OPERADIC KOSZUL DUALITY

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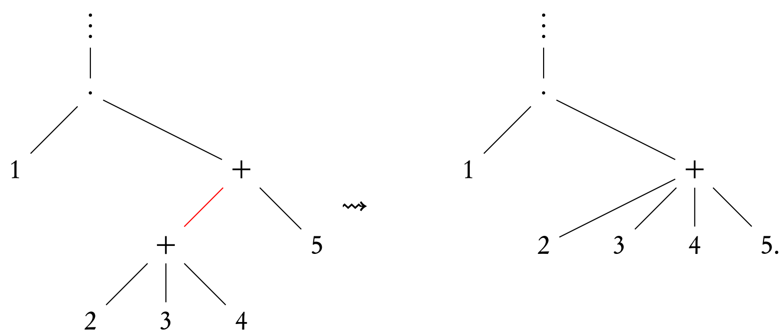
1. Operads

Our goal in this talk is to give a sort of categorified version of "Koszul duality". One of the primary motivations for us to do so is to take the classical results about Koszul duality, which form a convoluted and complex body of literature, and stretch them apart to see which assumptions power which parts of the theory. Classically, Koszul duality results are concerned chiefly with two cases: derived categories over modules and various forms of homotopy theory (e.g., rational nilpotent spaces). Our first goal will be to give a definition of "algebra" and "action" which encompass both of these situations.

The rough idea will be to take dreadfully seriously the notion of parse trees: the expression $1 \cdot ((2+3+4)+5)$ can be rendered graphically in a *parse tree* as



There is a root node representing the output of the computation; each leaf node contains some input from the base set $\mathbb{Z}$; and each internal vertex is labeled by some operation to be performed on the results of its child subtrees. Moreover, since the operation "+" is associative, we can contract the highlighted edge of the left tree to form the tree on the right. Altogether, these observations motivate "operads":

Definition. A *multicategory* C is a category where arrows are allowed to have more than one input: for every list of objects $a_1, \ldots, a_n \in \mathsf{C}$ and object $b \in \mathsf{C}$, we assign some collection of arrows $\mathsf{C}(a_1, \ldots, a_n; b)$. These come equipped with composition laws of the form

$$\mathsf{C}(b_1, \ldots, b_n; c) \otimes (\mathsf{C}(a_{1,1}, \ldots, a_{1,m_1}; b_1) \otimes \cdots \otimes \mathsf{C}(a_{n,1}, \ldots, a_{n,m_n}; b_n)) \to \mathsf{C}(a_{1,1}, \ldots, a_{n,m_n}; c),$$

corresponding to the concatenation of trees, and they satisfy generalizations of, e.g., the associativity axiom. An *operad* is a multicategory with one object.[1]

For an operad P, we abbreviate the Hom-object from n copies of the unique object to itself by $\mathsf{P}(n)$. These come with composition maps of the following form, corresponding to the contraction of trees:

$$\mathsf{P}(j) \otimes (\mathsf{P}(n_1) \otimes \cdots \otimes \mathsf{P}(n_j)) \to P(n_+).$$

If F is a set with $|F| = n$, we also write $\mathsf{P}(F)$ for this family; in fact, P can be taken to be a functor from finite sets with bijections to C, itself called a symmetric sequence, with these additional structure maps.

Remark. We will concern ourselves strictly with *reduced* operads, which are those for which $\mathsf{P}(1)$ is the monoidal unit of the enrichment category.

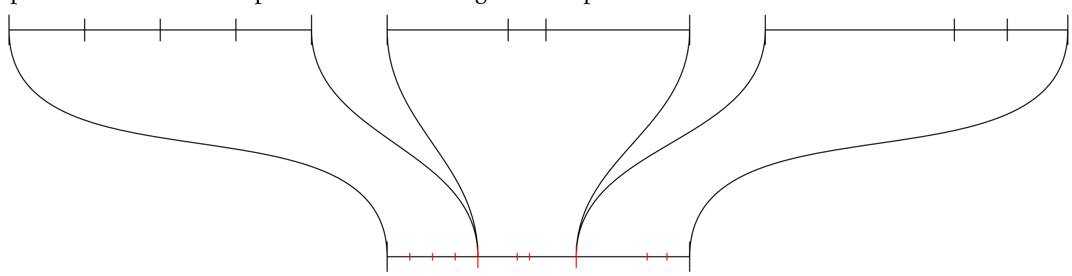
[1] Enriched multicategories and hence enriched operads can be similarly defined, and we will quietly use them.

Example. Now, for some examples, consider the following operads:

(1) Linear associative operad: Let $k\langle x_1,\dots,x_n\rangle$ denote the free associative k-algebra on generators $x_1, \dots, x_n$, and set $\mathsf{Ass}(n)$ to be the k-subspace spanned by monomials containing each variable exactly once. The operadic multiplication maps are given by expression substitution.
(2) Linear commutative operad: An operad Comm can be formed similarly, using instead the free commutative k-algebra $k[x_1,\dots,x_n]$.
(3) Linear Lie operad: An operad Lie can be formed similarly, using instead the free Lie algebra on generators $x_1,\dots,x_n$.
(4) Topological associative operad:[2] Let $\mathscr{A}_\infty(n)$ denote the configuration space of $(n-1)$ unequal (but indistinguishable) points on the unit interval $(0,1)$. The operadic product is given by "insertion"; here's an example portrait of the action of the map

$$\mathscr{A}_\infty(3)\otimes(\mathscr{A}_\infty(4)\otimes\mathscr{A}_\infty(3)\otimes\mathscr{A}_\infty(3))\to\mathscr{A}_\infty(10)$$

on a particular selection of points in these configuration spaces:



(5) Topological commutative operad: In general, an "$\mathscr{E}_\infty$ operad" is any operad for which $\mathscr{E}_\infty(n)$ is contractible (in analogy to $\mathsf{Comm}(n)=k$) and on which Σ_n acts freely — i.e., $\mathscr{E}_\infty(n)$ is a model for $E\Sigma_n$. Concrete models of such things can be given, such as the limit of May's "little cubes" operads, which generalize the configuration space model for $\mathscr{A}_\infty$ given above.

These operads give universal embodiments of these multiplicative structures in the following sense:

Definition. Associated to any object in a symmetric monoidal category C, there is an endomorphism operad $\mathsf{End}(A)$ whose n^{th} level is given by the mapping object

$$\mathsf{End}(A)(n)=\mathsf{C}(A^{\otimes n},A).$$

The structure of a P-*algebra* for such an object A is given by a map $\mathsf{P}\to\mathsf{End}(A)$, thought of as selecting the necessary family of multiplications inside of the endomorphisms of A.

One can quickly check that the classical notions of associative algebras, commutative algebras, and Lie algebras are exactly those algebras in k-modules for the associative, commutative, and Lie operads defined above. [3] Somewhat more exotically, the based loopspace $\Omega X=\mathrm{Maps}(S^1,X)$ forms an $\mathscr{A}_\infty$-algebra, where the multiplication maps concatenate lists of unit length loops, descaled according to the partition of the unit interval.

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[2]Caveat lector: this operad satisfies the unit axiom only up to homotopy.

[3]It's worth remarking that this simultaneous generalization to algebraic and topological settings is the point of working with operads. If we restrict ourselves to the k-linear setting, we're actually recovering nothing new, in the following precise sense. Given a P-algebra A, a module M for A can be defined as follows: there is an action operad consisting of maps $\mathsf{Act}(M;A)(n)=\mathsf{C}(M\otimes A^{\otimes(n-1)};M)$, and one requires a map $\mathsf{P}\to\mathsf{Act}(M;A)$ suitably extending the algebra structure map for A. Then, we can form the universal enveloping associative algebra $U(\mathsf{P},A)$, which is generated by symbols $X(\lambda;a_1,\dots,a_n)$ for $\lambda\in\mathsf{P}(n+1)$ and $a_1,\dots,a_n\in A$.[4] These are subject to a predictable list of relations so that the category of left $U(\mathsf{P},A)$-modules is equivalent to the category of operadic A-modules. To see one half of this equivalence, for such an operadic A-module M the symbol $X(\lambda;a_1,\dots,a_n)$ is made to act on $m\in M$ by the formula

$$X(\lambda;a_1,\dots,a_n)\cdot m=\lambda(a_1,\dots,a_n,m).$$

2. The operadic co/bar construction

Let's now proceed to the Koszul duality part of the talk. Previously, for a graded k-algebra A, we've defined its Koszul dual by considering the object $\mathrm{Tor}^A_{*,*}(k,k)$. This is defined on the level of chains by the simplicial bar construction $B(k;A;k)$, which is given by the coend formula

$$B(k;A;k) = \int^{\Delta^n \in \Delta} \Delta^n \otimes A^{\otimes n}.$$

This is interpreted to say that the bar complex is built from n-simplices labeled by n-tuples of elements of A, glued together according to some (implicit) rules. Our key idea is going to be to consider the points $0 \le t_1 \le \cdots \le t_n \le 1$ in the standard n-simplex as parametrizing branch lengths in the following rooted tree:

$$\text{root} \xrightarrow{\ t_1 - 0\ } a_1 \xrightarrow{\ t_2 - t_1\ } a_2 \xrightarrow{\qquad} \cdots \xrightarrow{\qquad} a_{n-1} \xrightarrow{\ t_n - t_{n-1}\ } a_n \xrightarrow{\ 1 - t_n\ } \text{leaf}.$$

The gluing maps are meant to handle length 0 edges (i.e., colliding vertices), and this is the picture we will generalize.

Suppose that we have an operad P in a category tensored over spaces, as well as left- and right-modules L and R for P. Fix a finite set F, and let $\mathsf{Trees}(F)$ denote the set of rooted trees T equipped with a surjection $\sigma : F \to \mathrm{Leaves}(T)$, subject to the condition that the root is the only vertex allowed to have one child. We label the leaves by points in the left-module L, the root by a point in the right-module R, and the internal vertices by points in the operad P. We also define a space $w(T)$ of edge weights, which are subject to the condition that the root is distance 1 from any leaf.

With these labels and lengths, our trees match how the partitioned line looks. Now we describe the gluing data, by putting a partial order on $\mathsf{Trees}(F)$ and stating how the labels transform:

(1) Root edge collapse: let e be a *root edge* of length 0, i.e., one of the endpoints of e is the root node. Then, we say that $T/e \le T$, where "/" denotes graph contraction. We relabel the root node using the right-action of P on R.
(2) Bud collapse: let e be a *leaf edge* of length 0, i.e., one of the endpoints of e is a leaf node. Let v denote the *other* node to which e is attached, and let $e_1, \ldots, e_k$ be an exhaustive list of the edges connecting v to a leaf node. Then, we say that $T/\{e_1, \ldots, e_k\} \le T$. We relabel the bud node using the left-action of P on L.
(3) Internal edge collapse: Let e be an *internal edge* of length 0, i.e., neither endpoint of e is a leaf or the root. Then, we say that $T/e \le T$, and we relabel the node using the operadic composition in P.

Using $i(v)$ to denote the child edges of a vertex v, we collect this data to define the *arboreal bar complex*:

$$B(\mathsf{R};\mathsf{P};\mathsf{L})(F) = \int^{T \in \mathsf{Trees}(F)} w(T)_+ \otimes \left(\mathsf{R}(i(\mathrm{root})) \otimes \bigotimes_{\text{internal vertices } v} \mathsf{P}(i(v)) \otimes \bigotimes_{\text{leaves } \ell} \mathsf{L}(\sigma^{-1}(\ell)) \right).$$

You'll note that the right- and left-modules play radically different roles in the arboreal bar construction. The role of the left-module is commonly filled as follows: if A is an algebra for the operad P, one can define a left-module by taking the constant value $\mathsf{L}(F) = A$ and letting P act by the algebra structure map. The role of the right-module is more confusing, and Fresse has a big, dense book about it; we will be content to take the unit operad $\mathsf{R} = \mathbf{1}$.

Example. For the three linear operads we've introduced, the operadic bar complex $B(\mathbf{1};\mathsf{P};A)$ computes the Hochschild homology of an associative algebra A, the Harrison homology of a commutative algebra A, and Chevalley–Eilenberg homology of a Lie algebra A. (This is not easy to verify now, but will follow from the Koszul resolutions we build in the next section. See Ginzburg–Kapranov's Theorem 4.2.4.)

2.1. **Co/bar duality.** There is actually a version of the operadic bar construction which looks near-identical to the usual bar construction: there is a sense in which operads are monoid-objects in the category of symmetric sequences, and the simplicial bar construction in that setting yields something weakly equivalent to the arboreal bar construction presented above. The benefit to doing things the arboreal way is that it visibly has the structure of a *cooperad*. Namely, for each partition $F = \coprod_{j\in J} F_j$ of F, we define as follows a map

$$B(\mathsf{R};\mathsf{P};\mathsf{L})(F) \to B(\mathsf{R};\mathsf{P};\mathbf{1})(J) \otimes \bigotimes_{j\in J} B(\mathbf{1};\mathsf{P};\mathsf{L})(F_j).$$

Select a tree T; if it can be "ungrafted" into a tree T_{root} and a sequence of tree $T_1, \ldots, T_j$ which graft to give T, then it does so uniquely. This gives us the value of the map on a particular point. If no such partitioning exists, then we map to the basepoint. Checking compatibilities, one sees that this rule determines precisely the desired map.

By taking $\mathsf{R} = \mathsf{L} = \mathbf{1}$, we find the structure of a cooperad on $B(\mathbf{1};\mathsf{P};\mathbf{1})$. By taking $\mathsf{R} = \mathbf{1}$ and $\mathsf{L} = A$ for a P-algebra A, this produces the structure of a left $B(\mathbf{1};\mathsf{P};\mathbf{1})$-comodule structure on $B(\mathbf{1};\mathsf{P};\mathsf{L})$. This can also all be done dually, using an appropriately defined end to construct a cobar complex[5] $\Omega(\mathsf{L};\mathsf{Q};\mathsf{R})$ for a cooperad Q. It carries a map

$$\Omega(\mathsf{L};\mathsf{Q};\mathbf{1})(J) \otimes \bigotimes_{j\in J} \Omega(\mathbf{1};\mathsf{Q};\mathsf{L})(F_j) \to \Omega(\mathsf{R};\mathsf{Q};\mathsf{L})(F).$$

Meta-Theorem (Co/bar duality)**.** *Let* $\mathbf{1}$ *denote the symmetric monoidal unit of a self-enriched category, suppose it is the target of a monoidal functor from spaces, and let* $DX = F(X, \mathbf{1})$ *denote the internal Spanier–Whitehead dualizing functor. Then, for operads which are levelwise dualizable and which satisfy* $D(\mathsf{P}(n) \wedge \mathsf{P}(m)) \simeq D\mathsf{P}(n) \wedge D\mathsf{P}(m)$*, there is a natural equivalence of operads*

$$K(\mathsf{P})(n) := D(B(\mathbf{1};\mathsf{P};\mathbf{1})(n)) \simeq \Omega(\mathbf{1}; D\mathsf{P}; \mathbf{1})(n).$$

It turns out not to be easy to prove such theorems in bulk, but there is such a theorem for operads in the dg-category of chain complexes and in the self-enriched category of spectra.[6]

3. The algebraic case: Koszulality as efficiency

The arboreal bar construction of an operad P comes with a natural filtration, analogous to the filtration of the bar construction of a graded algebra by internal degree: it is filtered by the number of internal vertices to the trees. We will specialize to levelwise finite rank operads in chain complexes, so that we have access to co/bar duality.

Definition. The operad P is said to be *Koszul* if the internal vertex and leaf filtrations agree homologically, i.e., if the internal vertex filtration spectral sequence is concentrated along the diagonal.

This definition has three immediate consequences:

1. The filtration spectral sequence must collapse at E_2, as the diagonal is not capable of self-interacting.
2. This same filtration shows that these homology groups form a chain complex which can be used to compute the homology of a general two-sided bar complex.[7]
3. This also enforces the *quadradicity* of the operad: it is freely generated by some binary operations, modulo an ideal generated by ternary operations. Using a generators-relations-and-syzygies description of homology, one can see that the operad is generated by $\mathsf{P}(2)$ if and only if the Koszul condition holds for first homology, and under sufficient flatness hypotheses it is furthermore quadratic if the Koszul condition holds for second homology.

Example. The operads Ass, Lie, and Comm are all Koszul; this follows from classical results concerning the vanishing of Hochschild, Chevalley–Eilenberg, and Harrison homologies of free algebras.[8]

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[5]The notation "Ω" is justified, from the standpoint of a topologist: the based loopspace of a space X can be taken to be the cobar complex on X with the diagonal map $X \to X \times X$ used as its coassociative comultiplication.

[6]It ought to be the case that the operadic co/bar constructions can be used to build the functors witnessing the equivalence of the derived categories of co/modules for dual co/algebras, but I've been unable to figure this out precisely. Word on the street is that the right thing to do is to consider a module for an operadic algebra as a sort of square-zero extension of the algebra, pass through the arboreal construction, and try to identify what comes out on the far side as some other kind of co-square co-zero co-extension. Gosh.

[7]This agrees with the complex defined in 4.2.1 of Ginzburg–Kapranov, though we have defined it slightly differently.

[8]See (the easy part of) Ginzburg–Kapranov's Theorem 4.2.5.

Example. In particular, this means that these three operads are all quadratic.[9] To be more specific, they are specified by the following quadratic data:

- Ass: $E = k^2$, with Σ_2 acting freely. The ideal R is spanned by the associators $x_i(x_j x_k) - (x_i x_j)x_k$.[10]
- Comm: $E = k$, with Σ_2 acting trivially. The ideal R is the 2-dimensional subspace spanned by the shuffles $x_i(x_j x_k) - x_k(x_i x_j)$. This carries the 2-dimensional irreducible Σ_3-representation.
- Lie: $E = k$, with Σ_2 acting by sign. The ideal R is the 1-dimensional subspace spanned by $[x_1,[x_2,x_3]] + [x_3,[x_1,x_2]] + [x_2,[x_3,x_1]]$, corresponding to the Jacobi identity. It carries the sign Σ_3-representation.

Finally, given that a Koszul operad can be specified by quadratic data, it is natural to ask for a similar expression of its Koszul dual, rather than having to pass through the whole mess of the arboreal bar construction every time. Some calculation in the first and second homology gives the following description:

Theorem. *Suppose that the Koszul operad* P *is specified by the quadratic data* (E,R), *where* E *is a finite-dimensional* k-$k^{\otimes 2}$*-bimodule and* R *is a* Σ_3*-invariant ideal in* $\mathsf{Free}(E)(3)$. *Then, the Koszul dual* $K(\mathsf{P})$ *is specified by the quadratic data* $(E^\vee, R^\perp)$, *where* $E^\vee$ *is the* k*-linear dual of* E *and* $R^\perp$ *is the subspace of* $\mathsf{Free}(E^\vee)(3) = (\mathsf{Free}(E)(3))^\vee$ *perpendicular to* R *under the natural* k*-pairing.*

Theorem. $K(\mathsf{Ass}) \cong \mathsf{Ass}$, $K(\mathsf{Comm}) \cong \mathsf{Lie}$, *and* $K(\mathsf{Lie}) \cong \mathsf{Comm}$.

Proof. This follows from calculating the orthogonal subspaces $R^\perp$ in the above examples. □

[9] While we're on the subject of quadratic data, there is also a quadratic linear operad Pois whose algebras are Poisson algebras. I don't think it can be formed using free algebras as done far above, but you can generate it along the lines of the examples that follow this footnote. There is also a kind of topological Lie operad: it is the Koszul dual of the operad in spectra which is constant at the sphere spectrum. Its ordinary homology ("homology" here is meant as spectra, not as operads in Spectra) is equivalent to the Lie operad in chain complexes, which motivates the description.

[10] One can introduce a scalar product so that all distinct monomials are mutually orthogonal, the left-associated ones pair with themselves to 1, and the right-associated ones pair with themselves to -1; then one quickly calculates that R is its own annihilator, and hence $K(\mathsf{Ass}) \cong \mathsf{Ass}$.