & \mathbb{RP}^{\infty}_{-n-1} & \longleftarrow & \mathbb{RP}^{\infty}_{-n-2} & \longleftarrow & \cdots,
\end{array}$$

which is equivalent data to a map $\mathbb{S}^{-1} \to \mathbb{RP}^{\infty}_{-\infty} := \lim_k \mathbb{RP}^{\infty}_{-k}$.

Theorem (Lin). *This map is a 2-adic equivalence.*

Words cannot describe how bizarre this theorem is. It's proven not conceptually, but by considering the Adams spectral sequence for $\mathbb{RP}^{\infty}_{-\infty}$: as you extend the pattern of $H^*\mathbb{RP}^\infty$ further and further down, the Adams E^2-page begins to look more and more like a shift of that of the sphere. The associated spectral sequence is even more bizarre: it takes the ouroborean form

$$E^1_{s,t} = \pi_{s+t}\mathbb{S}^\wedge_2 \Rightarrow \pi_{s+1}\mathbb{S}^\wedge_2.$$

Any class $\alpha \in \pi_*\mathbb{S}$ must be represented by some element (plus indeterminacy) on the E^1-page, and that element is called its *root invariant* $R(\alpha)$. Many interesting basic calculations of the root invariant are known, and it appears to have "redshifting properties" in the language of chromatic homotopy theory. Understanding the cellular structure of $\mathbb{RP}^\infty$ would also have a lot to say about the behavior of this spectral sequence and hence of the root invariant.

As a final example of how important $\mathbb{RP}^\infty$ is, consider the following: there is a map of fiber sequences

$$\begin{array}{ccccccc}
\Omega^n S^n & \xrightarrow{E} & \Omega^{n+1} S^{n+1} & \xrightarrow{H} & \Omega^{n+1} S^{2n+1} & \xrightarrow{P} & \Omega^{n-1} S^n \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
Q\mathbb{RP}^{n-1} & \longrightarrow & Q\mathbb{RP}^n & \longrightarrow & QS^n & \longrightarrow & Q\Sigma\mathbb{RP}^{n-1}.
\end{array}$$

This induces a map from the EHP spectral sequence (which itself ties the computation of stable homotopy groups of spheres to their unstable homotopy groups) to the Atiyah–Hirzebruch spectral sequence

$$\widetilde{H}_*(\mathbb{RP}^\infty; \pi_*\mathbb{S}) \Rightarrow \pi_*\Sigma^\infty\mathbb{RP}^\infty.$$

This induces an interesting array of periodicities in the EHP spectral sequence, and knowledge of the cellular structure of $\mathbb{RP}^\infty$ helps produce differentials in the EHP spectral sequence. It's also worth remarking that this is a truncation of the spectral sequence considered in Lin's theorem.[3]

2. The cell structure of Thom spectra

[4]Having hopefully convinced you that the cellular structure of $\mathbb{RP}^\infty$ is highly interesting, I'd like to pose the following question — which you should now agree is so interesting that it cannot have a good answer:

Question. *Given all these interrelationships among the $\mathbb{RP}^n_k$, can we recursively determine their cellular structures?*

Here is one attempt at making this more precise:

(1) Since the 0-cell of $\mathbb{RP}^{n+1}_0$ splits off, we can easily form $\mathbb{RP}^{n+1}_0 \simeq \mathbb{S}^0 \vee \mathbb{RP}^{n+1}_1$.
(2) Atiyah's theorem says $\mathrm{Thom}(\mathbb{RP}^n; \mathscr{L}) \cong \mathbb{RP}^{n+1}_1$.
(3) This spectrum $\mathbb{RP}^{n+1}_1$ must be a suspension spectrum; the bundle $\mathscr{L}$ on $\mathbb{RP}^n$ must extend to $\mathbb{RP}^{n+1}$; and the truncations $\mathbb{RP}^{n+1}_k$ must satisfy James's theorem.

[3]Almost all of these statements have analogues for BG where G is finite cyclic or finite symmetric (like $G = \mathbb{Z}/2 = \Sigma_2$ gives $\mathbb{RP}^\infty$). There is also an interesting question of which results extend to describe a profinite version of $\mathbb{CP}^\infty \simeq BS^1$.

[4]Much of the rest of this was told to me by Tyler Lawson, in `http://mathoverflow.net/a/138228/1094`. It is my fault if I've garbled it.

The second point is the most interesting among them.

Question. *More generally: given a virtual vector bundle V over a CW complex X, what information is needed to determine the cellular structure of the spectrum* $\mathrm{Thom}(X;V)$*?*

In supreme generality, a Thom space is constructed as follows: let G be a group with a map $G \to \operatorname{Aut} S^n$, and let P be a principal G-bundle on a space X, classified by a map $\alpha : X \to BG$. The Thom space of this bundle is defined by the fiberwise smash: $\mathrm{Thom}(X;\alpha) := P_+ \wedge_G S^n$. We produce a cellular structure on the Thom space as follows: let $X^{(*)}$ be a cellular filtration of X. Each n-cell $D^n \to X$ can be lifted to a map $D^n \to P$, and because P is principal this extends to a G-cell $G \times D^n \to P$. Altogether, this gives P the structure of a G-cell complex. Because smashing (over G) commutes with colimits, this induces a cell structure on the Thom space, where the cells match those of X, but whose attaching maps have been twisted by the structure of P and the action of G.

Because passing to suspension spectra also commutes with colimits, this discussion also produces stable cell structures.[5] Stably, however, we can make the interesting comparison we are seeking: let $\mathbb{S}[G] = \Sigma^\infty_+ G$ be the stable group-algebra on G, so that $\mathbb{S}[G]$ acts on $Y = \Sigma^\infty_+ P$. Then, let $\mathbb{S}$ denote the stable sphere with trivial $\mathbb{S}[G]$-action, $S^0 = \Sigma^{-n}\Sigma^\infty S^n$ the stable sphere with natural $\mathbb{S}[G]$-action. There are the following two equalities:

$$\mathrm{Thom}(X;\alpha) = Y \wedge_{\mathbb{S}[G]} S^0, \qquad \Sigma^\infty_+ X = Y \wedge_{\mathbb{S}[G]} \mathbb{S}.$$

The filtrations on Y itself and these two spectra give rise to the following three spectral sequences:

(1) Taking homotopy groups of the cellular filtration of Y gives a chain complex $C_d = \pi_*(\Sigma^{-d} Y / Y^{d-1})$ of free modules over $\pi_*\mathbb{S}[G]$, and so a spectral sequence $H_* C_* \Rightarrow \pi_* Y$.

(2) Tensoring the filtration with $\mathbb{S}$ over $\mathbb{S}[G]$ tensors the complex C with $\pi_*\mathbb{S}$ over $\pi_*\mathbb{S}[G]$ (which acts trivially). This chain complex becomes the cellular complex for $\Sigma^\infty_+ X$, and its homology becomes the Atiyah–Hirzebruch spectral sequence for its stable homotopy:

$$\left\{ \begin{array}{c} E^1 = C_*(X;\pi_*\mathbb{S}), \\ E^2 = H_*(X;\pi_*\mathbb{S}) \end{array} \right\} \Rightarrow \pi_* \Sigma^\infty_+ X.$$

(3) Tensoring the filtration with S^0 over $\mathbb{S}[G]$ tensors the complex C with $\pi_*\mathbb{S}$ over $\pi_*\mathbb{S}[G]$ (which acts nontrivially). This chain complex becomes the cellular complex for $\mathrm{Thom}(X;\alpha)$, and its homology becomes the Atiyah–Hirzebruch spectral sequence for its stable homotopy:

$$\left\{ \begin{array}{c} E^1 = C_*(X;\pi_*\mathbb{S}) \\ E^2 = H_*(\mathrm{Thom}(X;\alpha);\pi_* S) \end{array} \right\} \Rightarrow \pi_* \mathrm{Thom}(X;\alpha).$$

Our goal is to understand how the differentials in the second spectral sequence can be modified to give those in the third. For a simple example of such a theorem, orientability can be so interpreted: an $H\mathbb{Z}$-orientation of α gives an identification $H_*(X;\pi_* S) \cong H_*(\mathrm{Thom}(X;\alpha);\pi_* S)$, i.e., it shows that their d^1-differentials agree.

3. Examples

3.1. **$G = \operatorname{Aut} S^n$, $X = S^k$.** In this case, we can give a complete description of the possible Thom spectra. Giving X the (unbased) cell structure of one 0-cell and one k-cell, the principal bundle P is then formed by attaching $G \times D^k$ to G along a G-equivariant map $G \times S^{k-1} \to G$, i.e., a class $\alpha \in \pi_{k-1} \operatorname{Aut} S^n$. In the case $k > 1$,[6] the space $\mathrm{Thom}(X;\alpha)$ is given as the cofiber

$$S^{n+k-1} \xrightarrow{\alpha} S^n \to \mathrm{Thom}(X;\alpha).$$

[5] Slightly more than this can be said: the set of "allowable twists" of the attaching maps in the twisted cell structures are a torsor for the J-groups of X. This carries some useful intuition: for instance, whatever twists are performed should also be able to be un-performed.

[6] In the case $k = 1$, some care has to be taken, since then we're using the disconnected space $S^0 = \{+, -\}$. You end up multiplying by the difference $1 - \alpha$, corresponding to the values of the points $+$ and $-$.

3.2. $G = O(1)$ **and** $G = U(1)$**.** In these cases, G is both a sphere and an H-space, with Hopf maps $h = 2$ and $h = \eta$ respectively. The homotopy-theoretic group algebra can then be calculated to be $\pi_*(\mathbb{S}[G]) = \pi_*[d]/(d^2 - hd)$. On homotopy groups, the two tensor products $\mathbb{S} \wedge_{\mathbb{S}[G]} -$ and $S^0 \wedge_{\mathbb{S}[G]} -$ act by sending d to either 0 or h, respectively.

It's simple to determine the lowest order differentials in the spectral sequences for BG, which in the case of $G = O(1)$ gives $\mathbb{R}\mathrm{P}^\infty$. The G-complex fibering over the space $X = BG$ is $Y = \Sigma^\infty_+ EG$, and the corresponding complex C is an acyclic resolution of $\mathbb{S}$ by free G-modules. In fact, a model for C exists with one cell in every dimension divisible by $(1 + \dim G)$ and whose differentials are given by alternating multiplication by d and by $d - h$. We deduce that the lowest order differentials on the suspension spectrum induce alternating multiplication by 0 and by $-h$, and on the Thom spectrum by h and by 0.

3.3. **Higher order complexes.** The formula we've indicated for describing d_1 in the case of these G is the single easy step; all the higher differentials are vastly more complex, as we have to study equivariant cells which have been lifted across multiple filtration layers. For instance, here is the Atiyah–Hirzebruch spectral sequence for $\mathbb{R}\mathrm{P}^4_1$:

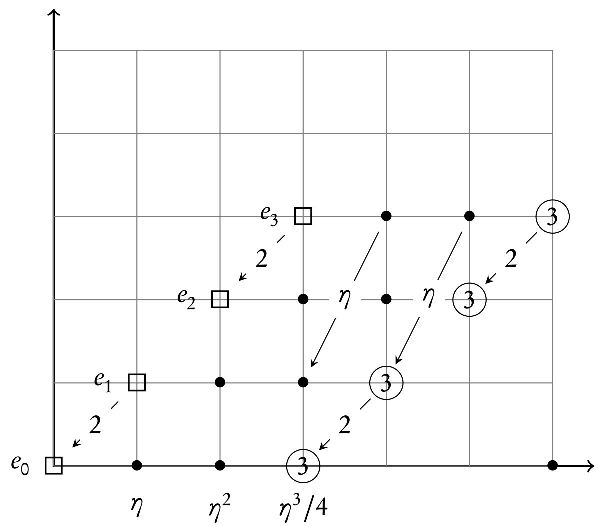

FIGURE 4. Spectral sequences for $\Sigma^{-1}\mathbb{R}\mathrm{P}^3_1$, $\mathrm{Thom}(\mathbb{R}\mathrm{P}^3; 0) \simeq (\mathbb{R}\mathrm{P}^3_1)_+$, and $\mathrm{Thom}(\mathbb{R}\mathrm{P}^3, \mathscr{L} - 1) \simeq \Sigma^{-1}\mathbb{R}\mathrm{P}^4_1$.

The two main features of this picture that we've explained so far are the the Atiyah–Hirzebruch spectral sequence for $\mathbb{R}\mathrm{P}^3_1$ includes into the bottom three rows, and that the d_1-differential on the top row is determined by some Tor formula.

However, where this differential $d_2(\eta e_3) = \eta^2 e_1$ comes from (in terms of this story) is something of a mystery.[7] In other contexts, similar questions are addressed by thinking about Massey products (e.g., $\eta^2 = \langle 2, \eta, 2 \rangle$ seems potentially relevant), but it's unclear how to adapt that to this situation.[8,9]

[7] It is worth pointing out that this differential is part of the cell structure of $\mathbb{R}\mathrm{P}^4_1$ but is *not* a primary attaching map. The primary attaching maps corresponding precisely to the action of the differentials on the classes e_i themselves, i.e., the first nonzero differential off of a given row.

[8] Perhaps this differential can also be thought of as $2e_2 = \langle \eta, 2, e_0 \rangle$.

[9] Also worth mentioning: the existence of these recurrences tell us something about the infinite loopspace structure of $b\mathrm{gl}_1\mathbb{S}$. For instance, for $\eta \in \pi_2 b\mathrm{gl}_1\mathbb{S}$, there the existence of $\mathbb{C}\mathrm{P}^2$ forces relation $\eta \cdot \eta = 0$.