Robert Langlands: Abel Laureate 2018

Eric Peterson

May 2nd, 2018

Context

Fields medal

- Single accomplishment
- Limited to 40 years old

Stereotypes:

- “Experimentalist”
- Prover

Abel prize

- Lifetime of work
- No age limit (avg. 72)

Stereotypes:

- “Theorist”
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Langlands, letter to Weil, January 1967

“In response to your invitation to come and talk I wrote the enclosed letter. After I wrote it I realized there was hardly a statement in it of which I was certain. If you are willing to read it as pure speculation I would appreciate that; if not—I am sure you have a waste basket handy.”

About this talk

Philosophy of mathematics

"Mathematics is the art of giving the same name to different things."

Henri Poincaré

"...The Langlands Program is a Grand Unified Theory of mathematics."

Edward Frenkel

"You should give a talk about this." Jeff Cordova

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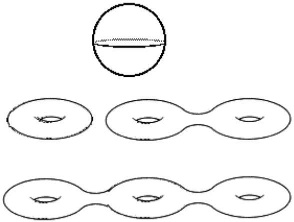
Warning to mathematicians

My target audience is experimental physicists. Theorem statements will elide details, especially as I aim to avoid mentioning the adèles. Forgive me.

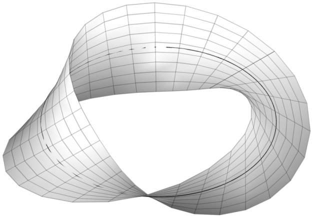
Complex geometry

A *Riemann surface* is a closed surface which is built by gluing together (pieces of) copies of the complex plane so that "rotation by *i*" agrees on each overlap.



Riemann surfaces

(underdetermined)



no bueno

Complex geometry

All functions are assumed to have series expansions.

Example

Functions to the sphere = functions with poles. $\leftarrow$ "*meromorphic functions*"

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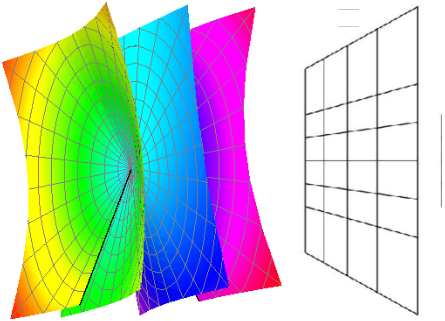
Anatomy of a meromorphic function



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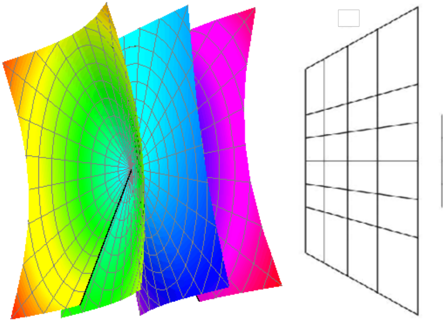

$w = z^3$

- Three sheeted cover
- Bad point: "*ramification*"
- Wandering preimages: "*monodromy*"

Complex geometry

Theorem

The set of meromorphic functions totally determines the Riemann surface.

meromorphic functions

$$\{\text{Riemann surfaces}\} \rightleftarrows \left\{\begin{array}{c}\text{complex fields}\\ \text{w/ one “transcendant” family}\end{array}\right\}$$

$$\{\mathbb{P}^1 \xrightarrow{w=z^3} \mathbb{P}^1\} \longmapsto \{\mathbb{C}(z) \supseteq \mathbb{C}(w)\}\,.$$

Glossary:

- $\mathbb{P}^1$: the sphere
- $\mathbb{C}(z)$: fractions of polynomials in z
- $w = z^3$

Algebraic number theory

More fields

- A *number field* K is the rationals + some extra numbers that serve as distinct roots of some given polynomials.
- Its *small Galois group*, gal(K), tracks the ways to reassign those extra numbers among each other.

Examples

Roots	one root of $x^2 - 2$	one root of $x^3 - 2$	all roots of $x^3 - 2$
gal(K)	C_2	trivial	Σ_3

Algebraic number theory

The *relative Galois group* of a pair of number fields $K \subseteq L$ is the part of the small Galois group of L that fixes K. The *big Galois group* of a number field K, $\mathrm{Gal}(K)$, is the union of all the Galois groups for $K \subseteq L$, L large.

Theorem

If $\mathrm{Gal}(K)$ surjects onto G, there is a number field $L \supseteq K$ with Galois group G.

Maxim

Complicated $\mathrm{Gal}(K)$ $\rightsquigarrow$ lots of surjections $\rightsquigarrow$ lots of fields $\rightsquigarrow$ complicated K.

Algebraic geometry

$$z^3 \left.\right\} \text{ is ramified at } \left\{ w = 0 \right\} \text{ because of the repeated root.}$$

Algebraic geometry

$$\left.\begin{matrix} z^3 \\ x^3 - 2 \end{matrix}\right\} \text{ is ramified at } \left\{\begin{matrix} w = 0 \\ 2 = 0 \end{matrix}\right\} \text{ because of the repeated root.}$$

Weil's Rosetta stone

↙ Riemann surface		↘ number fields
monodromy		Galois group
ramification	⇜ ⇝	ramification
⋮		⋮

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Weil's Rosetta stone

Diophantine equation
(e.g., $0 = x^3 - 2$)

$\swarrow$ $\searrow$

Riemann surface		number fields
monodromy		Galois group
ramification	$\leftrightsquigarrow$	ramification
$\vdots$		$\vdots$

These are related through their common source.

Analytic number theory

Prime factorization in analysis

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \cdots$$

$\zeta(s)$ diverges at $s = 1$, converges to the right of this point.

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$$= \prod_{p} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right).$$

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Goofy example

There are infinitely many prime numbers:

$$\log \zeta(s) = \log \left(\prod_p \frac{1}{1 - p^{-s}}\right) = \sum_p \frac{1}{p^s} + \mathrm{bdd}$$

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$$\infty \xleftarrow{s\to 1^+} \log \zeta(s) = \log\left(\prod_p \frac{1}{1-p^{-s}}\right) = \sum_p \frac{1}{p^s} + \text{bdd} \xrightarrow{s\to 1^+} \sum_p \frac{1}{p} + \text{const.}$$

Harmonic analysis

A *character* of a group G is a function $\chi\colon\ G \to \mathbb{C}$ satisfying $\chi(gg') = \chi(g)\chi(g')$.

Theorem

A nice complex function on a nice **commutative** group can be written as a sum of characters. There are "enough" characters to form a basis of all functions.

The function space has an inner product:

$$f = \sum_{\chi} \langle \chi | f \rangle \cdot \chi; \quad \langle \chi | f \rangle = \int_G \chi(g^{-1}) f(g)\, \mathrm{d}\mu.$$

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Examples

- Functions on $\mathbb{R}$: $\chi_a(x) = e^{2\pi i \cdot a \cdot x}$, $a \in \mathbb{R}$. *Fourier transform*: $\mathcal{F}\{f\}(a) = \langle \chi_a | f \rangle$.
- Functions on $\mathbb{R}^{\times}_{>0}$: $\chi_a(x) = x^a$, $a \in \mathbb{R}$. *Mellin transform*: $\mathcal{M}\{f\}(a) = \langle \chi_a | f \rangle$.

Harmonic number theory

Direct calculation

$\mathcal{M}\{e^{-\pi n^2 z}\} = \pi^{-s}\Gamma(s)n^{-2s}$ for some fixed function $\Gamma(s)$.

$$\mathcal{M}\left\{\sum_{n=1}^{\infty} e^{-\pi n^2 z}\right\}(s/2) = \pi^{-s/2}\Gamma(s/2)\zeta(s) =: Z(s).$$

- Multiplicative Fourier transform of $e^{2\pi i n^2 z}$, an additive character.

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- Multiplicative Fourier transform of $e^{2\pi i n^2 z}$, an additive character.
- Fourier self-duality:

$$\left[\sum_{n\in\mathbb{Z}} \psi(n) = \sum_{n\in\mathbb{Z}} \mathcal{F}\{\psi\}(n)\right] + \left[\mathcal{F}\{e^{\pi i\cdot n\cdot z}\}(s) = e^{\pi i\cdot n\cdot s}\right] \quad \rightsquigarrow \quad Z(s) = Z(1-s).$$

- Riemann hypothesis: $\mathcal{F}\{\text{Dirac comb at zeroes of } Z\} \approx \text{Dirac comb at } \log p$
- Tate's thesis: $Z = Z_\infty \cdot \prod_p Z_p$; $Z_p(s) = (1-p^s)^{-1}$; $Z_\infty(s) = \pi^{-s/2}\Gamma(s/2)$, all coming from analogues of Mellin transforms of characters.

Harmonic number theory

Other ζ–functions

- A *Dirichlet character* χ is a character with domain $(\mathbb{N}, \times)$ **(Not a group!)** with values in roots of unity.

$$\zeta_\chi(s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \frac{1}{1 - \chi(p)p^{-s}}$$

Theorem: Nontrivial character $\Rightarrow \zeta_\chi(1)$ converges.
Application: ∞-ly many sol'ns to $p = nd + r$ for fixed coprime d, r.

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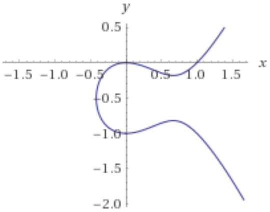
- *Artin ζ–functions*: Replace a character's target $\mathbb{C}$ with $n \times n$ matrices, called a *representation*. For $G = \mathrm{Gal}(K)$, can produce a ζ–function.
    - Lots of examples. (Main ones: cohomology groups.)
    - Seems to contain interesting information.
    - Mysterious: only behaves well for representations "coming from geometry".
    - Almost impossible to prove theorems. ($\zeta_\rho(1)$?)

ζ–function for an elliptic curve

Example elliptic curve: $C = \{y^2 + y = x^3 - x^2\}$



$$\zeta_C(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

Properties of ζ_C:

- The number of points in C over $\overline{\mathbb{F}}_p$ is $p - a_p$.

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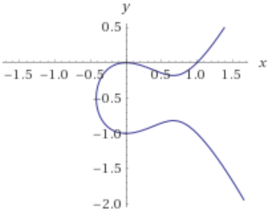

$$\zeta_C(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \overset{\mathcal{M}}{\leftsquigarrow} f_C(q) = \sum_{n=1}^{\infty} a_n q^n = q \prod_{n=1}^{\infty} (1 - q^n)^2 (1 - q^{11n})^2.$$

$$= q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 + \cdots$$

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- For certain C, there is a related f_C with $\mathcal{M}\{f_C(e^{2\pi i z})\} = \zeta_C(s)$.
- The series f_C converges on the unit disk.
- f_C is *modular*: scales predictably under Möbius transformations ($SL_2(\mathbb{Z})$).

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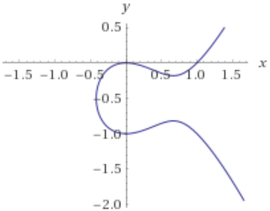

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- The series f_C converges on the unit disk.
- f_C is *modular*: scales predictably under Möbius transformations ($SL_2(\mathbb{Z})$).
- (Shimura–Taniyama conjecture / modularity theorem / Abel Prize 2016:) For every C, there is an f_C. If C, C' have the same f, they're equivalent.

The Langlands Program

$$\left\{ \text{d-dim'l Galois rep's} \right\} \overset{\text{noncomm. } \mathcal{M}}{\longleftrightarrow} \left\{ \text{automorphic rep's of } d \times d \text{ matrices} \right\}$$

$$\begin{array}{ccc} \downarrow \text{Artin} & & \downarrow \\ \{\zeta\text{–functions}\} & = & \{\zeta\text{–functions}\} \end{array}$$

Main point: The correspondence from the previous slide is "general behavior".

Wrinkles: Special behavior of f_C got stuck into another representation.

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$$\left\{\begin{array}{c} d\text{-dim'l Galois rep's} \\ \text{with extra structure} \end{array}\right\} \xleftrightarrow{\text{noncomm. } \mathcal{M}} \left\{\begin{array}{c} \text{automorphic rep's of } d \times d \text{ matrices} \\ \text{with } \textit{different} \text{ extra structure} \end{array}\right\}$$

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Wrinkles: Special behavior of f_C got stuck into another representation. Different extra structures.

Examples of Langlands duals

left structure, G	SL_n	$SO(2n+1)$	$Spin(2n)$	$SO(2n)$	$SU(n)$	E_8	$\cdots$
right structure, $G^\vee$	PGL_n	$Sp(2n)$	$SO(2n)/Z$	$SO(2n)$	$SU(n)/Z$	E_8	$\cdots$

General formula: replace $\mathfrak{g}$ + root system by $\mathfrak{g}^\vee$ + coroots.

Geometric Langlands

Harmonic analysis (f)
Analytic NT (ζ)

Algebraic NT (Gal(K), $\prod_p \zeta_p$)
Rep'n Theory ($\rho, \bigotimes_p \rho_p$)

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Langlands duality in complex geometry

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Quantum field theory

Electromagnetic duality

$$\{(E, B) \text{ satisfying Maxwell's}\} \longleftrightarrow \{(B, -E) \text{ satisfying Maxwell's}\}$$

$$\text{electric monopole, charge } e \longleftrightarrow \text{magnetic monopole, charge } 1/e$$

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Nonabelian E-M duality (Montonen–Olive / Kapustin–Witten)

$$\left\{ \begin{array}{c} \mathcal{N} = 4 \text{ SUSY Yang–Mills} \\ \text{gauge group } G \\ \text{coupling } \tau = \theta/2\pi + 4\pi i/g^2 \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \mathcal{N} = 4 \text{ SUSY Yang–Mills} \\ \text{gauge group } G^\vee \\ \text{coupling } -1/n_G\tau \end{array} \right\}$$

In general: "S–duality"

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In general: “S–duality”

Physical number theory?

Irreducible Galois rep's are the “fundamental particles” of number theory.

Thank you!