Two Qubit Circuits and the Monodromy Polytope

Int'l Workshop on Quantum Compilation
November 7th, 2019 in Westminster, CO
Which programs can be encoded efficiently onto a given target?

- Topological constraints
- Expression of the input program
- Target instruction set

- Execution characteristics, e.g. fidelity
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### Gate Matrices

- **CZ**
  \[
  \begin{pmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & -1 \\
  \end{pmatrix}
  \]

- **CX**
  \[
  \begin{pmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 1 \\
  0 & 0 & 1 & 0 \\
  \end{pmatrix}
  \]

- **CPHASE**
  \[
  \begin{pmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & e^{i\varphi} \\
  \end{pmatrix}
  \]

- **ISWAP**
  \[
  \begin{pmatrix}
  1 & 0 & 0 & 0 \\
  0 & 0 & i & 0 \\
  0 & -i & 0 & 0 \\
  0 & 0 & 0 & 1 \\
  \end{pmatrix}
  \]

- **XY**
  \[
  \begin{pmatrix}
  1 & 0 & 0 & 0 \\
  0 & \cos \varphi & -i \sin \varphi & 0 \\
  0 & -i \sin \varphi & \cos \varphi & 0 \\
  0 & 0 & 0 & 1 \\
  \end{pmatrix}
  \]

- **fSim**
  \[
  \begin{pmatrix}
  1 & 0 & 0 & 0 \\
  0 & \cos \varphi & -i \sin \varphi & 0 \\
  0 & -i \sin \varphi & \cos \varphi & 0 \\
  0 & 0 & 0 & e^{i\theta} \\
  \end{pmatrix}
  \]

- **B**
  \[
  \begin{pmatrix}
  1 & 0 & 0 & -i \\
  0 & -1 & -i & 0 \\
  0 & -i & -1 & 0 \\
  -i & 0 & 0 & 1 \\
  \end{pmatrix}
  \]

- **MS**
  \[
  \begin{pmatrix}
  1 & 0 & 0 & i \\
  0 & 1 & -i & 0 \\
  0 & -i & 1 & 0 \\
  i & 0 & 0 & 1 \\
  \end{pmatrix}
  \]
Cartan decomposition:

- Any 2Q gate $U$ can be written as $U = K_1 A K_2$.
- $K_1$ and $K_2$ are local, $A$ "feels diagonal": commute, determined by eigenvalues, ... .
- $U, V$ satisfy $U = K V K'$ exactly when $U, V$ have the same $A$ component.
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- $U$, $V$ satisfy $U = K V K'$ exactly when $U$, $V$ have the same $A$ component.

Kraus-Cirac: “Draw (the logarithms of) the eigenvalues of $A$”, as in
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**Equivalent question:** How does one rewrite AKA as KAK?
1Q analogue: How do YZY-Euler decompositions multiply?

For a, c known and b unknown, what can be said about e in

\[ RZ(a) \ RY(b) \ RZ(c) = RY(d) \ RZ(e) \ RY(f) \]
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For $a$, $c$ known and $b$ unknown, what can be said about $e$ in

$$RZ(a) \ RY(b) \ RZ(c) = RY(d) \ RZ(e) \ RY(f) ?$$

$$\cos(e) = \cos(a) \cos(c) - \cos(b) \sin(a) \sin(c).$$

Observations: $e$ is nonlinear as a function of $b$.

The main trick is that 1Q operators have few eigenvalues.
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$$\cos(e) = \cos(a) \cos(c) - \cos(b) \sin(a) \sin(c),$$

or,

$$|a - c| \leq e \leq \pi - |a + c - \pi|.$$

Observations: $e$ is nonlinear as a function of $b$.

The “allowable $e$’s form a line segment.

The main trick is that 1Q operators have few eigenvalues.
**Theorem:** The set of 2Q programs of the form $K_1 \text{CX} K_2 \text{CX} K_3$ is the triangle connecting I, CZ, and ISWAP.

**Observations:** The set is again a “convex body”. The main trick is that CX has a simple formula.

(The relationship between $K_2$ and A actually is linear—a happy accident.)
Theorem, abbrev.: Fix $A_1, A_2$, and consider the set of $A_3$ satisfying
$$A_1 K A_2 = K A_3 K .$$
In the Kraus-Cirac picture, it becomes the union of two convex polytopes.

Proof: Nonabelian Yang-Mills, Riemann surface, symplectic reduction, principal $G$-bundle with $g$-valued
connection, monodromy, moment map, moduli of curves, parabolic bundle, semistability, Grassmannian,
Schubert classes, quantum cohomology ring, intersection form, Gromov-Witten invariants, ...
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Observations: Proof is hard, but the result can be wielded by a computer. $A_1$ and $A_2$ can range over polytopes themselves, still OK. Iterable: describe the space of circuits of depth 3, 4, 5, ... . Does not produce circuit decompositions.
**Theorem:** The set of 2Q programs using 2 ISWAPs is the same as with CX. The set of 2Q programs using 3 ISWAPs is the same as with CX.

**Observations:** No substantial difference for efficient compilation.

ISWAP is simple enough that one can extract circuits.
**Theorem:** The set of 2Q programs using 2 CPHASEs is the same as with CX. The set of 2Q programs using 3 CPHASEs is the same as with CX.

**Observations:** No substantial difference for efficient compilation!! Don’t bother extracting CPHASE circuits—just use CZ.
Theorem: The set of 2Q programs using $3, 4, 5$ $\sqrt{CZ}$s is as in the picture:

Observations: This feels “worse” than CZ, which needs $\leq 3$ applications.
Theorem: The set of 2Q programs using $3, 4, 5\sqrt{CZ}$s is as in the picture:

Observations: This feels “worse” than CZ, which needs $\leq 3$ applications. Motivates expected depth: $\langle \sqrt{CZ} \text{ depth} \rangle = 3.60416$, versus $\langle CZ \text{ depth} \rangle = \langle ISWAP \text{ depth} \rangle = \langle CPHASE \text{ depth} \rangle = 3$. 
Theorem: The set of 2Q programs using 2 XYs is as in the picture:

Observations: $\langle XY \text{ depth} \rangle = 2.16$. Substantial improvement over CZ.
Theorem: The set of 2Q programs using 2 XY(3π/4)s is as in the picture:

Observations: Looks like the XY figure, but with some corners trimmed. \( \langle \text{XY depth} \rangle < \langle \text{XY(3π/4) depth} \rangle = 2.25 < \langle \text{CZ depth} \rangle. \)
Open questions / further directions

**Approximations:** A factor determines the average-case-best approximation. For CZ, the best approximation is simple to compute. General closed formula?

**Decompositions:** We know only *when* there is a decomposition. For CZ, one can easily build circuits. For ISWAP, less easily. In general, even numerical methods look interesting.

**Errors:** Depth 2 polytope is not flat ⇒ local errors become nonlocal.

**Many-body:** This analysis can’t be obviously extended to ≥3Q. Recursive decompositions of ≥3Q into 2Q specifically use CNOT.
Thank You!

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