Spectra and $G$–spectra

Eric Peterson

September 23, 2015

http://math.harvard.edu/~ecp/latex/talks/intro-to-spectra.pdf
A *cell structure* on a pointed space $X$ is an inductive presentation by iteratively attaching $n$–disks:

$$
\begin{array}{c}
\bigvee S^{n-1} \longrightarrow X^{(n-1)} \\
\downarrow \\
\bigvee D^n \longrightarrow X^{(n)}.
\end{array}
$$

Suspension $\Sigma$ is an operation on spaces which preserves gluing squares, and $\Sigma S^{n-1} \simeq S^n$ and $\Sigma D^n \simeq D^{n+1}$. So, $\Sigma$ is a "shift operator" on cell structures.

**Theorem ("Stability")**

$$
H_n(X; A) \sim = H_{n+1}(\Sigma X; A),
$$

$$
\Sigma H^*(X; A) \sim = H^*(\Sigma X; A).
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Cell structures

Definition

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Suspension $\Sigma$ is an operation on spaces which preserves gluing squares, and $\Sigma S^{n-1} \simeq S^n$ and $\Sigma D^n \simeq D^{n+1}$. So, $\Sigma$ is a “shift operator” on cell structures.

Theorem (“Stability”)

$$H^n(X; A) \cong H^{n+1}(\Sigma X; A),$$

$$\Sigma H^*(X; A) \cong H^*(\Sigma X; A).$$
Suspension: Freudenthal’s theorem

Calculation: $\pi_\ast$ of a suspension

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Theorem (Freudenthal)

- $X$: $s$–connected space ($\pi_{*\leq s} X = 0$)
- $Y$: $t$–dimensional space (no cells above dimension $t$)

Then

$$F(Y, X) \to F(\Sigma Y, \Sigma X)$$

is a $(2s - t)$–equivalence.
**Suspension: Freudenthal’s theorem**

**Calculation: \( \pi_* \) of a suspension**

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**Theorem (Freudenthal)**

- **\( X \):** \( s \)-connected space \( (\pi_{\leq s} X = 0) \)
- **\( Y \):** \( t \)-dimensional space (no cells above dimension \( t \))

Then

\[
F(Y, X) \to F(\Sigma Y, \Sigma X)
\]

is a \((2s - t)\)-equivalence.

**Corollary**

The 2 matters: \( \pi_n F(\Sigma^m Y, \Sigma^m X) \) is independent of \( m \gg n \).
Definition

Call “$\Sigma^\infty X$” the suspension spectrum of $X$.

$$[\Sigma^\infty Y, \Sigma^\infty X] = \text{colim}_m [\Sigma^m Y, \Sigma^m X]$$
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$$[\Sigma^\infty Y, \Sigma^\infty X] = \operatorname{colim}_m[\Sigma^m Y, \Sigma^m X]$$
$$= \operatorname{colim}_m[Y, \Omega^m \Sigma^m X]$$
$$= [Y, \operatorname{colim}_m \Omega^m \Sigma^m X] =: [Y, QX].$$
Suspension spectra

**Definition**

Call "\(\Sigma^\infty X\)" the suspension spectrum of \(X\).

\[
\begin{align*}
[\Sigma^\infty Y, \Sigma^\infty X] &= \colim_m [\Sigma^m Y, \Sigma^m X] \\
&= \colim_m [Y, \Omega^m \Sigma^m X] \\
&= [Y, \colim_m \Omega^m \Sigma^m X] =: [Y, QX].
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Good news: stable invariants

\[ \pi_* \Sigma^\infty X = [\Sigma^\infty S^*, \Sigma^\infty X] \] is a stable invariant of \( X \).
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On the other side, the sequence \( Q \Sigma^* X \) represents a stable functor. This is because \( Q \Sigma X \) deloops \( QX \): \( \Omega(Q \Sigma X) = QX \). Hence,

\[ [\Sigma Y, Q \Sigma^* X] = [Y, \Omega Q \Sigma^* X] = [Y, Q \Sigma^{*-1} X]. \]
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\[ [\Sigma Y, Q\Sigma^* X] = [Y, \Omega Q\Sigma^* X] = [Y, Q\Sigma^{*-1} X]. \]

Bad news: not all stable invariants

\( K(A, n) \) represents a stable functor too:

\[ [Y, K(A, n)] = H^n(Y; A). \]

\( K(A, n + 1) \) deloops \( K(A, n) \), but \( K(A, n) \neq QX \) for any \( X \).
The Eilenberg–Mac Lane spectrum

\[ \pi_\ast \Sigma^\infty K(A, n) = \begin{cases} 
A & \text{if } \ast = n, \\
0 & \text{if } \ast \leq 2n, \ast \neq n, \\
\text{mystery groups} & \text{if } \ast > 2n.
\end{cases} \]

So, “\( \text{colim}_n \Sigma^{-n} \Sigma^\infty K(A, n) \)” has the right homotopy groups.
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So, “\( \text{colim}_n \Sigma^{-n} \Sigma^\infty K(A, n) \)” has the right homotopy groups.

**Definition (Boardman, more or less)**

A *spectrum* is an ind-diagram of things like \( \Sigma^{-n} \Sigma^\infty X \).

The Eilenberg–Mac Lane spectrum is presented by the ind-system

\[ HA := \{ \Sigma^{-n} \Sigma^\infty K(A, n) \}. \]
Theorem (Boardman)
The smash product \( \wedge \) lifts from spaces to spectra:

\[
\{ \Sigma^{n_\alpha} \Sigma^\infty X_\alpha \} \wedge \{ \Sigma^{m_\beta} \Sigma^\infty Y_\beta \} =: \{ \Sigma^{n_\alpha + m_\beta} \Sigma^\infty (X_\alpha \wedge Y_\beta) \}.
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It has an adjoint, the function spectrum: \([Z \wedge Y, X] \simeq [Z, X^Y]\).
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Theorem

\[
X \mapsto \pi_\ast (HA \wedge \Sigma^\infty X) \quad \text{and} \quad X \mapsto \pi_{-\ast} (HA \Sigma^\infty X)
\]

satisfy the axioms of ordinary (co)homology with $A$ coefficients.
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Theorem

$$X \mapsto \pi_*(HA \wedge \Sigma^\infty X) \quad \text{and} \quad X \mapsto \pi_{-*}(HA \Sigma^\infty X)$$

satisfy the axioms of ordinary (co)homology with $A$ coefficients.

Theorem (Brown, Atiyah)

For $E_*(-)$ and $E^*(-)$ generalized (co)homology theories, there is a spectrum $E$ such that

$$\tilde{E}_*(X) \cong \pi_*(E \wedge \Sigma^\infty X) \quad \text{and} \quad \tilde{E}^*(X) = \pi_{-*}(E \Sigma^\infty X).$$
Moral

Spectra are an enrichment of homology theories where homotopy theory can be done.
Gluing homology theories

Moral
Spectra are an enrichment of homology theories where homotopy theory can be done.

Example: Quotient sequences
The quotient sequence $S \rightarrow S \rightarrow S/2$ induces an exact sequence

$$0 \rightarrow \pi_2 S \rightarrow \pi_2 S/2 \rightarrow \pi_1 S \rightarrow 0$$

$$0 \rightarrow \mathbb{Z}/2 \rightarrow \mathbb{Z}/4 \rightarrow \mathbb{Z}/2 \rightarrow 0.$$

Spectra guarantee that these problems have consistent solutions.
Ring-valued cohomology theories induce multiplication maps on representing spectra. In homotopy theory, associativity is a structure rather than a property:

\[(a \circ (b \circ c)) \rightarrow ((a \circ b) \circ c)\]
Ring-valued cohomology theories induce multiplication maps on representing spectra. In homotopy theory, associativity is a structure rather than a property:

$$(a \circ (b \circ c)) \rightarrow ((a \circ b) \circ c) \rightsquigarrow S^0 \rightarrow F(E \wedge^3, E)$$
Ring spectra

Ring-valued cohomology theories induce multiplication maps on representing spectra. In homotopy theory, associativity is a structure rather than a property:

$$(a \circ (b \circ c)) \rightarrow ((a \circ b) \circ c) \quad \xrightarrow{\sim} \quad S^0 \rightarrow F(E^\wedge 3, E)$$

$$((a \circ b) \circ c) \circ d \rightarrow (a \circ b) \circ (c \circ d)$$

$$\downarrow$$

$$(a \circ (b \circ c)) \circ d$$

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Ring-valued cohomology theories induce multiplication maps on representing spectra. In homotopy theory, associativity is a structure rather than a property:

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Leads to quasicategories and \(A_\infty\)-rings ("coherently associative"). It pays off: \(A_\infty\)-rings have a good theory of modules, ...
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Leads to quasicategories and $A_\infty$–rings ("coherently associative"). It pays off: $A_\infty$–rings have a good theory of modules, ...
Theorem (Atiyah–Hirzebruch)

Let $E$ be a generalized homology theory and $X$ a cellular space.

$$E_{p,q}^1 = C_p^{\text{cell}}(X; E_q) \Rightarrow E_{p+q}X.$$
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A cell structure suspends to a presentation of $\Sigma^\infty X$ by shifts of wedges of $\mathbb{S}$. Applying $E \wedge -$ to these diagrams give a presentation of $E \wedge \Sigma^\infty X$ by shifts of wedges of $E$. 

For $E = HA$, there is a sense in which $HA \wedge \Sigma^\infty X \simeq C^* (X; A)$. 

$E \wedge \Sigma^\infty X \leftrightarrow \text{"}E\text{"-chains on }X\text{"}$. 

In good cases, this is "base change" from $\mathbb{S}$ to $E$. 

Eric Peterson Spectra and $G$-spectra
**Theorem (Atiyah–Hirzebruch)**

Let $E$ be a generalized homology theory and $X$ a cellular space.

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**Generalized cellular chains**

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**Spectra and $G$–spectra**

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**Eric Peterson**
Intermission
Where spaces had points, $G$–spaces have orbits:

$$G/H \xrightarrow{\text{equivariant}} X.$$ 

Different choices of $H \leq G$ stratify the space:

$$G/H \mapsto F_G(G/H_+, X) = X^H.$$
Where spaces had points, $G$–spaces have orbits:

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**Definitions**

$$\pi_n(X) : G/H \mapsto [G/H_+ \wedge S^n, X]_G = \pi_nX^H$$

A weak equivalence of $G$–spaces is a $G$–map which is a $\pi_*$–iso. That is, for each $H$

$$\pi_*X^H \xrightarrow{\sim} \pi_*Y^H.$$
Definition

A \textit{G–cell structure} on a pointed \(G\)–space \(X\) is a presentation by iteratively attaching \(n\)–disks of the form \(G/H_+ \wedge D^n\) along images of \(G/H_+ \wedge S^{n−1}\).
Equivariant obstruction theory

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We would like a cohomology theory that controls the obstructions to extending maps of $G$–cell complexes across a new cell, analogous to the role of ordinary cohomology.
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\[
C^n(X; M) : G/H \mapsto \text{Hom}(H_n((X^H)^n, (X^H)^{n-1}), M(G/H)).
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Satisfies the “obvious” Eilenberg–Steenrod axioms.
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Satisfies the “obvious” Eilenberg–Steenrod axioms.

**Warning**

This works, but it’s not great. No Poincaré duality, for instance.
Sphere could also mean $S^V := V^+$ for $V$ a $G$–representation. Spheres grade cohomology theories: $S^n \leftrightarrow H^n$.

When can a representation be put in for $\ast$ in $H^\ast(X; M)$?
Question

Sphere could also mean $S^V := V^+$ for $V$ a $G$–representation. Spheres grade cohomology theories: $S^n \sim H^n$. When can a representation be put in for $\ast$ in $H^\ast(X; M)$?

Answer

Exactly when $M$ is a Mackey functor:

for any $G$–map $f : G/H \to G/K$
we choose a “transfer map” $t(f) : M(G/H) \to M(G/K)$
satisfying a “double coset formula” reminiscent of character theory. (The definition is set up so that $G/H \leftrightarrow \text{Rep}(H)$ fits.)
### Question

Sphere could also mean \( S^V := V^+ \) for \( V \) a \( G \)-representation. Spheres grade cohomology theories: \( S^n \leftrightarrow H^n \).

When can a representation be put in for \( \ast \) in \( H^*(X; M) \)?)

### Answer

Exactly when \( M \) is a Mackey functor:

For any \( G \)-map \( f : G/H \to G/K \), we choose a “transfer map” \( t(f) : M(G/H) \to M(G/K) \) satisfying a “double coset formula” reminiscent of character theory. (The definition is set up so that \( G/H \leftrightarrow \text{Rep}(H) \) fits.)

These are great: Poincaré duality and everything else you could hope for.
Define suspension $G$–spectra by

$$[\Sigma^\infty_G Y, \Sigma^\infty_G X]_G = [Y, \text{colim}_V \Omega^V \Sigma^V X]_G.$$ 

Equivariant Freudenthal says this colimit is degenerate. $G$–spectra are ind-systems of $S^V$–desuspensions of suspension $G$–spectra.

Theorem, redux

For $E^\star(\ast)$ and $E^\ast(\ast)$ Bredon (co)homology theories (i.e., $\star$ is allowed), there is a $G$–spectrum $\tilde{E}$ such that $\tilde{E}^\star(X) \sim \pi^\star(E \wedge \Sigma^\infty_G X)$ and $\tilde{E}^\ast(X) = \pi^\ast(E \Sigma^\infty_G X)$.

Theorem, redux

For any Mackey functor $M$, there is an Eilenberg–Mac Lane $G$–spectrum $HM$ presenting Bredon cohomology $H^\star(\ast; M)$. 

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Spectra and $G$–spectra
Definitions, redux

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For $E_\ast(-)$ and $E^\ast(-)$ generalized Bredon (co)homology theories (i.e., $\ast = V$ is allowed), there is a $G$–spectrum $E$ such that

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Definitions, redux

Define suspension $G$–spectra by

$$[\Sigma_{G}^{\infty} Y, \Sigma_{G}^{\infty} X]_{G} = [Y, \text{colim}_{V} \Omega^{V} \Sigma^{V} X]_{G}.$$ 

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Theorem, redux

For any Mackey functor $M$, there is an Eilenberg–Mac Lane $G$–spectrum $HM$ presenting Bredon cohomology $H^{\ast}(-; M)$. 

Eric Peterson | Spectra and $G$–spectra
Stable fixed points

We built $G$–spaces so that they carry fixed point data: “$X^H$”. This splits into three notions of fixed points for $G$–spectra:

- **Geometric:**
  \[
  \Phi^H(\Sigma^\infty G X) = \Sigma^\infty X^H, \\
  \Phi^H(\text{colim}_\alpha \{X_\alpha\}) = \text{colim}_\alpha \{\Phi^H X_\alpha\}, \\
  \Phi^H(X \wedge Y) = \Phi^H(X) \wedge \Phi^H(Y).
  \]

- **Categorical:**
  \[
  [E, X^H] = [E, X]_H, \quad \pi_n(X) : G/H \mapsto \pi_n X^H.
  \]

- **Homotopical:**
  \[
  X^{hH} = F_H(EH_+, X).
  \]

There is a map of fiber sequences if $H = \mathbb{C}p^q$.
Stable fixed points

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  \Phi^H(\Sigma_\infty^G X) = \Sigma_\infty^X^H, \\
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  [E, X^H] = [E, X]^H, \quad \pi_n(X) : G/H \mapsto \pi_n X^H.
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  \[
  X^{hH} = F_H(EH_+, X).
  \]

There is a map of fiber sequences

\[
\begin{array}{ccc}
? & \longrightarrow & X^H \\
\downarrow & & \downarrow \\
X_{hH} & \longrightarrow & X^{hH}
\end{array}
\]

\[
\begin{array}{ccc}
& & \longrightarrow & \Phi^H(X) \\
& & \downarrow & \\
& & \downarrow \\
& & \Phi^H(X) & \longrightarrow \pi_n X^H
\end{array}
\]

Generally, this is the best we can say.
Stable fixed points

We built $G$–spaces so that they carry fixed point data: \( \chi^H \). This splits into three notions of fixed points for $G$–spectra:

- **Geometric:**
  \[
  \Phi^H(\Sigma_{\infty} G \chi) = \Sigma_{\infty} \chi^H,
  \Phi^H(\text{colim}_{\alpha}\{\chi_{\alpha}\}) = \text{colim}_{\alpha}\{\Phi^H \chi_{\alpha}\},
  \Phi^H(\chi \wedge Y) = \Phi^H(\chi) \wedge \Phi^H(Y).
  \]

- **Categorical:**
  \[
  [E, \chi^H] = [E, \chi]^H, \quad \pi_n(\chi) : G/H \mapsto \pi_n \chi^H.
  \]

- **Homotopical:**
  \[
  \chi^{hH} = F_H(EH_+, \chi).
  \]

There is a map of fiber sequences

\[
\begin{array}{c}
X^{hH} \text{ if } H = C_p \\
\downarrow \\
“\text{homotopy orbits”} \\
X^{hH} \xrightarrow{\text{“transfer”}} \chi^{hH} \xrightarrow{\Phi^H} \chi^H \\
\downarrow \\
X^{tH}.
\end{array}
\]

Generally, this is the best we can say.
$KU$ exists as a $C_2$–spectrum with action by complex conjugation.

$\begin{array}{ccc}
? & \rightarrow & X^H \\
\downarrow & & \downarrow \\
X_{hH} & \rightarrow & X^{hH} \\
\end{array}$

$\begin{array}{ccc}
\Phi^H(X) & \rightarrow & \Phi^H(X) \\
\downarrow & & \downarrow \\
X^{tH} & \rightarrow & \Phi^H(X) \\
\end{array}$

$X = KU$  \hspace{1cm} H = C_2$

$\begin{array}{ccc}
KO & \rightarrow & KO \\
\downarrow & & \downarrow \\
KO & \rightarrow & KO \\
\end{array}$
\( KO \simeq KU^{hC_2} \)

\( KU \) exists as a \( C_2 \)-spectrum with action by complex conjugation.

\[
\begin{array}{c}
? \\ \downarrow \\
X_h^H \\ \downarrow \\
X_{hH} \\
\end{array} \quad \begin{array}{c}
\longrightarrow \quad X^H \\ \longrightarrow \\
\Phi^H(X) \\
\downarrow \\
X^{hH} \\
\downarrow \\
X^{tH}.
\end{array} \quad \begin{array}{c}
KO \\ \longrightarrow \\
KO \\ \longrightarrow \\
\ast.
\end{array}
\]

\( X = KU \)

\( H = C_2 \)

\( KO \) \( KO \) \( KO \) \( KO \) \( KO \) \( KO \) \( KO \) \( KO \) \( KO \) \ast.

**Homotopy fixed point spectral sequence:**

\( H^*_{gp}(C_2; \pi_* KU) \Rightarrow \pi_* KO \)
Slice spectral sequence (Dugger)

You can also get the homotopy groups as Mackey functors.
Theorem (McCarthy)

Let $f : R \to S$ be a surjection of rings with nilpotent kernel. Then there is a pullback square

$$
\begin{array}{ccc}
K(R)^\wedge_p & \xrightarrow{\text{"trace"}} & TC(R)^\wedge_p \\
\downarrow & & \downarrow \\
K(S)^\wedge_p & \xrightarrow{\text{"trace"}} & TC(S)^\wedge_p,
\end{array}
$$

where

$$
TC(R) = \text{fib} \left( \lim_{n \to \infty} THH(R)_{C_p^n} \xrightarrow{R-\text{id}} \lim_{n \to \infty} THH(R)_{C_p^n} \right)
$$

and $THH$ is the subject of this (and the Thursday) seminar.
Theorem (McCarthy)

Let $f : R \to S$ be a surjection of rings with nilpotent kernel. Then there is a pullback square

$$
\begin{array}{ccc}
K(R)_p & \xrightarrow{\text{"trace"}} & TC(R)_p \\
\downarrow & & \downarrow \\
K(S)_p & \xrightarrow{\text{"trace"}} & TC(S)_p,
\end{array}
$$

where

$$
TC(R) = \text{fib} \left( \lim_{n \to \infty} THH(R)^{C_{p^n}} \xrightarrow{R - \text{id}} \lim_{n \to \infty} THH(R)^{C_{p^n}} \right)
$$

and $THH$ is the subject of this (and the Thursday) seminar.

There are lots of theorems along these lines, relating equivariant structure on $THH$ to sundry things in algebraic $K$–theory.