INTRODUCTION TO E–THEORY

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These are notes for a sequence of three lectures delivered at the University of Pittsburgh in June 2015 as part of the workshop *Flavors of Cohomology*. The goal of the lectures is to advertise a family of cohomology theories called Morava E–theories. Though these spectra do not appear on the first page of any textbook in algebraic topology, they arise naturally in a few different contexts. Our initial goal will be to show how they arise from the theory of complex-oriented spectra, which will take us on an extended tour of the role of algebraic geometry in the study of homology theories. Secondly, we will investigate applications suggested by this construction, including the appearance of E–theory in the study of finite spectra and in the classification of homology theories with Künneth isomorphisms. Finally, we will talk about the behavior of the E–local categories and their role in understanding behaviors in the finite stable category.

The notes are meant to be read by a graduate student with a mild background in algebraic topology: someone with some familiarity with the stable category, with extraordinary cohomology theories, and with simplicial methods. We also expect some comfortability with basic constructions in algebraic geometry, but by and large we will only encounter the most polite affine schemes and we won't manipulate them in any serious way.

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1. Day 1: Quillen's theorem

Abstract. For certain ring spectra E, we describe a construction of a very rich algebro-geometric category in which E–homology is valued, called the *context* for E. We also give a tour of the theory of Thom spectra and announce Quillen's description of the context for the Thom spectrum of the complex J–homomorphism.

1.1. **Homology cooperations and their structure.** Let's get right to the task advertised in the abstract: for a ring spectrum E, we're looking to use algebraic geometry to capture as much of the structure of the output of E_* and E^*. Consider first the case $E = H\mathbb{F}_2$ of ordinary mod–2 cohomology, where $H\mathbb{F}_2^*$ is naturally valued in modules for the "Steenrod algebra":

$$\mathscr{A}^* \otimes H\mathbb{F}_2^*(X) \to H\mathbb{F}_2^*(X).$$

This action is very useful, but $\mathscr{A}^*$ has the unfortunate feature of being a highly *noncommutative* ring, which makes it a clumsy object from the perspective of algebraic geometry. However, the Steenrod algebra is actually a Hopf algebra, and its linear dual $\mathscr{A}_*$ is *commutative* and it *coacts* on homology:

$$(H\mathbb{F}_2)_* X \to (H\mathbb{F}_2)_* X \otimes \mathscr{A}_*.$$

A theorem of Milnor gives a concise description of this dual Hopf algebra:

Theorem (Milnor)**.** *There is an isomorphism of rings*

$$\mathscr{A}_* \cong \mathbb{F}_2[\xi_1, \xi_2, \ldots, \xi_n, \ldots]$$

with diagonal

$$\Delta \xi_n = \sum_{j=0}^{n} \xi_j \otimes \xi_{n-j}^{2^j}.$$

This is a very reasonable commutative ring, so that we might hope to leverage algebraic geometry, and Δ is expressed by a very reasonable formula, so we might also hope to express arguments with it slickly.

Stable homotopy theorists are also interested in many other ring spectra E, but to generalize this story away from $H\mathbb{F}_2$ we will need to more carefully identify its cast of characters by names internal to topology. After all, taking E_*–linear duals is unlikely to be well–behaved in general. The dual Steenrod algebra arises as the homotopy of $H\mathbb{F}_2 \wedge H\mathbb{F}_2$ and the diagonal map has the signature

$$\begin{tikzcd} \mathscr{A}_* \arrow[r, "\Delta"] \arrow[d, equal] & \mathscr{A}_* \otimes_{\mathbb{F}_2} \mathscr{A}_* \arrow[d, equal] \\ \pi_*(H\mathbb{F}_2 \wedge H\mathbb{F}_2) \arrow[r] & \pi_*(H\mathbb{F}_2 \wedge H\mathbb{F}_2 \wedge H\mathbb{F}_2). \end{tikzcd}$$

Together with the ring structure and a healthy obsession with simplicial objects, this is clue enough as to what we should be investigating for general E:

$$\mathscr{D}_E(X) := \left\{ \begin{array}{ccccccc} & & & & & & \longrightarrow \\ & & & \longrightarrow & E & \longleftarrow & \\ & \xrightarrow{\eta_L} & E & \longleftarrow & \wedge & \longrightarrow & \\ E & \xleftarrow{\mu} & \wedge & \xrightarrow{\Delta} & E & \longleftarrow & \\ \wedge & \xrightarrow{\eta_R} & E & \longleftarrow & \wedge & \longrightarrow & \cdots \\ X & & \wedge & \longrightarrow & E & \longleftarrow & \\ & & X & & \wedge & \longrightarrow & \\ & & & & X & & \end{array} \right\}.$$

The leftward arrows come from E–multiplication and the rightward arrows come from the unit $\mathbb{S} \to E$.[1]

[1] Incidentally, this cosimplicial ring spectrum has a name: the descent coring for the map $\mathbb{S} \to H\mathbb{F}_2$. In terms of descent theory, if the map $\mathbb{S} \to E$ is "of effective descent", meaning the homotopy limit of this diagram exists and agrees with $\mathbb{S} \wedge X$, then the coskeletal spectral sequence gives a way to compute the homotopy of X, starting from its homology. This is the *E–Adams spectral sequence*.

This object is interesting because of its layers. The homotopy of the 0^{th} level recovers the homology groups E_*X. The maps η_L and η_R from the 0^{th} level to the 1^{st} level give maps

$$E_*X \xrightarrow{E_*\eta_L, E_*\eta_R} (E \wedge E)_*X \xleftarrow{\bigstar} E_*E \otimes_{E_*} E_*X,$$

but in general $\bigstar$ will not be an isomorphism, inhibiting our discovery of a "coaction map". In good cases, however, this can be repaired:

Definition. Take E_*E to be an E_*–module using the left-unit map. We will say that E satisfies **FH**, the **F**latness **H**ypothesis, when the right-unit map $E_* \to E_*E$ is a flat map of E_*–modules.

If E satisfies **FH**, then $\bigstar$ becomes an isomorphism! In fact, iterating this gives an isomorphism

$$\pi_*\mathscr{D}_E(X)[j] = \pi_*(E^{\wedge(j+1)} \wedge X) \xleftarrow{\bigstar} (E_*E)^{\otimes_{E_*} j} \otimes_{E_*} E_*X \cong \pi_*\mathscr{D}_E[1]^{\otimes_{\pi_*\mathscr{D}_E[0]} j} \otimes_{\pi_*\mathscr{D}_E[0]} \pi_*\mathscr{D}_E(X)[0],$$

i.e., the cosimplicial ring $\pi_*\mathscr{D}_E$ is 1–truncated and the module $\pi_*\mathscr{D}_E(X)$ is determined by its 0^{th} level.[2]

Now that I've subjected you to a flurry of "co-"s, I'd like to take some of them back by finally appealing to algebraic geometry.

Definition. E satisfies **CH**, the **C**ommutativity **H**ypothesis, when $\pi_*E^{\wedge j}$ is commutative for all $j \geq 1$.

In the case that E satisfies **CH**, we can study the simplicial scheme

$$\mathscr{M}_E := \operatorname{Spec} \pi_*\mathscr{D}_E,$$

and the cosimplicial object $\pi_*\mathscr{D}_E(X)$ determines a quasicoherent sheaf $\mathscr{M}_E(X)$ over $\mathscr{M}_E$.

Definition. The object $\mathscr{M}_E$ is called the *context* of E. The construction $\mathscr{M}_E(X)$ describes E–homology as a functor

$$E_* : \mathsf{Spaces} \to \mathsf{QCoh}(\mathscr{M}_E).$$

If E satisfies **FH**, $\mathscr{M}_E$ takes values in groupoids.

This is a lot of fancy words for some simple cooperations, but I claim that the conceptual payoff is worth the hassle. For instance, return to the example $E = H\mathbb{F}_2$, so that $\mathscr{M}_E[0] = \operatorname{Spec} \mathbb{F}_2$ is a point and $\mathscr{M}_E[1] = \operatorname{Spec} \mathscr{A}_*$ is the spectrum of the infinite polynomial algebra from before. In order to justify the utility of this language, we should give a geometric description of $\operatorname{Spec} \mathscr{A}_*$. Consider the generating function

$$F(t) = \sum_{j=0}^{\infty} \xi_j x^{2^j}.$$

The composition of two such series F' and F'' in $\mathscr{A}_* \otimes \mathscr{A}_*$ takes the form

$$F'(F''(t)) = \sum_{j=0}^{\infty} \xi'_j \left(\sum_{k=0}^{\infty} \xi''_k t^{2^k} \right)^{2^j} = \sum_{n=0}^{\infty} \left(\sum_{j+k=n} \xi'_j (\xi''_k)^{2^j} \right) x^{2^n},$$

and so power series composition exactly captures the Milnor diagonal. The power series F can be identified as the generic mod–2 power series satisfying the homomorphism property $F(x' + x'') = F(x') + F(x'')$, and so we identify $\operatorname{Spec} \mathscr{A}_*$ with $\underline{\operatorname{Aut}}(\widehat{\mathbb{G}}_a)$.[3] Finally, because $H\mathbb{F}_2$ satisfies **FH**, we learn that

$$\mathscr{M}_{H\mathbb{F}_2} \simeq \operatorname{Spec} \mathbb{F}_2 /\!/ \underline{\operatorname{Aut}}(\widehat{\mathbb{G}}_a).$$

This last line embodies the utility of contexts: starting with this isomorphism, you can unpack that $H\mathbb{F}_2$–homology is valued in $\mathbb{F}_2$–modules with a coaction by a Hopf algebra whose formulas you can write out from memory alone.

[2] We should further emphasize that even when $X = \mathbb{S}$ for a general E the left- and right-units $E_* \to E_*E$ may differ, making **FH** have real content. In the case of $E = H\mathbb{F}_2$, this was not the case, simply because there can't be many maps $\mathbb{F}_2 \to \mathscr{A}_*$ (and so $H\mathbb{F}_2$ automatically satisfies **FH**). For more complicated rings than $\mathbb{F}_2$, all sorts of behavior can arise.

[3] If this notation makes you uncomfortable, check the end of the talk for an explanation of "formal group laws".

1.2. **A general Thom isomorphism.** Today's punchline theorem is about the context $\mathscr{M}_{T(J)}$ of a certain ring spectrum $T(J)$ coming from the theory of Thom spectra. Once I explain the notation, some of you might recognize this as the complex bordism spectrum, but I don't think I can count on that to quickly supply us with the background we need to recognize $\mathscr{M}_{T(J)}$. Instead, I'll construct $T(J)$ from scratch in a way that gives us the statements we need for free. Additionally, this takes us through some interesting tools available to a "modern" homotopy theorist — where "modern" primarily means "geometrically uninclined".

Given an S^n–bundle over a space X

$$S^n \to E \xrightarrow{\xi} X$$

its Thom spectrum[4] $T(\xi)$ is the stable cofiber

$$\Sigma^{-n-1}\Sigma^\infty_+ E \xrightarrow{\Sigma^{-n-1}\Sigma^\infty_+ \xi} \Sigma^{-n-1}\Sigma^\infty_+ X \xrightarrow{\text{cofiber}} T(\xi).$$

Though simple to define, this construction has a number of pleasant properties that indicate it's worth studying:

(1) If ξ is the trivial bundle, then $T(\xi)$ recovers the suspension spectrum $\Sigma^\infty_+ X$ of X. In general, then, a twisted bundle ξ should be thought of as giving a *twisted suspension* $T(\xi)$ of X.

(2) A map of spherical bundles gives rise to a map of Thom spectra, i.e., T is a *functor*

$$T : \mathsf{SphericalBundles} \to \mathsf{Spectra}.$$

In particular, this gives rise to a definition of the Thom spectrum for a stable spherical bundle, by taking the colimit over the maps among the stages.

(3) Given a vector bundle V, we can restrict to the spherical subbundle of unit-length vectors $J(V)$.

(4) Finally, J and T are both *monoidal*. The spherical subbundle $J(V \oplus W)$ is the fiberwise join $J(V) \hat{*} J(W)$ of the individual spherical subbundles, and there is an equivalence $T(\xi \hat{*} \zeta) \simeq T(\xi) \wedge T(\zeta)$.[5]

We will now deduce the Thom isomorphism theorem from these properties. The first foothold is that classifying spaces abound: stable spherical bundles are classified by a space BF and stable vector bundles are classified by BU. The fiberwise join and the direct sum constructions imbue BF and BU with the structure of H–spaces (in fact, E_∞–spaces), compatible with the induced map

$$J : BU \to BF.$$

The second foothold is that the shearing[6] map σ is an equivalence for any group G:

$$\sigma : (x, y) \mapsto (xy^{-1}, y).$$

Now, we put these two things next to each other. That J respects product structures is summarized by the commutative diagram

$$\begin{tikzcd}
BU \times BU \arrow[r, "{\sigma, \simeq}"] \arrow[drr, bend right] & BU \times BU \arrow[r, "\mu_{BU}"] \arrow[d, "J \times J"] \arrow[dr] & BU \arrow[d, "J"] \\
& BF \times BF \arrow[r, "\mu_{BF}"] & BF,
\end{tikzcd}$$

in which we've also drawn the shearing map σ. The long composite takes the form

$$J \circ \mu_{BU} \circ \sigma(x, y) = J \circ \mu_{BU}(xy^{-1}, y) = J(xy^{-1}y) = J(x).$$

It follows that the second coordinate plays no role, and that the Thom spectrum of the long composite agrees with the Thom spectrum of the map $0 \times J$.[7] Stringing together the properties above, we get:

$$T(J) \wedge T(J) \simeq T(J \times J) \overset{\sigma}{\simeq} T(J \times 0) \simeq T(J) \wedge T(0) \simeq T(J) \wedge \Sigma^\infty_+ BU.$$

It's then easy to extract a more general statement from the one at hand:

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[4] One might prefer the name "reduced Thom spectrum", because of the dimension shift in the definition.

[5] Incidentally, naturality and monoidality mean that Thom spectra associated to group maps like J have the induced structure of ring spectra.

[6] This is closely related to a categorical definition of G–torsors: a G–set X is a G–torsor when $(g, x) \mapsto (x, gx)$ is an equivalence.

[7] This is to say that $\mu \circ (0 \times J)$ is homotopic to the long composite, but $(0 \times J)$ is *not* homotopic to $(J \times J) \circ \sigma$.

Theorem (Thom, proof by Mahowald)**.** *If $f : G \to BF$ is a group map, $T(f) \to E$ is a ring map, and $\xi : X \to G$ classifies a spherical bundle factoring through f, then there is an equivalence*

$$E \wedge T(\xi) \simeq E \wedge \Sigma^\infty_+ X.$$

This is called "the Thom isomorphism", and we should take a moment to ponder its significance. The role of the smash product in stable homotopy theory is that it's used to form homology:

$$E_*(X) := \pi_*(E \wedge \Sigma^\infty_+ X).$$

So, this equivalence is a homotopical form of the assertion that $T(\xi)$ and $\Sigma^\infty_+ X$ have the same E–homology. Additionally, because we have this topological statement, we can extract a slightly stronger moral: the twisted suspension embodied by the spherical bundle ξ is *invisible* to the homology theory E.

1.3. **Statement of Quillen's theorem.** We've gone far too long without giving an example. Let $\mathbb{C}\mathrm{P}^\infty \simeq BU(1)$ be the classifying space for line bundles, and using $U(1) \simeq S^1$ pass to its circle–bundle to get

$$\begin{tikzcd} U(1) \arrow[r] \arrow[d, equal] & EU(1) \arrow[r, "\mathscr{L}"] \arrow[d, equal] & BU(1) \arrow[d, equal] \\ S^1 \arrow[r] & * \arrow[r, "J(\mathscr{L})"] & \mathbb{C}\mathrm{P}^\infty. \end{tikzcd}$$

Since $EU(1)$ is contractible, we see $T(J(\mathscr{L})) \simeq \Sigma^{-2}\Sigma^\infty \mathbb{C}\mathrm{P}^\infty$. Given a J–oriented spectrum $\varphi : T(J) \to E$, the Thom isomorphism machinery above furnishes us with isomorphisms

$$E^*\mathbb{C}\mathrm{P}^\infty \cong \tilde{E}^{*+2}\mathbb{C}\mathrm{P}^\infty, \qquad E^*\mathbb{C}\mathrm{P}^n \cong \tilde{E}^{*+2}\mathbb{C}\mathrm{P}^{n+1}.$$

Pushing the canonical class $1 \in E^0\mathbb{C}\mathrm{P}^0$ across this isomorphism, we can inductively deduce[8]

$$E^*\mathbb{C}\mathrm{P}^\infty \cong_\varphi E^*[\![x]\!].$$

As a responsible homotopy theorist, I should admit that spectra are generally very nasty objects, and successfully computing some cohomology ring is actually a pretty big deal. If we're in a situation where we can *reliably* compute something, it's very important to get all we can from it. To address this, I'm now going to take off my homotopy theorist hat and put my algebraic geometer hat back on.

This day is strangely pace very hodge-po Hm.

As the classifying space for line bundles, $BU(1)$ has a product structure induced by tensoring. This begets a map

$$\begin{tikzcd} E^*BU(1) \arrow[r] \arrow[d, equal, "\cong_\varphi"] & E^*BU(1) \otimes_{E^*} E^*BU(1) \arrow[d, equal, "\cong_\varphi"] \\ E^*[\![t]\!] \arrow[r] & E^*[\![x,y]\!] \end{tikzcd}$$

which is determined by the image of t, some bivariate power series $x +_\varphi y$. This notation for this series is useful because it helps us remember what axioms it satisfies:

(1) Unitality: $x +_\varphi 0 = x$ and $0 +_\varphi y = y$. (Consider tensoring with the trivial line bundle.)
(2) Symmetry: $x +_\varphi y = y +_\varphi x$. (Tensoring is commutative.)
(3) Associativity: $(x +_\varphi y) +_\varphi z = x +_\varphi (y +_\varphi z)$. (Tensoring is associative.)

Such a power series is called a *formal group law*.[9] The universal such power series is represented by an affine scheme $\mathscr{M}_{\mathbf{fgl}}$, and the identity orientation of $T(J)$ gives a map $\mathscr{M}_{T(J)}[0] \to \mathscr{M}_{\mathbf{fgl}}$. Moreover, $T(J) \wedge T(J)$ is the universal ring spectrum with two J–orientations (coming from the left- and right-units) and a transposition relating them:

$$T(J) \wedge T(J) \xrightarrow{\text{twist}} T(J) \wedge T(J).$$

It follows that the induced formal group laws $x +_{\eta_L} y$ and $x +_{\eta_R} y$ must be related by some "formal group law isomorphism" $f(t) \in (T(J) \wedge T(J))_*[\![t]\!]$, i.e., a power series f satisfying

$$f(x +_{\eta_L} y) = f(x) +_{\eta_R} f(y).$$

[8] More miraculously, a piece of vector bundle geometry called the "splitting principle" shows that the converse holds: if E is a ring spectrum with a x so that $\mathbb{S} \to \Sigma^{-2}\Sigma^\infty \mathbb{C}\mathrm{P}^\infty \xrightarrow{x} E$ factors the unit map $\mathbb{S} \to E$, then it can be shown that E has a unique J–orientation selecting that class.

[9] All the formal group laws we'll consider will implicitly be commutative and 1–dimensional.

Theorem (Quillen's theorem)**.** *The spectrum $T(J)$ satisfies **FH** and **CH**. Moreover, the maps*

$$\operatorname{Spec} T(J)_* \to \mathscr{M}_{\mathbf{fgl}},$$
$$\operatorname{Spec} T(J)_* T(J) \to \mathscr{M}_{\mathbf{fgl}} \times \mathscr{M}_{\mathbf{ps}}^{\mathrm{gpd}},$$
$$\mathscr{M}_{T(J)} \to \mathscr{M}_{\mathbf{fgl}} // \mathscr{M}_{\mathbf{ps}}^{\mathrm{gpd}} =: \mathscr{M}_{\mathbf{fg}}$$

described above are all equivalences.

This is a pretty powerful theorem.[10] In our discussion of $T(J)$, we've been so hands off that we've had essentially no control over its behavior. Nonetheless, this theorem puts $T(J)$ on almost even footing with $H\mathbb{F}_2$: just as the compact description of $\mathscr{M}_{H\mathbb{F}_2}$ given above lets you totally unpack the category in which $H\mathbb{F}_2$–homology is valued, Quillen's description of $\mathscr{M}_{T(J)}$ gives you complete access to the structure theorems governing the category in which $T(J)$–homology is valued. We will do our best to leverage this tomorrow.

[10]This situation has a strange feature worth remarking on: the ring maps $T(J) \wedge T(J) \to E$ act transitively on the set of ring maps $T(J) \to E$, i.e., the "(decoordinatized) formal group" associated to E is determined totally by E. This is very different from the algebraic case, where a given ring can support many non-isomorphic formal group laws.

2. DAY 2: E–THEORY AND PERIODIC SELF-MAPS

ABSTRACT. We outline a program for studying the functor $\mathcal{M}_{T(J)}(X)$ by first studying the local structure of $\mathcal{M}_{\mathbf{fg}}$. After a brief tour of the arithmetic literature on formal group laws, we deduce the existence of certain homology theories: the Morava E– and K–theories. We then give examples of local-to-global methods in algebraic topology: for instance, a condition for detecting non-nilpotent self-maps.

2.1. **Some philosophy on flat maps.** Yesterday, we developed a rich target for $T(J)$–homology: sheaves over an algebro-geometric object $\mathcal{M}_{T(J)}$. Furthermore, Quillen's theorem gave an identification $\mathcal{M}_{T(J)} \simeq \mathcal{M}_{\mathbf{fg}}$. Our initial goal for today is to outline a program by which we can leverage this to study $T(J)$. Abstractly, one can hope to study any sheaf, including $\mathcal{M}_{T(J)}(X)$, by analyzing its stalks. The main utility of Quillen's theorem is that it gives us access to a concrete model of $\mathcal{M}_{T(J)}$, so that we can determine where to even look for those stalks.

With this in mind, given a map

$$\operatorname{Spec} R \xrightarrow{f} \mathcal{M}_{\mathbf{fg}},$$

life would be easiest if the R–module determined by $f^*\mathcal{M}_{T(J)}(X)$ were itself the value of a homology theory $R_*(X) = T(J)_*X \otimes_{T(J)_*} R$. After all, the pullback of some arbitrary sheaf along some arbitrary map has no special behavior, but homology functors do have familiar special behaviors which we could hope to exploit. Generally, this is unreasonable to expect: homology theories are functors which convert cofiber sequences of spectra to long exact sequences of groups, but base–change from $\mathcal{M}_{\mathbf{fg}}$ to $\operatorname{Spec} R$ preserves exact sequences exactly when f is *flat*. In that case, this gives the following theorem:

Theorem (Landweber, part 1)**.** *For any diagram*

$$\begin{tikzcd}
\operatorname{Spec} R \arrow[r, "i"] \arrow[dr, "\text{flat}"'] & \mathcal{M}_{\mathbf{fgl}} \arrow[r, equal] \arrow[d] & \mathcal{M}_{T(J)}[0] \arrow[r, equal] \arrow[d] & \operatorname{Spec} T(J)_* \\
& \mathcal{M}_{\mathbf{fg}} \arrow[r, equal] & \mathcal{M}_{T(J)} &
\end{tikzcd}$$

such that the diagonal arrow is flat, the functor

$$R_*(X) := T(J)_*(X) \otimes_{T(J)_*} R$$

determines a homology theory.

In the course of proving this theorem, Landweber devised a method to recognize flat maps. Recall that a map f is flat exactly when for any closed substack $i : A \to \mathcal{M}_{\mathbf{fg}}$ with ideal sheaf $\mathcal{I}$ there is an exact sequence

$$0 \to f^*\mathcal{I} \to f^*\mathcal{O}_{\mathcal{M}_{\mathbf{fg}}} \to f^* i_* \mathcal{O}_A \to 0.$$

Landweber classified the closed substacks of $\mathcal{M}_{\mathbf{fg}}$, thereby giving a method to check maps for flatness.

This appears to be a moot point, however, as it is unreasonable to expect this idea to apply to computing stalks: the inclusion of a closed substack (and so, in particular, a closed point Γ) is flat only in highly degenerate cases. This can be repaired: the inclusion of the formal completion of a closed substack of a Noetherian[11] stack is flat, and so we naturally become interested in the infinitesimal deformation spaces of the closed points Γ on $\mathcal{M}_{\mathbf{fg}}$. If we can analyze those, then Landweber's theorem will produce homology theories called E_Γ. Moreover, if we find that these deformation spaces are *smooth*, it will follow that their deformation rings support regular sequences. In this excellent case, by taking the regular quotient we will be able to recover a *homology theory* K_Γ which plays the role of computing the stalk of $\mathcal{M}_{T(J)}(X)$ at Γ.[12]

[11] $\mathcal{M}_{\mathbf{fg}}$ is not Noetherian, but we will find that each closed point except $\widehat{\mathbb{G}}_a$ lives in an open substack that happens to be Noetherian.

[12] Incidentally, this program has no content when applied to $\mathcal{M}_{H\mathbb{F}_2}$, as $\operatorname{Spec} \mathbb{F}_2$ is simply too small.

2.2. **Local structure of $\mathcal{M}_{\mathrm{fg}}$.** Motivated by the program above, we now set out to describe the local structure of $\mathcal{M}_{\mathrm{fg}}$. Noting that formal group laws arise as analytic germs of multiplication laws on Lie groups, we will first take a cue from Lie theory and attempt to define exponential and logarithm functions for a given formal group law F over a ring R. In Lie theory, this is accomplished by studying left–invariant differentials: a 1–form $f(x)dx$ is said to be left–invariant under F when

$$f(x)dx = f(y +_F x)d(y +_F x) = f(y +_F x)\frac{\partial(y +_F x)}{\partial x}dx.$$

Restricting to the origin by setting $y = 0$, we deduce the condition

$$f(0) = f(x) \cdot \left.\frac{\partial(y +_F x)}{\partial x}\right|_{y=0}.$$

If R is a $\mathbb{Q}$–algebra, then setting the boundary condition $f(0) = 1$ and integrating against x yields

$$\log_F(x) = \int \left(\left.\frac{\partial(y +_F x)}{\partial x}\right|_{y=0} \right)^{-1} dx.$$

To see that the series $\log_F$ has the claimed homomorphism property, note that

$$\frac{\partial \log_F(y +_F x)}{\partial x} = f(y +_F x)d(y +_F x) = f(x)dx = \frac{\partial \log_F(x)}{\partial x},$$

so $\log_F(y +_F x)$ and $\log_F(x)$ differ by a constant. Checking at $x = 0$ shows that the constant is $\log_F(y)$, hence

$$\log_F(x +_F y) = \log_F(x) + \log_F(y).$$

about rescal- Should you nest and call $\mathcal{M}_{\mathrm{fg}}^{(1)}$?

We thus deduce that $\mathcal{M}_{\mathrm{fg}} \times \operatorname{Spec}\mathbb{Q}$ is contractible: every formal group law is uniquely isomorphic to $\widehat{\mathbb{G}}_a$.

However, if R is not a $\mathbb{Q}$–algebra, then we may not be able to perform power series integration. Nonetheless, thinking of the $\mathbb{Q}$–algebra restriction as localization at (0), this inspires us to work arithmetically locally at a prime p and consider $\mathcal{M}_{\mathrm{fg}} \times \operatorname{Spec}\mathbb{Z}_{(p)}$. This task is eased considerably by the following fundamental theorem of Lazard:

Theorem (Lazard, part 1)**.** *The ring of functions on $\mathcal{M}_{\mathrm{fgl}}$ is polynomial in infinitely many variables.*[13]

As a direct consequence, if $f : S \to R$ is a surjective map of rings and F_R is any formal group law on R, then there exists a formal group law F_S on S with $f^*F_S = F_R$. We can thus reduce to the case where R is a torsion–free (or $\mathbb{Z}$–flat) ring for most of our theorems.

Theorem (Hazewinkel)**.** *Every formal group law F over a $\mathbb{Z}_{(p)}$–algebra is isomorphic to some F' whose rational logarithm has the form*

$$\log_{F'}(x) = \sum_{n=0}^{\infty} \ell_n x^{p^n}.$$

It follows that the radius of convergence of $\log_{F'}$ must be p^d for some d.[14] *The integer d is called the* height *of F'. It is an isomorphism invariant and it is insensitive to lifts along surjective maps from torsion–free $\mathbb{Z}_{(p)}$–algebras.*

Theorem (Lazard, part 2: classification of closed points)**.** *Over an algebraically closed field of characteristic p, there is a unique formal group law up to isomorphism for each height. Moreover, there is a representative Γ_d of each isomorphism class with coefficients in $\mathbb{F}_p$ whose logarithm satisfies*

$$\log_{\Gamma_d}(x) \equiv x \pmod{x^{p^d}}.$$

Theorem (Landweber, part 2: classification of closed substacks)**.** *Let BP_* be the ring classifying formal group laws with p–typical logarithms.*

(1) *It has the form $BP_* \cong \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_d, \ldots]$, where $v_d \equiv p\ell_d$* (mod *decomposables*).

[13]His proof does not give a canonical presentation. Rationally, these are the coordinate functions selecting the logarithm coefficients.
[14]If F is additive, then d can be infinite.

(2) *The unique closed substack of* $\mathcal{M}_{\mathbf{fg}} \times \operatorname{Spec}\mathbb{Z}_{(p)}$ *of codimension* d *is selected by* $BP_*/(p, v_1, \ldots, v_{d-1})$*, and its complementary open substack of dimension* d *is selected by either of* $v_d^{-1}BP_*$ *or* $v_d^{-1}\mathbb{Z}_{(p)}[v_1, \ldots, v_d]$.[15]

(3) *A* BP_**–module* M *gives a flat sheaf on* $\mathcal{M}_{\mathbf{fg}}$ *exactly when* $(p, v_1, v_2, \ldots, v_{d-1}, \ldots)$ *is a regular sequence* M *too.*

(4) *In particular,* BP_* *is itself such a module, and so gives rise to a homology theory* BP *with* $\mathcal{M}_{BP} \simeq \mathcal{M}_{\mathbf{fg}} \times \operatorname{Spec}\mathbb{Z}_{(p)}$.

Theorem (Lubin–Tate: description of deformation spaces)**.** *The deformation space of any height* $d < \infty$ *law* Γ *over a perfect field* k *of characteristic* p *is smooth of geometric dimension* $(d-1)$*. That is, it is noncanonically isomorphic to* $\mathbb{W}(k)[\![u_1, \ldots, u_{d-1}]\!]$*. For* $\Gamma = \Gamma_d$*, the coordinates can be taken to be* $v_{0 \le n < d}$.

Having stood on the shoulders of all these arithmetic geometers, we can now put our program into practice. We have a list of the closed points Γ_d of $\mathcal{M}_{\mathbf{fg}} \times \operatorname{Spec}\mathbb{Z}_{(p)}$, and their deformation spaces lift to $\mathcal{M}_{\mathbf{fgl}}$ as smooth formal subschemes. It follows from Landweber's theorem that we can construct homology theories E_{Γ_d} for each of these formal groups. Additionally, we can find regular sequences $(p, u_1, \ldots, u_{d-1}) \in (E_{\Gamma_d})_*$, and hence we can construct the regular quotient[16]

$$K(\Gamma_d) := E_{\Gamma_d}/(p, u_1, \ldots, u_{d-1}).$$

In the case that we pick the lift of Γ_d with p–series $[p](x) = x^{p^d}$, these objects are typically written E_d and $K(d)$, called Morava E–theory and Morava K–theory.

2.3. **E–theories and periodic self-maps.** Having constructed these "stalk" homology theories, I want to show that you can actually perform analyses of the kind I was describing at the beginning of today. Our example case is a famous theorem: the solution of Ravenel's nilpotence conjectures by Devinatz, Hopkins, and Smith. Their theorem concerns spectra which "detect nilpotence" in the following sense:

Definition. A ring spectrum E *detects nilpotence* if, for any ring spectrum R, the kernel of the Hurewicz homomorphism $E_* : \pi_* R \to E_* R$ consists of nilpotent elements.

First, a word about why one would care about such a condition. The following theorem is classical:

Theorem (Nishida)**.** *Every homotopy class* $\alpha \in \pi_{\ge 1}\mathbb{S}$ *is nilpotent.*

However, people studying K–theory in the '70s discovered the following phenomenon:

Theorem (Adams)**.** *Let* $M_{2n}(p)$ *denote the mod–*p *Moore spectrum with bottom cell in degree* $2n$*. Then there is an index* n *and a map* $v : M_{2n}(p) \to M_0(p)$ *such that* $KU_* v$ *acts by multiplication by the* n^{th} *power of the Bott class.*[17]

In particular, this means that v cannot be nilpotent, since a null-homotopic map induces the zero map in any homology theory. Just as we took the non-nilpotent endomorphism p in $\pi_0 \operatorname{End}\mathbb{S}$ and coned it off, we can take the endomorphism v in $\pi_{2p-2}\operatorname{End} M_0(p)$ and cone it off to form a new spectrum called $V(1)$.[18] Ravenel's burning question was whether the pattern continues: does $V(1)$ have a non-nilpotent self-map, and can we cone it off to form a new such spectrum with a new such map? Can we then do that again, indefinitely? In order to study this question, we are motivated to find spectra E as above — and in fact, we found one yesterday.

Theorem (Devinatz–Hopkins–Smith, hard)**.** *The spectrum* $T(J)$ *detects nilpotence.*

They also show that the $T(J)$ is the universal object which detects nilpotence, in the sense that any other ring spectrum can have this property checked stalkwise on $\mathcal{M}_{T(J)}$:

Theorem (Hopkins–Smith, easy)**.** *A ring spectrum* E *detects nilpotence if and only if* $K(d)_* E \neq 0$ *for all* $0 \le d \le \infty$ *and for all primes* p.

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[15] It's worth pointing out how strange this is. In Euclidean geometry, open subspaces are always top-dimensional, and closed subspaces can drop dimension.

[16] We think of $K(\Gamma_d)_* X$ as being a model for the stalk of $\mathcal{M}_{T(J)}(X)$ at Γ_d, though if $(E_{\Gamma_d})_* X$ has torsion this may not agree with $\Gamma_d^* \mathcal{M}_{T(J)}(X)$.

[17] The minimal such n is given by the formula $n = \begin{cases} p-1 & \text{when } p \ge 3, \\ 4 & \text{when } p = 2. \end{cases}$

[18] $V(1)$ actually means a finite spectrum with $BP_* V(1) \cong BP_*/(p, v_1)$. At $p = 2$ this spectrum doesn't exist and this is a misnomer.

Proof. If $K(d)_* E = 0$ for some d, then the non-nilpotent map $\mathbb{S} \to K(d)$ lies in the kernel of the Hurewicz homomorphism for E, so E fails to detect nilpotence.

Hence, for any d we must have $K(d)_* E \neq 0$. Because $K(d)_*$ is a field, it follows by picking a basis of $K(d)_* E$ that $K(d) \wedge E$ is a nonempty wedge of suspensions of $K(d)$. So, for $\alpha \in \pi_* R$, if $E_* \alpha = 0$ then $(K(d) \wedge E)_* \alpha = 0$ and hence $K(d)_* \alpha = 0$. So, we need to show that if $K(d)_* \alpha = 0$ for all n and all p then α is nilpotent. Taking Devinatz–Hopkins–Smith as given, it would suffice to show merely that $T(J)_* \alpha$ is nilpotent. This is equivalent to showing that the ring spectrum $T(J) \wedge R[\alpha^{-1}]$ is contractible or that the unit map is null:

$$\mathbb{S} \to T(J) \wedge R[\alpha^{-1}].$$

Pick a prime p and recall the regular sequence of Landweber's theorem. We define a spectrum $P(d+1)$ to be the regular quotient of BP by $(p, v_1, \ldots, v_d)$. A nontrivial result of Johnson and Wilson shows that if $T(J)_* X = 0$ for any X, then for any d we have $K([0,d])_* X = 0$ and $P(d+1)_* X = 0$.[19] Taking $X = R[\alpha^{-1}]$, have assumed all of these are zero except for $P(d+1)$. But $\operatorname{colim}_d P(d+1) \simeq H\mathbb{F}_p \simeq K(\infty)$, and $\mathbb{S} \to K(\infty) \wedge R[\alpha^{-1}]$ is assumed to be null as well. By compactness of $\mathbb{S}$, that null-homotopy factors through some finite stage $P(d+1) \wedge R[\alpha]$ with $d \gg 0$. $\square$

As another example of the primacy of these methods, we can show the following interesting result. Say that R is a field spectrum when every R-module (in the homotopy category) splits as a wedge of suspensions of R. It is easy to check (as mentioned in the proof above) that $K(d)$ is an example of such a spectrum.

Theorem. *Every field spectrum R splits as a wedge of Morava K-theories.*

Proof. Set $E = \bigvee_{\text{primes } p} \bigvee_{d \in [0,\infty]} K(d)$, so that E detects nilpotence. The class 1 in the field spectrum R is non-nilpotent, so it survives when paired with some K-theory $K(d)$, and hence $R \wedge K(d)$ is not contractible. Because both R and $K(d)$ are field spectra, the smash product of the two simultaneously decomposes into a wedge of $K(d)$s and a wedge of Rs. So, R is a retract of a wedge of $K(d)$s, and picking a basis for its image on homotopy shows that it is a sub-wedge of $K(d)$s. $\square$

This is interesting in its own right, because field spectra are exactly those spectra which have Künneth isomorphisms. So, even if you weren't neck-deep in algebraic geometry, you might still have struck across these homology theories just if you like to compute things, since Künneth formulas make things computable.

[19] It is immediate that $T(J)_* X = 0$ forces $P(d+1)_* X = 0$ and $v_{d'}^{-1} P(d')_*(X) = 0$ for all $d' < d$. What's nontrivial is showing that $v_{d'}^{-1} P(d')_*(X) = 0$ if and only if $K(d')_*(X) = 0$.

3. DAY 3: CHROMATIC LOCALIZATIONS

ABSTRACT. We now try to superimpose some of the structure seen yesterday in $\mathscr{M}_{\mathrm{fg}}$ directly onto the category of finite spectra. This summons certain Bousfield localizations, and we describe their primary application to the stable category.

3.1. Classification of thick subcategories.

3.1. Classification of thick subcategories. Our first goal for today is to apply these local methods once more to get a positive answer to Ravenel's question about finite spectra and periodic self-maps. The solution to this problem passes through some now-standard machinery for triangulated $\otimes$–categories.

Definition. A subcategory of the category of a triangulated category (e.g., p–local finite spectra) is *thick* if it is closed under weak equivalences, it is closed under retracts, and it has a 2-out-of-3 property for cofiber sequences.

Examples of thick subcategories include:

- The category C_d of p–local finite spectra which are $K(d-1)$–acyclic. (For instance, if $d=1$, the condition of $K(0)$–acyclicity is that the spectrum have purely torsion homotopy groups.) These are called "finite spectra of type at least d".
- The category D_d of p–local finite spectra F which have a self-map $v : \Sigma^N F \to F$, $N \gg 0$, inducing multiplication by a unit in $K(d)$–homology. These are called "v_d–self–maps".

Hopkins and Smith show the following classification theorem:

Theorem (Hopkins–Smith, easy). *Any thick subcategory C of p–local finite spectra must be C_d for some d.*

Proof. It is sufficient to show that any object $X \in \mathsf{C}$ with $X \in \mathsf{C}_d$ induces an inclusion $\mathsf{C}_d \subseteq \mathsf{C}$. Let $Y \in \mathsf{C}_d$ be any other spectrum of type at least d. Consider the endomorphism ring spectrum $R = F(X, X)$ and the fiber $f : F \to \mathbb{S}$ of its unit map. The action of f under $K(n)$–homology is an isomorphism exactly when X is $K(n)$–acyclic, and because the $K(n)$–acyclicity of X implies the $K(n)$–acyclicity of Y, it follows that $1 \wedge f : Y \wedge F \to Y \wedge \mathbb{S}$ is always null on $K(n)$–homology for all n. By a small variant of the local nilpotence detection theorem, it follows that

$$Y \wedge F^{\wedge j} \xrightarrow{1 \wedge f^{\wedge j}} Y \wedge \mathbb{S}^{\wedge j}$$

is null for $j \gg 0$, and hence that

$$\operatorname{cofib}\left(Y \wedge F^{\wedge j} \xrightarrow{1 \wedge f^{\wedge j}} Y \wedge \mathbb{S}^{\wedge j}\right) \simeq Y \wedge \operatorname{cofib} f^{\wedge j} \simeq Y \vee (Y \wedge \Sigma F^{\wedge j}),$$

so that Y is a retract. However, using $\operatorname{cofib}(f) = X \wedge DX \in \mathsf{C}$ and a smash version of the octahedral axiom

$$F \wedge F^{\wedge(j-1)} \xrightarrow{f \wedge 1} \mathbb{S} \wedge F^{\wedge(j-1)} \xrightarrow{1 \wedge f^{\wedge(j-1)}} \mathbb{S} \wedge \mathbb{S}^{\wedge(j-1)} \quad \Rightarrow \quad F \wedge \operatorname{cofib} f^{\wedge(j-1)} \to \operatorname{cofib} f^{\wedge j} \to \operatorname{cofib} f \wedge \mathbb{S}^{\wedge(j-1)}$$

one can inductively show that $\operatorname{cofib}(f^{\wedge j})$, hence $Y \wedge \operatorname{cofib}(f^{\wedge j})$, and hence Y all belong to C as well. □

They also show the *considerably* harder theorem:

Theorem (Hopkins–Smith, hard). *A p–local finite spectrum is $K(d-1)$–acyclic exactly when it admits a v_d–self–map.*

Executive summary of proof. Given the classification of thick subcategories, if a property is closed under thickness then one need only exhibit a single spectrum with the property to know that all the spectra in the thick subcategory it generates also all have that property. Inductively, they manually construct finite spectra $M_0(p^{i_0}, v_1^{i_1}, \ldots, v_{d-1}^{i_{d-1}})$ for sufficiently large[20] indices i_* which admit a self-map v governed by a commuting square

$$\begin{array}{ccc}
BP_* M_{|v_d| i_d}(p^{i_0}, v_1^{i_1}, \ldots, v_{d-1}^{i_{d-1}}) & \xrightarrow{v} & BP_* M_0(p^{i_0}, v_1^{i_1}, \ldots, v_{d-1}^{i_{d-1}}) \\
\| & & \| \\
\Sigma^{|v_d| i_d} BP_*/(p^{i_0}, v_1^{i_1}, \ldots, v_{d-1}^{i_{d-1}}) & \xrightarrow{- \cdot v_d^{i_d}} & BP_*/(p^{i_0}, v_1^{i_1}, \ldots, v_{d-1}^{i_{d-1}}).
\end{array}$$

These maps are guaranteed by very careful study of Adams spectral sequences. □

[20] Compare this asymptotic condition with the assertion yesterday that there is no root of $v : M_8(2) \to M_0(2)$.

3.2. **Balmer spectra and chromatic localization.** As part of a broad attempt to analyze a geometric object through its modules, Paul Balmer has demonstrated the following theorem:

Definition. Given a triangulated $\otimes$–category C, define a thick subcategory $\mathsf{C}' \subseteq \mathsf{C}$ to be a $\otimes$–*ideal* when it has the additional property that $x \in \mathsf{C}'$ forces $x \otimes y \in \mathsf{C}'$ for any $y \in \mathsf{C}$. Moreover, C' is said to be *prime* when $x \otimes y \in \mathsf{C}'$ forces at least one of $x \in \mathsf{C}'$ or $y \in \mathsf{C}'$. Define the *spectrum* of C to be its collection of prime $\otimes$–ideals, topologized so that $U(x) = \{\mathsf{C}' \mid x \in \mathsf{C}'\}$ form a basis of opens.

Theorem (Balmer)**.** *The spectrum of $D^{\mathrm{perf}}(\mathsf{Mod}_R)$ is naturally homeomorphic to the Zariski spectrum of R.*

Balmer's construction applies much more generally. The category $\mathsf{Spectra}$ can be identified with $\mathsf{Modules}_{\mathbb{S}}$, and so one can attempt to compute the Balmer spectrum of $\mathsf{Modules}_{\mathbb{S}}^{\mathrm{perf}} = \mathsf{Spectra}^{\mathrm{fin}}$. In fact, we just finished this.

Theorem. *The Balmer spectrum of $\mathsf{Spectra}_{(p)}^{\mathrm{fin}}$ consists of the thick subcategories C_d, and $\{\mathsf{C}_n\}_{n=0}^{d}$ are its open sets.*

Proof. Using the characterization of C_d as the kernel of $K(d-1)_*$, we see that it is a prime $\otimes$–ideal:

$$K(d-1)_*(X \wedge Y) \cong K(d-1)_* X \otimes_{K(d-1)_*} K(d-1)_* Y$$

is zero exactly when at least one of X and Y is $K(d-1)$–acyclic. $\square$

In fact, our favorite functor[21] $T(J)_* : \mathsf{Spectra} \to \mathsf{QCoh}(\mathscr{M}_{T(J)})$ induces a homeomorphism of the Balmer spectrum of $\mathsf{Spectra}^{\mathrm{fin}}$ to that of $\mathscr{M}_{\mathbf{fg}}$. However, Balmer's construction gives only a topological space, and not anything like a locally ringed space (or a space otherwise equipped locally with algebraic data).[22] Recalling Landweber's theorem from yesterday, Bousfield's theory of homological localization allows us to extend it as follows:

Theorem (Bousfield)**.** *Let R_* denote the homology theory associated to a flat map $j : \operatorname{Spec} R \to \mathscr{M}_{\mathbf{fg}}$ by Landweber's theorem. There is then a diagram*[23]

$$\begin{tikzcd}
\mathsf{Spectra}_R \arrow[rr, "R_* \quad \text{conservative}"] \arrow[d, shift right, "i"'] \arrow[d, phantom, "\dashv"] & & \mathsf{QCoh}(\operatorname{Spec} R) \arrow[d, shift right, "j_*"'] \arrow[d, phantom, "\dashv"] \\
\mathsf{Spectra} \arrow[u, shift right, "L_R"'] \arrow[urr, "R_*"] \arrow[rr, "T(J)_*"] & & \mathsf{QCoh}(\mathscr{M}_{T(J)}), \arrow[u, shift right, "j^*"']
\end{tikzcd}$$

such that i is left-adjoint to L_R, j^* is left-adjoint to j_*, i and j_* are inclusions of full subcategories, the red composites are all equal, and R_* is conservative on $\mathsf{Spectra}_R$.

In the case when R models the inclusion of the deformation space around the point Γ_d, we will denote the localizer by

$$\mathsf{Spectra} \xrightarrow{\widehat{L}_d} \mathsf{Spectra}_{\Gamma_d}.$$

In the case when R models the inclusion of the open complement of the unique closed substack of codimension d, we will denote the localizer by

$$\mathsf{Spectra} \xrightarrow{L_d} \mathsf{Spectra}_d = \mathsf{Spectra}_{\mathscr{M}_{\mathbf{fg}}^{\le d}}.$$

We have set up our situation so that the following properties of these localizations either have easy proofs or are intuitive from the algebraic analogue of $j^* \vdash j_*$:

[21] However, this functor is *not* a map of triangulated categories, so this has to be interpreted lightly.

[22] We will address this in our situation, but in general this is an open question: given a ring spectrum R, how can one recognize these local categories of spectra in terms of R, without reference to auxiliary spectra like $T(J)$? Or, just as importantly: what makes $T(J)$ a special $\mathbb{S}$–algebra?

[23] The meat of this theorem is in overcoming set-theoretic difficulties in the construction of $\mathsf{Spectra}_R$. Bousfield accomplished this by describing a model structure on $\mathsf{Spectra}$ for which R–equivalences create the weak–equivalences.

(1) There is an equivalence
$$L_d X \simeq (L_d \mathbb{S}) \wedge X,$$
analogous to $j^* M \simeq R \otimes M$ in the algebraic setting. Because $L_{K(d)}$ is associated to the inclusion of a formal scheme (i.e., an ind-finite scheme), it has the formula
$$\widehat{L}_d X \simeq \lim_I \left(M_0(v^I) \wedge L_d X \right)$$
analogous to $j^* M \simeq \lim_j (R/I^j \otimes M)$ in the complete algebraic setting.

(2) Because the open substack of dimension d properly contains both the open substack of dimension $(d-1)$ and the infinitesimal deformation neighborhood of the closed point of height d, there are natural factorizations
$$\mathrm{id} \to L_d \to L_{d-1}, \qquad \mathrm{id} \to L_d \to \widehat{L}_d.$$
In particular, $L_d X = 0$ implies both $L_{d-1} X = 0$ and $\widehat{L}_d X = 0$.

(3) The inclusion of the open substack of dimension $d-1$ into the one of dimension d has relatively closed complement the point of height d. Algebraically, this gives a gluing square (or Mayer-Vietoris square), and this is reflected in homotopy theory by a homotopy pullback square (the chromatic fracture square):
$$\begin{tikzcd} L_d \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \widehat{L}_d \arrow[d] \\ L_{d-1} \arrow[r] & L_{d-1}\widehat{L}_d. \end{tikzcd}$$

3.3. **Chromatic dissembly.** There are also considerably more complicated facts known about these functors:

Theorem (Hopkins–Ravenel)**.** *The homotopy limit of the tower*
$$\cdots \to L_d F \to L_{d-1} F \to \cdots \to L_1 F \to L_0 F$$
recovers the p-local homotopy type of any finite spectrum F.[24]

This suggests a productive method for analyzing the homotopy groups of spheres: study the homotopy groups of each $L_d \mathbb{S}$ and perform the reassembly process encoded by this inverse limit. Using the fracture square, one sees that it is also profitable to consider the homotopy groups of $\widehat{L}_d \mathbb{S}$. In fact, the spectral version of $\mathscr{M}_E(F)$ considered on the first day furnishes us with a tool by which we can approach this:

Theorem (Bousfield, et al.)**.** *The coskeletal filtration of $\mathscr{D}_E(F)$ gives a spectral sequence converging to the homotopy of its totalization, $F^\wedge_E$.*[25] *When F is finite and E models either of the cases above, this spectral sequence converges to $\pi_* L_E F$. Furthermore, there is a line bundle ω on $\mathscr{M}_E$ such that*[26]
$$E_2^{*,*} = H^*_{\mathrm{stack}}(\mathscr{M}_E; \mathscr{M}_E(F) \otimes \omega^{\otimes *}) \Rightarrow \pi_* L_E F.$$

The utility of this theorem is in the identification with stack cohomology. In the case $E = E_{\Gamma_d}$, recall that $\mathscr{M}_{E_{\Gamma_d}}[0]$ is a smooth infinitesimal thickening of the spectrum of a field, so that
$$\mathscr{M}_{E_{\Gamma_d}} = \left(\mathscr{M}_{\mathrm{fg}}\right)^\wedge_{\Gamma_d} \simeq \widehat{\mathbb{A}}^{d-1}_{\mathbb{W}(k)} /\!/ \underline{\mathrm{Aut}}(\Gamma_d)$$
as in the first example of $E = H\mathbb{F}_2$ on the first day. But, in this specific case, there is an identification of stack cohomology with group cohomology:
$$H^*_{\mathrm{stack}}(* /\!/ \underline{G}; \mathscr{M}) = H^*_{\mathrm{group}}(G; M).$$

[24]Spectra satisfying this limit property are said to be *chromatically complete*, which is closely related to being *harmonic*, i.e., being local with respect to $\bigvee_{d=0}^\infty K(d)$. (I believe this a joke about "music of the spheres".) It is known that nice Thom spectra (and in particular every suspension and finite spectrum) is harmonic, that every finite spectrum is chromatically complete, and that there exist some harmonic spectra which are not chromatically complete.

[25]There is a subtlety here: the object $\mathscr{D}_E(F)$ must be able to be formed as a homotopy coherent diagram in order to produce the totalization. Essentially, this forces E to be an A_∞–ring spectrum. This holds for all the examples of ring spectra we have discussed.

[26]The identification of the E_2–page as computing stack cohomology is the first place where we really mean to employ the full technology of stacks in this talk. Everywhere else, we have been essentially content to speak of simplicial presheaves.

Another theorem from the arithmetic geometry literature gives

$$\mathrm{Aut}(\Gamma_d) \cong \left(\mathbb{W}(k)\langle S \rangle \middle/ \left(\begin{array}{c} Sw = w^{\varphi} S, \\ S^d = p \end{array} \right) \right)^{\times},$$

and so we have reduced the computation of all of the stable homotopy groups of spheres to a very difficult problem in profinite group cohomology — but one which is arithmetically founded, so that arithmetic geometry might continue to lend a hand.

Example (Adams). In the case $d = 1$, $\mathrm{Aut}(\Gamma_1) = \mathbb{Z}_p^{\times}$ and it acts on $\pi_* E_1 = \mathbb{Z}_p[u^{\pm}]$ by $\gamma \cdot u^n \mapsto \gamma^n u^n$. At odd primes p (so that p is coprime to the torsion part of $\mathbb{Z}_p^{\times}$), one computes

$$H^s(\mathrm{Aut}(\Gamma_1); \pi_* E_1) = \begin{cases} \mathbb{Z}_p & \text{when } s = 0, \\ \bigoplus_{j=2(p-1)k} \mathbb{Z}_p\{u^j\}/(pku^j) & \text{when } s = 1, \\ 0 & \text{otherwise.} \end{cases}$$

This, in turn, gives the calculation

$$\pi_t \widehat{L}_1 \mathbb{S}^0 = \begin{cases} \mathbb{Z}_p & \text{when } t = 0, \\ \mathbb{Z}_p/(pk) & \text{when } t = t|v_1| - 1, \\ 0 & \text{otherwise.} \end{cases}$$

These groups are familiar to homotopy theorists: the J–homomorphism $J : BU \to BF$ described on the first day selects exactly these elements (for nonnegative t).