TANGENT SPACES OF CERTAIN SPECTRA

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ABSTRACT. We outline a construction in derived algebraic geometry which produces elements of the $K(n)$-local Picard group. As an example, we produce a new model of a spectrum called the determinantal sphere, and as time permits we discuss newly indicated patterns in $K(n)$-local homotopy theory.

1. K-LOCAL INVERTIBLE SPECTRA

Throughout, fix a prime p (often it is useful for p to be odd, or even "large") and a finite positive integer height n. We in chromatic homotopy theory are very interested in a certain sequence of extraordinary complex-oriented homology theories called Morava K-theories. There is one such K-theory for each choice of n and p, and it is valued in graded vector spaces over the graded field $K_*(\mathrm{pt}) = \mathbb{F}_{p^n}[v_n^{\pm}]$. One of our primary goals is to understand what these functors are telling us about $\mathsf{Spectra}$. The theory of localization shows us that these Morava K-theories aren't really telling us about $\mathsf{Spectra}$ per se, but rather about the full subcategory of K-local spectra, written $\mathsf{Spectra}_K$. This subcategory comes with a reflector L_K, which has the properties that $K_*(X) = K_*(L_K X)$ and that K_* reflects isomorphisms in $\mathsf{Spectra}_K$.

Each of these three categories has a symmetric monoidal product: $\mathsf{Spectra}$ has a smash product $\wedge$, $\mathsf{VectorSpaces}_{K_*}$ has a tensor product $\otimes$, and $\mathsf{Spectra}_K$ has a localized smash product given by the formula $X \hat{\wedge} Y = L_K(X \wedge Y)$. Because K_* is a graded field, the homology theory K_* has Künneth isomorphisms, meaning that it is a monoidal functor, and by definition of $\hat{\wedge}$, L_K is also monoidal. So, in our quest to understand K_* and $\mathsf{Spectra}_K$, it is profitable to consider what information we can glean from these products.

One invariant of these categories determined by this data is their subcategories of invertible objects; after replacing objects with their isomorphism classes, these are called the "Picard groups" of these categories. Using the monoidalness of our various functors, we record what we have so far in a diagram:

$$\begin{tikzcd}
\mathsf{Spectra} \arrow[r, "L_K"] \arrow[rr, bend left, "K_*"] & \mathsf{Spectra}_K \arrow[r, "K_*"] & \mathsf{VectorSpaces}_{K_*}. \\
\mathrm{Pic} \arrow[r] \arrow[u] & \mathrm{Pic}_n \arrow[r] \arrow[u] & \mathsf{Lines}_{K_*} \arrow[u]
\end{tikzcd}$$

The Picard subcategory of $\mathsf{VectorSpaces}_{K_*}$ is easy to identify: the tensor product satisfies $\dim(V \otimes W) = \dim V \cdot \dim W$, and so the invertible vector spaces are those which are 1-dimensional, or which are "lines". The Picard group Pic of $\mathsf{Spectra}$ is easy enough to *describe*: it is $\mathbb{Z}$, generated by $\mathbb{S}^1$ the 1-sphere.[1] The Picard group Pic_n of $\mathsf{Spectra}_K$ is extremely complex (and hence interesting); essentially the only quantitative thing we know about it in general is that it is an extension of a finite group by a profinite p-group. There are explicit computations for $n = 1$ and for $n = 2$ with $p \geq 3$, but otherwise it remains very mysterious. However, we do at least have one powerful tool at our disposal to identify when a spectrum is K-locally invertible: a theorem of Hopkins, Mahowald, and Sadofsky asserts that the right-hand square above is a pullback. Since the vertical arrows are inclusions, we may translate this theorem as asserting that a spectrum X is K-locally invertible exactly when $K_*(X)$ is a line.

If Pic is isomorphic to $\mathbb{Z}$, and $\mathbb{Z}$ isn't a profinite p-group, then obviously there are new elements in Pic_n. What do they look like? Here's one cute and easy new family — start by considering the following horizontal system of cofiber sequences:

[1]Making this identification is harder that it sounds; the proof I've read requires knowing the degree-wise finite generation of the stable homotopy groups of spheres.

$$
\begin{tikzcd}
\cdots \arrow[r] & \mathbb{S}^{-1} \arrow[r] \arrow[d, "p^j"] & \mathbb{S}^{-1} \arrow[r] \arrow[d, "p^{j+1}"] & \cdots \arrow[r] & \mathbb{S}^{-1} \arrow[d] \\
\cdots \arrow[r] & \mathbb{S}^{-1} \arrow[r, "p"] \arrow[d] & \mathbb{S}^{-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathbb{S}^{-1}[p^{-1}] \arrow[d] \\
\cdots \arrow[r] & M^0(p^j) \arrow[r] & M^0(p^{j+1}) \arrow[r] & \cdots \arrow[r] & M^0(p^\infty).
\end{tikzcd}
$$

The colimit, pictured on the far right, is also a cofiber sequence. The topmost object is the colimit of a sequence of identity morphisms, so is simply $\mathbb{S}^{-1}$. The middle object is the colimit along iterates of the map p, so is, by definition, the spectrum $\mathbb{S}^{-1}[p^{-1}]$ with the p-self-map inverted. Lastly, the spectrum on the bottom does not have a familiar name, so we call it $M^0(p^\infty)$.

Working K-locally, however, we can see that the middle spectrum is contractible: the map p on $\mathbb{S}^{-1}[p^{-1}]$ is exactly multiplication by p in K_*-homology, but since the coefficient ring K_* is of characteristic p, this is the zero map. On the other hand, p is required to be invertible, which can only mean $K_*\mathbb{S}^{-1}[p^{-1}] = 0$. In turn, this means that the going-around map $M^0(p^\infty) \to \mathbb{S}^0$ is a K-local equivalence, and so $M^0(p^\infty)$ is an invertible spectrum, albeit not a very interesting one.[2]

A different way of at least detecting that $M^0(p^\infty)$ is K-locally invertible is to apply K-homology to the bottom row: each object in the sequence becomes a 2-dimensional K_*-vector space, and each map from one to the next is $- \cdot p = - \cdot 0$ on the (-1)-graded piece and the identity on the 0-graded piece — this is exactly what the diagram of cofiber sequences is recording. Hence, the homology of the colimit is a K_*-line, and the HMS lemma assures us we have an invertible spectrum.

$$
\begin{tikzcd}
\cdots \arrow[r] & K_*M^0(p^j) \arrow[r] & K_*M^0(p^{j+1}) \arrow[r] & \cdots \arrow[r] & K_*M^0(p^\infty) \\
\cdots \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \cdots \arrow[r] & \bullet \\
 & \bullet & \bullet & &
\end{tikzcd}
$$

This suggests a way we can modify this construction: if we insert other maps which are K_*-homology isomorphisms, then we will not harm this proof that the colimit is an invertible spectrum. In particular, each spectrum $M^0(p^j)$ is type 1 and admits an Adams v_1-self-map $v_1^{p^{j-1}} : M^0(p^j) \to M^{-|v_1|p^{j-1}}(p^j)$. With these maps in hand, select your favorite p-adic integer $a_\infty = \sum_{j=0}^\infty c_j p^j$ with $0 \le c_j < p$, and construct the system:

$$
M^0(p) \to \cdots \to M^{-|v_1|a_{j-1}}(p^j) \to M^{-|v_1|a_{j-1}}(p^{j+1}) \xrightarrow{v_1^{p^j c_j}} M^{-|v_1|a_j}(p^{j+1}) \to M^{-|v_1|a_j}(p^{j+2}) \to \cdots \to \mathbb{S}^{-|v_1|a_\infty}.
$$

Hopkins, Mahowald, and Sadofsky go on to check that this assignment $\mathbb{Z}_p^\wedge \to \mathrm{Pic}_1$ is an injective, continuous homomorphism, and for $p > 2$ its cosets are represented by $\mathbb{S}^1, \ldots, \mathbb{S}^{|v_1|}$.

Unfortunately, this is essentially the only sample family of exotic invertible spectra we know.[3] To produce other examples, we at least now know what to look for: line bundles. Now, my favorite sort of vector bundles (for the purposes of this talk) are tangent bundles, and in particular when looking at a smooth point on a 1-dimensional variety, the stalk of the tangent bundle there is a 1-dimensional free module. There are certain spectra which chromatic homotopy theory tells us to think of in analogy to 1-dimensional formal affine lines, like $\Sigma^\infty_+ \mathbb{C}\mathrm{P}^\infty$, and so one thing we could hope for is a dashed arrow completing the following commuting square:

$$
\begin{tikzcd}
\{\text{appropriate spectra}\} \arrow[r, dotted, "T_\eta"] \arrow[d, "K_*"'] & \mathrm{Pic}_n \arrow[d, "K_*"] \\
\left\{ \begin{array}{c} \text{formal affine varieties} \\ \text{of dimension 1} \end{array} \right\} \arrow[r, "T_0"] & \mathsf{VectorSpaces}_{K_*}
\end{tikzcd}
$$

This is our goal for the rest of the talk.

[2]This is a homotopical version of the statement that the p-primary part of the circle group S^1 is exactly the p-Prüfer group $\mathbb{Z}/p^\infty = \mathrm{colim}_j \mathbb{Z}/p^j$.

[3]This construction can be re-done using generalized Moore spectra to give a similar inclusion $\mathbb{Z}_p^\wedge \to \mathrm{Pic}_n$ for $n > 1$.

2. Coalgebras

So, what we want is some spectrum-level operation which mimics the construction of the algebro-geometric tangent space. The primary obstacle to realizing this dream is a variance issue: classical algebraic geometry is done with rings, but we are concerned with covariant K-homology. So, while most typically investigate the formal scheme $X_K = \operatorname{Spf} K^*X$, we should work out how to use K_*X instead.

As a stepping stone, let's start by thinking about spaces X such that K^*X is finite as a K^*-module, i.e., X_K is a finite scheme.[4] In this case, K_*X is simple to describe: it is the K_*-linear vector space dual of K^*X. Accordingly, where K^*X had the structure of a K^*-algebra, K_*X has the structure of a K_*-coalgebra. You can quickly check that the three categories of finite K^*-algebras, finite K_*-coalgebras, and finite schemes are all (co/contravariantly) equivalent. Diagrammatically, here are the functors available to us:

$$\begin{tikzcd} \mathsf{FiniteCoalgebras} \arrow[r, shift left, "(-)^\vee"] \arrow[rr, bend left, "\mathrm{Sch}"] & \mathsf{FiniteAlgebras} \arrow[l, shift left, "(-)^\vee"] \arrow[r, shift left, "\mathrm{Spec}"] & \mathsf{FiniteSchemes} \arrow[l, shift left, "\mathcal{O}"] \end{tikzcd}$$

For a test ring T, the functor of interest Sch is given by the formula

$$(\mathrm{Sch}\, C)(T) = \left\{ u \in C \otimes T \;\middle|\; \begin{array}{c} \Delta u = u \otimes u \in (C \otimes T) \otimes_T (C \otimes T) \\ \varepsilon u = 1 \in T \end{array} \right\}.$$

Now, let's approach the infinite case. It is no longer the case that we can get away with naively applying linear algebraic duality to turn algebras into coalgebras, because...

(1) ... the double-dual of an infinite-dimensional vector space is not isomorphic to the original.
(2) ... the inequality $(A \otimes A)^\vee \not\cong A^\vee \otimes A^\vee$ prevents us from dualizing the multiplication on a ring to a comultiplication on its dual.

Instead, there is a structure theorem for coalgebras which indicates what is happening: every finite dimensional vector subspace of a coalgebra can be finitely enlarged to a finite dimensional subcoalgebra — and, as a corollary, every coalgebra is ind-finite.

In actuality, this ind-finiteness is

In actuality, this ind-finiteness has been in the picture of algebraic-geometry-in-algebraic-topology all along: the formula $X_K = \operatorname{Spf} K^*X$ means that we should consider the directed system of compact subspaces X_α of X, then set $X_K = \operatorname{colim}_\alpha \operatorname{Spec} K^*X_\alpha$. This is what we mean when we say we're studying formal schemes: formal schemes are ind-finite schemes, meaning that they arise from pro-finite systems of rings or from ind-finite systems of coalgebras. And, in fact, X_K can be modeled as $X_K = \operatorname{Sch} K_*X$ for arbitrary X. This extends our diagram like so:

$$\begin{tikzcd} \mathsf{FiniteCoalgebras} \arrow[r, shift left, "(-)^\vee"] \arrow[rr, bend left, "\mathrm{Sch}"] \arrow[d] & \mathsf{FiniteAlgebras} \arrow[l, shift left, "(-)^\vee"] \arrow[r, shift left, "\mathrm{Spec}"] \arrow[d] & \mathsf{FiniteSchemes} \arrow[l, shift left, "\mathcal{O}"] \arrow[d] \\ \mathsf{Coalgebras} \arrow[r, shift left] \arrow[rr, bend right, "\mathrm{Sch}"'] & \mathsf{ProfiniteAlgebras} \arrow[l, shift left] \arrow[r, shift left, "\mathrm{Spf}"] & \mathsf{FormalSchemes} \arrow[l, shift left] \end{tikzcd}$$

So, this new setting is very nearly just the old setting with one extra duality thrown in to the formula, and hence we can translate over any construction we want by dualizing appropriately. For a point $x : K^*X \to K^*$ in the affine scheme X_K, we would build the cotangent space by taking the corresponding ideal $I = \ker(K^*X \to K)$ and considering the quotient $T_x X_K = \operatorname{coker}(I \otimes_{K^*X} I \to I)$. In the new setting, a point corresponds to a map $x : K_* \to K_*X$, from which we build the (K_*X)-comodule $M = \operatorname{coker}(K_* \to K_*X)$, and then the tangent space

[4]This is a pretty interesting case already, since it encompasses all manifolds, for instance.

$T_x X_K = \ker(M \to M \square_{K_* X} M)$. Here $\square$ is the cotensor product, defined by dualizing the diagram for a tensor product below:

$$\begin{tikzcd} I \otimes_A J & I \otimes_k J \arrow[l] & & M \square_C N \arrow[r] \arrow[d] & M \otimes_k N \arrow[d, "1 \otimes \psi_N"] \\ I \otimes_k J \arrow[u] & I \otimes_k A \otimes_k J, \arrow[l, "1 \otimes \alpha_J"'] \arrow[u, "\alpha_I \otimes 1"'] & & M \otimes_k N \arrow[r, "\psi_M \otimes 1"] & M \otimes_k C \otimes_k N. \end{tikzcd}$$

This is all supposed to indicate what to do on the level of spectra: for an appropriate pointed coalgebra spectrum $\eta : \mathbb{S} \to X$, we build the intermediate spectrum $M = \operatorname{cofib}(\mathbb{S} \to X)$ and set $T_\eta X = \operatorname{fib}(M \to M \square_X M)$. The remaining challenge is to decide what the cotensor square $M \square_X M$ should mean. A quick calculation in K-homology will show that the pullback square above is not sufficient, essentially because the fiber of spectra will also detect the cokernel of the map $K_* M \to K_*(M \square_X M)$ in odd degrees. Instead we should consider the derived cotensor product, defined by the totalization of the two-sided cobar construction $M \square_X M = \operatorname{Tot} B^*(M; X; M)$.[5] This comes with a coskeletal filtration spectral sequence

$$E^2_{*,*} = \operatorname{Cotor}^{K_* X}_{*,*}(K_* M, K_* M) \Rightarrow K_*(M \square_X M),$$

which has bad convergence properties in general, but in the case that $X_K \cong \hat{\mathbb{A}}^1$ the following can be shown:

(1) $\operatorname{Cotor}^{K_* X}_{*,*}(K_* M, K_* M)$ vanishes for nonzero Cotor-degree.
(2) The spectral sequence collapses at the E^2-page.
(3) It converges strongly, with no extension problems, to $K_*(M \square_X M)$.

In all, this means that applying K_*-homology to all these spectra results in the coalgebraic version of the construction of the tangent space given above — and hence $T_\eta X = \operatorname{fib}(M \to M \square_X M)$ is a K-locally invertible spectrum.

3. Applications

Having assembled this technology, it's time to try out some examples. As mentioned in the introduction, one such spectrum X is $X = \Sigma^\infty_+ \mathbb{C}\mathrm{P}^\infty$, pointed by the inclusion of the disjoint basepoint. Appealing to $H\mathbb{Z}$-homology rather than K-homology, we see that $H\mathbb{Z}_* T_+ \Sigma^\infty_+ \mathbb{C}\mathrm{P}^\infty$ is a single $\mathbb{Z}$ concentrated in degree 2, and so it must be the case that $T_+ \Sigma^\infty_+ \mathbb{C}\mathrm{P}^\infty \simeq \mathbb{S}^2$.

That's kind of cool, but not very exciting — after all, the whole idea was to get new elements of Pic_n, and we already knew about $\mathbb{S}^2$. Toward that end, here's a more exotic example of a suitable spectrum X: Ravenel and Wilson show that $K_* \underline{H\mathbb{Z}/p^\infty}_q$ is the qth exterior power of $K_* \underline{H\mathbb{Z}/p^\infty}_1$ as a Hopf algebra. This means that each $\underline{H\mathbb{Z}/p^\infty}_q$ begets a formal group of dimension $\binom{n-1}{q-1}$, hence for $q = n$ we find a formal variety of dimension 1:

$$\operatorname{Sch} K_* \underline{H\mathbb{Z}/p^\infty}_n \cong \hat{\mathbb{A}}^1.$$

Our tangent space machine then immediately gives us some element of Pic_n, and it would be helpful to know more about it — in particular, it would be nice to know if it is also a standard sphere or something new.

One thing you may have noticed is that Lines_{K_*} is not a very interesting category: it has one isomorphism class, for instance. It turns out that there is a refinement of the map $\operatorname{Pic}_n \to \mathsf{VectorSpaces}_{K_*}$ through the category of line bundles over a certain formal variety LT_n, equivariant against the action of a certain p-adic analytic group $\mathbb{S}_n$:

$$\operatorname{Pic}_n \xrightarrow{\mathscr{K}} \mathsf{LineBundles}_{\mathbb{S}_n}(LT_n) \xrightarrow{i^*} \mathsf{VectorSpaces}_{K_*}.$$

Though it's a low bar to clear, this middle category records much more information than Lines_{K_*}. Hopf ring techniques allow us to identify the image of $T_+ \Sigma^\infty_+ \underline{H\mathbb{Z}/p^\infty}_n$ in this middle group: it is the determinantal bundle $\Omega^{n-1}_{LT_n/\mathbb{Z}_p^\wedge}$! This in particular means that it is not a standard sphere, and so we have uncovered a genuinely new invertible spectrum. Other authors have called similar spectra with this property "the determinantal sphere."

Now that we've accomplished our original goal of producing an interesting invertible spectrum, I want to tell you about some other things that come out of these methods. The first, most basic point is that this construction can be iterated, yielding the following diagram of fiber sequences:

[5] Accordingly, X will have to be in some sense a coA_∞-coalgebra spectrum. Whatever this is, suspension spectra definitely have this property.

$$\begin{tikzcd}
C = M^{\square_C 0} \arrow[r] & M = M^{\square_C 1} \arrow[r] & M^{\square_C 2} \arrow[r] & M^{\square_C 3} \arrow[r] & M^{\square_C 4} \arrow[r] & \cdots \\
A_0 = \mathbb{S} \arrow[u] & A_1 = T_\eta X \arrow[u, "\beta"'] & A_2 \arrow[u] & A_3 \arrow[u] & A_4 \arrow[u] & \cdots .
\end{tikzcd}$$

In the case of $X = \Sigma^\infty_+ \mathbb{C}\mathrm{P}^\infty$, this recovers the cellular decomposition of $\mathbb{C}\mathrm{P}^\infty$. In the case of $X = \Sigma^\infty_+ \underline{H\mathbb{Z}/p^\infty}_n$, this recovers something that feels identical to the cellular decomposition of $\mathbb{C}\mathrm{P}^\infty$, but where "cell" is extended to allow attaching maps along arbitrary elements of Pic_n. In particular, one can show that the spectrum D_j is the jth smash power of D_1, as is the case for $\mathbb{C}\mathrm{P}^\infty$.

Given this striking similarity between $\mathbb{C}\mathrm{P}^\infty$ and $\underline{H\mathbb{Z}/p^\infty}_n$, one can ask what other things are lying around that are also similar. For instance, the map $\mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^\infty$ is an element of homotopy called the Bott element β, and we have an analogous map $\beta : T_+\Sigma^\infty_+ \underline{H\mathbb{Z}/p^\infty}_n \to \Sigma^\infty \underline{H\mathbb{Z}/p^\infty}_n$. Moreover, Snaith showed that the spectrum $\Sigma^\infty_+ \mathbb{C}\mathrm{P}^\infty[\beta^{-1}]$ is weakly equivalent to the K-theory spectrum KU, and we can formally construct a Snaith-type determinantal K-theory spectrum $R = \Sigma^\infty_+ \underline{H\mathbb{Z}/p^\infty}_n[\beta^{-1}]$. Westerland has shown a couple of remarkable features of this spectrum:

1. R is weakly equivalent to the fixed point spectrum $E^{hS\mathbb{S}_n}$, where $S\mathbb{S}_n$ is the subgroup of "special" elements of the stabilizer group, i.e., those in the kernel of the determinant.
2. The space $\Omega^\infty R\langle 1\rangle$ supports an analogous map to $BGL_1\mathbb{S}$, yielding a determinantal theory of Thom spectra. In particular, there is a Thom spectrum analogous to MU, which classifies those spectra which are appropriately oriented against $\underline{H\mathbb{Z}/p^\infty}_n$.
3. ...There is even an analogue of the image of J...

Westerland answers a lot of questions and opens a lot of doors, and there are a lot of things left to investigate. Though there are many others, here are two yet-unanswered questions I'd like to leave you with:

1. Analogues of the spaces $\Omega SU(m)$ can be constructed, but analogues of the spaces $BU(m)$ are unknown. What should the analogue of a rank m bundle be, where $\underline{H\mathbb{Z}/p^\infty}_n$ classifies "rank 1 bundles"?
2. The tangent space analysis also applies 2-adically to $\mathbb{R}\mathrm{P}^\infty$, where it's found that $T_+\Sigma^\infty_+ \mathbb{R}\mathrm{P}^\infty \simeq \mathbb{S}^1$. In both the real and complex cases, it is the case that the tangent space agrees with $\Sigma O(1)$ and $\Sigma U(1)$ respectively — that's because these spaces can be realized as bar constructions. Does $\Sigma^{-1} T_+ \Sigma^\infty_+ \underline{H\mathbb{Z}/p^\infty}_n$ have an interpretation as a base case of a Lie group? Does it have an unstable realization, as $O(1)$ and $U(1)$ do? How does the annular tower compare to the bar filtration?