COTANGENT SUMS AND THE G-SIGNATURE THEOREM

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AFTER HIRZEBRUCH-ZAGIER

At the first *Many Cheerful Facts* talk this semester, Ben McMillan gave an overview of differential geometry by explaining curvature, ending with the Gauss-Bonnet theorem. An audience member asked whether this had any ramifications by allowing certain integrals to be computed that had been previously unaccomplishable, and no one had any examples for him. To construct a manifold with a prescribed curvature form while maintaining enough control to read off its genus some other way — what a thought! As it turns out, this is exactly the goal of the Hirzebruch-Zagier manuscript, which we'll give a light-hearted tour of here.

1. G-signature theorem

First, let's recall the nonequivariant signature theorem. Define the $\mathcal{L}$-class of a complex vector bundle using the Hirzebruch formalism discussed in class and the power series $\frac{x}{\tanh x}$. The signature theorem then asserts an equality

$$\operatorname{Sign}(X) = \langle \mathcal{L}(X), [X] \rangle,$$

where $\mathcal{L}(X) = \mathcal{L}(TX \otimes_{\mathbb{R}} \mathbb{C})$ and $\operatorname{Sign}(X) = p_+ - p_-$ is the number of positively signed eigenvalues minus the negatively signed eigenvalues of the operator $B(x, y) = \langle x \smile y, [X] \rangle$ defined on the middle cohomology of X (relative to the boundary, if X has one). By Poincaré duality, this operator B is "the same" as the intersection form on the homology of X.

Now, we wish to take into account an orientable action of a compact Lie group G on X. For any $g \in G$, we can consider the submanifold X^g of points fixed by g; this submanifold has a normal bundle N^g in X which splits as a sum

$$N^g = N^g_\pi \oplus \bigoplus_{0<\theta<\pi} N^g_\theta,$$

where for each θ, g acts by rotation by θ.[1] Correpsondingly, the $\mathcal{L}$-class can be modified to parametrize an $\mathbb{RP}^1$'s worth of classes: let $\mathcal{L}_\theta$ be defined by the power series $\coth(x + \frac{i\theta}{2})$. In particular, $\mathcal{L}_\pi = e/\mathcal{L}$, where e the Euler class is given by the power series x.

The appropriate thing to do for the other side of the signature theorem is to consider the G-action on the cohomology H^*X, which leaves invariant the form B defined above. So, picking an equivariant inner product, we can define a G-equivariant operator A by $\langle Ax, y \rangle = B(x, y)$, and provided that X satisfies $\dim X \equiv 0 \mod 4$ we compute that A is self-adjoint.[2] Then, V^+ is the direct sum of the positively-weighted eigenspaces and V^- as the sum of the negatively-weighted eigenspaces, so that $\operatorname{Sign}(g, V) = \operatorname{tr}(g|_{V^+}) - \operatorname{tr}(g|_{V^-})$. The G-equivariant statement of the index theorem is then: for X^g connected and orientable,

$$\operatorname{Sign}(g, X) = \left\langle \mathcal{L}(X^g)\mathcal{L}_\pi(N^g_\pi) \prod_{0<\theta<\pi} \mathcal{L}_\theta(N^g_\theta), [X^g] \right\rangle.$$

[1]This forces N^g_θ to be even-dimensional and to carry an almost-complex structure for $\theta \neq \pi$, but there is no information gained at $\theta = \pi$. This in turn means that $\mathcal{L}_\pi$ and $\mathcal{L}$ need to be defined for real bundles rather than just complex ones. This is possible by exploiting a connection between the Pontryagin class and the $\mathcal{L}$-class for complex bundles, then using the Pontryagin class, which makes sense for real bundles, to extend the definition of $\mathcal{L}$, and hence $\mathcal{L}_\pi$, to real bundles as well.

[2]Generally, one can compute $\langle A^* x, y \rangle = \pm \langle Ax, y \rangle$, depending upon $\dim X \equiv 0, 2 \mod 4$, so that B is either symmetric or skew-symmetric respectively. In the skew-symmetric case, there is additionally some care to be taken with regards to the handed-ness of the group action. See pg. 29-30 of Hirzebruch-Zagier for how to proceed.

2. An example and Rademacher reciprocity

Now, let G be finite. When G acts freely on X, we have $\operatorname{Sign}(X/G) = \frac{1}{|G|}\sum_{g\in G}\operatorname{Sign}(g,X) = \frac{1}{|G|}\operatorname{Sign}(X)$, using the fact that $H^*(X/G) \to (H^*X)^G$ is an isomorphism. The G-signature theorem tells us how to correct this statement when the G-action is not free; for a connected 4-manifold with a faithful[3] G-action, we instead have

$$|G|\operatorname{Sign}(X/G) = \operatorname{Sign}(X) + \sum_x \operatorname{def}_x + \sum_Y \operatorname{def}_Y,$$

where def denotes the trigonometric expressions[4]

$$\operatorname{def}_x = -\sum_{\substack{g\in G\\ gx=x}} \cot\frac{\alpha_{x,g}}{2}\cot\frac{\beta_{x,g}}{2}$$

$$\operatorname{def}_Y = \sum_{\substack{g\in G\\ g|_Y=\mathrm{id}_Y}} \left(1+\cot^2\frac{\theta_{Y,g}}{2}\right)(Y\circ Y),$$

where the angles measure the rotations induced by g on the normal bundle.[5] The def terms are so-named as to measure the "defect" in the group action away from freeness.

Let's calculate an example. Let $X = \mathbb{CP}^2$ with the usual presentation by homogeneous coordinates, and pick a triple of finite cyclic groups $C_a, C_b, C_c \subseteq S^1 \subseteq \mathbb{C}^\times$ with a, b, and c mutually coprime. Then, setting $G = C_a \times C_b \times C_c$, there is a G-action on $\mathbb{CP}^2$ given by

$$(g_0,g_1,g_2)\cdot[z_0:z_1:z_2] = [g_0z_0 : g_1z_1 : g_2z_2].$$

We can immediately pick off the fixed sets of this action: there are the sub-$\mathbb{CP}^1$s $Y_i = \{[z_0:z_1:z_2] \mid z_i = 0\}$, and there are the points $x_i = [\delta_{0i}:\delta_{1i}:\delta_{2i}]$. The subgroups that fix these sets are C_a, C_b, C_c, and G, G, G respectively. We compute[6] the defects for the set Y_0 to be

$$\operatorname{def}_{Y_0} = \sum_{\substack{e^{ia\theta}=1,\\ e^{i\theta}\neq 1}} (1+\cot^2(\theta/2)) = \frac{a^2-1}{3}$$

and for x_0 to be

$$\operatorname{def}_{x_0} = -bc\sum_{k=1}^{a-1}\cot\frac{\pi kb}{a}\cot\frac{\pi kc}{a} =: -4abc\cdot s(b,c;a).$$

Now, note that $G = C_a\times C_b\times C_c \cong C_{abc}$ embeds in S^1, as the orders are all coprime, and our definition of the G-action on $\mathbb{CP}^2$ extends to an action of S^1. Because S^1 is connected and $\operatorname{Sign}(g,X)$ is defined homologically, this shows that $\operatorname{Sign}(g,X) = \operatorname{Sign}(1,X) = \operatorname{Sign}(X)$ for all $g\in G$. This means that $\operatorname{Sign}(X/G) = \frac{1}{|G|}\sum_{g\in G}\operatorname{Sign}(g,X) = \operatorname{Sign}(X) = 1$. Assembling all this into an application of the defect formula, we see

$$\begin{aligned}
|G|\operatorname{Sign}(X/G) &= \operatorname{Sign}(X) + \sum_Y \operatorname{def}_Y + \sum_x \operatorname{def}_x\\
abc\cdot 1 &= 1 + \frac{a^2-1}{3} + \frac{b^2-1}{3} + \frac{c^2-1}{3} - 4abc(s(b,c;a)+s(c,a;b)+s(a,b;c))\\
\frac{a^2+b^2+c^2-3abc}{12abc} &= s(b,c;a)+s(c,a;b)+s(a,b;c).
\end{aligned}$$

This last identity is called Rademacher reciprocity for cotangent sums.

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[3]Faithfulness means 1 is the only element of G with codimension 0 fixed set.

[4]The switch to standard trig functions from hyperbolic ones is explained by the loss of the i factor. The two kinds of trig functions are related through some complex geometry; every projective quadratic can be made to look like $x^2+y^2=z^2$ (or whatever) through a linear change of coordinates.

[5]The class $Y\circ Y$ is the oriented cobordism class of the self-intersection manifold of Y, thought of as an integer by counting the algebraic number of points. In our example, $Y\circ Y$ will be 1, so don't worry about it.

[6]Here I have to handwave a bit, as these calculations actually take some nontrivial patience with trigonometric series identities. If you're curious, you can see the argument employed in 9.2 of Hirzebruch-Zagier, pgs. 178-181.

3. The moral

The point of Hirzebruch-Zagier is that this setup is general enough that it can be used to organize a great deal of trigonometric calculations arising in number theory, by finding a sufficiently friendly manifold and stratifying it by fixed sets of some compact Lie group action. You don't entirely get out of doing the work, as several steps in our computations of the defects requires some manipulation of trigonmetric sums, but the topology sort of tells you where you're going and how to get there. The usefulness of this shouldn't be underestimated when lost in a sea of number theoretic identities.

This method for cotangent sums arises chiefly because of the appearance of trigonometric functions in the Hirzebruch series for the $\mathcal{L}$-class, together with the role of the $\mathcal{L}$-classes in the signature theorem. Other index theorems will produce similar organizing frameworks for topology to exert some control on the calculational behavior of whatever Hirzebruch series their correction classes arise from. It is still not clear why, morally, these topological objects should exhibit such control on number theoretic identities; figuring this out has fallen in and out of fashion over the years. Hirzebruch-Zagier contains a partial list of references detailing what all had been done in the mid-1970s, and one can wander outward from there.