SPECTRAL SEQUENCES FROM CO/SIMPLICIAL OBJECTS

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Abstract. Simplicial complexes are much more useful and versatile than mere combinatorial models for topological spaces, the first context they're introduced in. The goal of this talk is to show how co/simplicial objects and their associated spectral sequences can be used to help organize your topological life generally. We will instantiate these objects in a variety of settings and compute some explicit examples to see what's what.

1. Simplicial objects

First things first: a simplicial object in a category C is an object of C for each integer n, representing the collection of n-simplices, together with maps that discard vertices from the n-simplex to form $(n-k)$-simplices and maps that form degenerate flat $(n+k)$-simplices. Formally, one considers the category $\mathbf{\Delta}$, whose objects are finite ordinals and whose morphisms are order-preserving maps of linearly ordered sets. A simplicial object in C is exactly a functor $S : \mathbf{\Delta}^{op} \to \mathsf{C}$; the functor S sends the ordinal $[n]$ to the collection of n-simplices, and the various maps among ordered sets correspond to these face and degeneracy maps.

Simplicial objects arise in a lot of ways. Here are three especially useful ones:

(1) Given a category C, we can quite obviously form the bottom two rungs of a simplicial set (i.e., a simplicial object in Sets), called $N(\mathsf{C})$, the nerve of C: the set of objects of the category is $N(\mathsf{C})[0]$ and the set of morphisms of the category is $N(\mathsf{C})[1]$. The domain and codomain information of a morphism gives the required face maps $N(\mathsf{C})[1] \rightrightarrows N(\mathsf{C})[0]$, and the identity arrows give the degeneracy map $N(\mathsf{C})[0] \to N(\mathsf{C})[1]$. Composition gives us the means to fill this out to a full simplicial object. The n-simplices, $N(\mathsf{C})[n]$, is the set of n-chains of composable arrows. Drawing out all the individual compositions reveals why these deserve to be considered n-simplices. For instance, a 2-simplex is a pair of composable morphisms $A \xrightarrow{f} B \xrightarrow{g} C$, and the composite $A \xrightarrow{gf} C$ fills out the diagram to a triangle. Similarly, all possible composites in a 3-chain yield a tetrahedron, and so on. The face maps are given by discarding one of the vertices in this simplex of composites, yielding a sub-simplex whose edges are still composable with the prescribed composites. The degeneracy maps are given by inserting the identity morphism and extending along it.

(2) There are a sequence of standard n-simplex topological spaces described by the following bounded hyperplane: the n-simplex is the subset of $\mathbb{R}^{n+1}$ satisfying $\sum_{i=0}^{n} x_i = 1$ and $0 < x_i < 1$ for all i. For giggles, denote this subspace Δ^n. Then, for any topological space X, one has a set of continuous maps $\Delta^n \to X$, which form the levels of a simplicial set $\Pi_\infty X$. The face and degeneracy maps are those induced by the inclusions of and projections onto faces of these standard simplices.

(3) Let $\{U_i\}_{i \in I}$ be a cover of a fixed space X. We can build a simplicial space (i.e., a simplicial object in the category of spaces) denoted $\check{\mathrm{C}}\mathrm{ech}\, U$, called the Čech complex, whose n-simplices are given by $(\check{\mathrm{C}}\mathrm{ech}\, U)[n] = \coprod_{|J|=n} U_{J_0} \cap \cdots \cap U_{J_n}$. The degeneracy maps are given by intersecting with more elements of the cover to raise the intersection degree, and the face maps are given by including into the parent spaces of lesser intersection degree.

2. The bar spectral sequence

By reading Hatcher, you've likely heard of a close cousin of simplicial sets: simplicial complexes. Indeed, given a simplicial set S, we can build a simplicial complex using the following formula:

$$|S| = \operatorname{colim}\left(\coprod_{n \geq 0} S[n] \times \Delta^n \xrightarrow{\coprod_n \coprod_{i \leq n} \coprod_{x \in S[n]} (S(s_i)(x) \times \Delta(s_i))} \coprod_{n+1 \geq 0} S[n+1] \times \Delta^{n+1} \right).$$

This is called the geometric realization of S. Consider, for instance, the following semisimplicial[1] set S: it has three 0-simplices a, b, and c, and three 1-simplices X, Y, and Z with face maps

$$\begin{aligned} s_0 X &= a, & s_0 Y &= b, & s_0 Z &= c, \\ s_1 X &= b, & s_1 Y &= c, & s_1 Z &= a. \end{aligned}$$

Then the formula above for $|S|$ consists of 3 copies of Δ^0, 3 copies of Δ^1, and the inclusions described by the various values given in the table above. In the end, the three Δ^1s glue together, end to end, to form a circle.

This construction is often interesting in the examples of simplicial sets given above. For example, when X is a CW-complex, the space $|\Pi_\infty X|$ has a natural[2] map $|\Pi_\infty X| \to X$ which is a weak equivalence. Now, fix a group G, and consider the category $G{-}\mathsf{Sets}$ of sets with a faithful and effective G-action, together with G-equivariant functions between sets. The realization $|G{-}\mathsf{Sets}|$ is a model for BG, the classifying space of G. To make this believable, suppose that we have a G-bundle Y over X, which has a unique-up-to-homotopy map $X \to BG$. Then, any point $x \in X$ (i.e., the image of Δ^0 along some map to X) has a preimage along the bundle map $Y \to X$ which is a G-set. Then, a path in X (i.e., the image of Δ^1 along some map to X) lifts to a path of G-sets in Y, and sending the left-endpoints to the right-endpoints gives a map of G-sets from the fiber over the left endpoint to the fiber over the right (though this lift may require a choice through a fibration). Continuing in this way, from Y we can produce a morphism of simplicial sets $\Pi_\infty X \to N(G{-}\mathsf{Sets})$, which realizes to a map of spaces $X \simeq |\Pi_\infty X| \to |N(G{-}\mathsf{Sets})|$, classifying Y. It is believable, then, that $|N(G{-}\mathsf{Sets})|$ is a classifying space for G-bundles, and this construction gives a nice, combinatorial model for it.

It's no surprise, though, that the simplicial set $N(G{-}\mathsf{Sets})$ is quite large, which makes it unwieldy for computations. To help deal with this, we can add complexity one level at a time to a realization $|S|$ by considering the skeleton simplicial sets $\operatorname{sk}^n S$, whose k-simplices are exactly the k-simplices of S for $k \le n$ and which contain only the necessary degenerate simplices for $k > n$. We can additionally omit the degenerate simplices to get a semisimplicial set, and the geometric realization of that object is homotopy weakly equivalent to the realization of the full simplicial set. The inclusion $\operatorname{sk}^{n-1} S \hookrightarrow \operatorname{sk}^n S$ gives an ascending filtration of $|S|$. The filtration quotient $|\operatorname{sk}^n S|/|\operatorname{sk}^{n-1} S|$ is exactly the nondegenerate n-simplices in S with their boundaries quotiented out.

We can apply this to our simplicial model for BG. The set $N(G{-}\mathsf{Sets})[n]$ consists of n-tuples of elements of G — i.e., elements of $G^{\times n}$ — and the nondegeneracy condition ensures that the identity element can't appear in the tuple, so we're left with the simplices parametrized by $G^{\wedge n}$. Hence, the nth filtration quotient takes the form $(\Delta^n/\partial\Delta^n) \wedge G^{\wedge n} = \Sigma^n G^{\wedge n}$. Applying E-homology to the filtration, we get a spectral sequence of the form

$$E^1_{p,q} = H_p \Sigma^q G^{\wedge q} \Rightarrow H_p BG.$$

Provided that H_* has a Künneth isomorphism, so that $H_* \Sigma^q G^{\wedge q} = (H_* \Sigma G)^{\otimes q}$, the d^1-differential is exactly that associated to the bar resolution of $H_* G$, and so we have an additional identification

$$E^2_{p,q} = \operatorname{Tor}^{H_* G}_{*,*}(H_*, H_*).$$

3. Homotopy co/limits and the Mayer-Vietoris spectral sequence

Quotienting is a miserably behaved operation under homotopy. The most common example are the pushouts $\operatorname{colim}(\mathrm{pt} \leftarrow X \to \mathrm{pt})$ and $\operatorname{colim}(CX \leftarrow X \to CX)$, two homotopy equivalent diagrams with very different colimits: pt and ΣX respectively. Homotopy comes with a "better" operation that has the same rough effect as quotienting together points: instead, we can add paths between them. So, for instance, the first diagram can be filled out to $\operatorname{colim}(\mathrm{pt} \leftarrow X \times I \to X \leftarrow X \times I \to \mathrm{pt}) = \Sigma X$ — this means that the suspension of X is the "right" answer, homotopically speaking. Taking this idea all the way to its conclusion, for a diagram $F : \mathsf{I} \to \mathsf{C}$, we can detect how much fat we have to add to the object $F(i)$ via the nerve $N(i/I)$.

[1] "Semisimplicial" means that the degenerate simplices are omitted.

[2] Indeed, this is the counit.

This is sewn together in the formula

$$\operatorname{hocolim}_I F = \operatorname{colim}\left(\coprod_{a:i\to j} F(i)\times|N(j/I)| \rightrightarrows \coprod_i F(i)\times|N(i/I)|\right).$$

As a simple example of how this works, for a simplicial set S we can form the composite $\tilde{S} = \mathbf{\Delta}^{op} \xrightarrow{S} \mathsf{Sets} \xrightarrow{discrete} \mathsf{Spaces}$, and there is then a homotopy equivalence

$$\operatorname{hocolim}_{\mathbf{\Delta}^{op}} \tilde{S} = |S|.$$

For a slightly less trivial example, we also have a variant on Milnor's theorem: if U_* is a cover of X with each k-fold intersection a cofibrant inclusion, then

$$|\operatorname{\check{C}ech} U_*| := \operatorname{hocolim}_{\mathbf{\Delta}^{op}} \operatorname{\check{C}ech} U_* \simeq X.$$

Just as with the bar spectral sequence, we can consider the skeletal filtration $|\operatorname{sk}^n \operatorname{\check{C}ech} U_*|$, which has filtration quotients $|\operatorname{sk}^n \operatorname{\check{C}ech} U_*|/|\operatorname{sk}^{n-1} \operatorname{\check{C}ech} U_*| = \Sigma^n\left(\coprod_J U_{J_0}\cap\cdots\cap U_{J_n}\right)$. Hence, we have a spectral sequence

$$E^1_{p,q} = H_p\Sigma^q\left(\coprod_{\substack{|J|=q\\ J_i\neq J_j}} U_{J_0}\cap\cdots\cap U_{J_q}\right) \Rightarrow H_pX,$$

and again the d^1-differential is exactly that associated to the underlying Moore complex[3].

Let's give this spectral sequence a spin for $\mathbb{C}\mathrm{P}^2$ covered by its affine charts; using the homogeneous coordinates $[x_0 : x_1 : x_2]$ for $\mathbb{C}\mathrm{P}^2$, the three charts are $U_i = \{[x_0 : x_1 : x_2] \in \mathbb{C}\mathrm{P}^2 \mid x_i \neq 0\}$. Their single intersections $U_i\cap U_j$ are described by $U_i \cap U_j = \{[x_0 : x_1 : x_2] \mid x_i = 1, x_j \neq 0\} \simeq S^1$, and the triple intersection is $U_0\cap U_1\cap U_2 = \{[x_0 : x_1 : x_2] \mid x_0 = 1, x_1 \neq 0, x_2 \neq 0\} \simeq S^1\times S^1$. The generators of $H_1(U_0 \cap U_1 \cap U_2)$ wrap around the holes in the x_1 and x_2 coordinates, so if we concern ourselves only with $U_i\cap U_j$ where $i < j$, the map $H_1(U_0\cap U_1\cap U_2) \to H_1(U_1\cap U_2)\oplus H_1(U_0\cap U_1)\oplus H_1(U_0\cap U_2)$ is described by the matrix $\begin{bmatrix}1&1\\1&0\\0&1\end{bmatrix}$. This $\mathbb{Z}$-linearly row-reduces to $\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}$, and so the d^1 differential leaves a $\mathbb{Z}$ in (p, q)-bidegree $(2, 0)$ and another $\mathbb{Z}$ in $(4, 1)$. This gives the homology groups in $H_*\mathbb{C}\mathrm{P}^2$.

4. Cosimplicial objects and the descent spectral sequence

Dual to these ideas, there is also a theory of cosimplicial objects, which are functors $C : \mathbf{\Delta} \to \mathsf{C}$. Cosimplicial objects have a sort of dual to realization, called totalization, given by the Eckmann-Hilton type formula

$$\operatorname{Tot} C = \lim\left(\prod_{n\geq 0} C[n]^{\Delta^n} \xleftarrow{\prod_n\prod_{i\leq n}\prod_{x\in C[n]} C(s_i)(x)^{\Delta(s_i)}} \prod_{n+1\geq 0} C[n+1]^{\Delta^{n+1}}\right).$$

The skeletal filtration of S also has a mirror image, the coskeletal filtration $\operatorname{cosk}^n C$ of C, where again the k-cosimplices of $\operatorname{cosk}^n C$ are exactly the k-cosimplices of C for $k \leq n$ and have "just enough" simplices to flesh out the codegeneracy and coface maps for $k > n$. (Formally, this construction is given by Kan extensions.) However, while geometric realization takes the skeletal filtration to a sequence of cofibration inclusions, totalization takes the coskeletal filtration to a tower of fibrations. Finally, there is a construction of a homotopy limit Eckmann-Hilton dual to the homotopy colimit, and we have the familiar formula

$$\operatorname{Tot} C = \operatorname{holim}_{\mathbf{\Delta}} \tilde{C}.$$

[3]... where the Moore complex is the chain complex C_* built from a semisimplicial group by taking the group of n-simplices as C_n and building the differential out of an alternating sum of face maps.

Why all the fuss — where do cosimplicial objects arise? Consider the classical definition of a sheaf as a set-valued functor F defined on the category of open sets of some space, satisfying the equalizer condition

$$F(X) = F\left(\operatorname{colim}\left(\coprod_{i,j} U_i \cap U_j \xrightarrow{i_i, i_j} \coprod_i U_i\right)\right) = \lim\left(\prod_{i,j} F(U_i \cap U_j) \leftarrow \prod_i F(U_i)\right)$$

for a cover U_* of X. Studying this formula in light of the discussions above, we see that what's really happening here is that the sheaf condition is a statement about the homotopy limits and colimits of the 1-skeleta of the Cech complex:

$$F(\operatorname{hocolim} \operatorname{sk}^1 \check{\mathrm{C}}\mathrm{ech}\, U_*) = \operatorname{holim} F(\operatorname{sk}^1 \check{\mathrm{C}}\mathrm{ech}\, U_*).$$

For homotopy theoretical purposes, however, sheaves are often not as useful as one might hope. For instance, there is no good way to use the original definition of sheaf to build a sheaf of spaces — the equivalence with the associated étale space breaks down, for instance, ruining much of the theory. The trouble, it turns out, is that we're discarding all of the homotopy theoretic information by restricting to the 1-skeleton, which was fine when we were working with sets as they don't have homotopy theoretic information anyway, but is important now. We instead define a sheaf of spaces (or other space-like objects living in some simplicially enriched and co/tensored model category where all this makes sense) to be a contravariant functor F on some ground site satisfying the revised sheaf axiom

$$F(\operatorname{hocolim} \check{\mathrm{C}}\mathrm{ech}\, U_*) = \operatorname{holim} F(\check{\mathrm{C}}\mathrm{ech}\, U_*).$$

If you haven't seen this definition before, then you probably haven't seen many examples either — this stronger condition is fairly hard to come by! Nevertheless, here's one: for a finite CW complex X, the functor $F(\operatorname{Spec} R \to \operatorname{Spec} \mathbb{Z}) = HR \wedge \Sigma^\infty_+ X$ defines a homotopy sheaf of spectra on the fppf site of affines over $\operatorname{Spec} \mathbb{Z}$. A "cover of $\operatorname{Spec} \mathbb{Z}$" in this Grothendieck topology means a collection of flat $\mathbb{Z}$-algebras R_i such that for any $\mathbb{Z}$-module M, the conditions that $M = 0$ and $M \otimes_{\mathbb{Z}} R_i = 0$ for all i are equivalent. For example, localizations of rings are also flat, and so the rings $\mathbb{Z}_{(2)}$ and $2^{-1}\mathbb{Z}$ together form an fppf cover of $\mathbb{Z}$, with intersection $\operatorname{Spec} \mathbb{Z}_{(2)} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} 2^{-1}\mathbb{Z} = \operatorname{Spec} \mathbb{Q}$.

Let's consider the space $\mathbb{R}\mathrm{P}^3$. The chain complex computing its unreduced cellular homology with R coefficients is

$$\cdots \to 0 \to R \xrightarrow{0} R \xrightarrow{2} R \xrightarrow{0} R \to 0.$$

For R ranging in $2^{-1}\mathbb{Z}$, $\mathbb{Z}_{(2)}$, and $\mathbb{Q}$, this gives the homology groups

$$\begin{aligned}
&H_3(\mathbb{R}\mathrm{P}^3; 2^{-1}\mathbb{Z}) = 2^{-1}\mathbb{Z}, && H_3(\mathbb{R}\mathrm{P}^3; \mathbb{Z}_{(2)}) = \mathbb{Z}_{(2)}, && H_3(\mathbb{R}\mathrm{P}^3; \mathbb{Q}) = \mathbb{Q}, \\
&H_2(\mathbb{R}\mathrm{P}^3; 2^{-1}\mathbb{Z}) = 0, && H_2(\mathbb{R}\mathrm{P}^3; \mathbb{Z}_{(2)}) = 0, && H_2(\mathbb{R}\mathrm{P}^3; \mathbb{Q}) = 0, \\
&H_1(\mathbb{R}\mathrm{P}^3; 2^{-1}\mathbb{Z}) = 0, && H_1(\mathbb{R}\mathrm{P}^3; \mathbb{Z}_{(2)}) = \mathbb{Z}/2, && H_1(\mathbb{R}\mathrm{P}^3; \mathbb{Q}) = 0, \\
&H_0(\mathbb{R}\mathrm{P}^3; 2^{-1}\mathbb{Z}) = 2^{-1}\mathbb{Z}, && H_0(\mathbb{R}\mathrm{P}^3; \mathbb{Z}_{(2)}) = \mathbb{Z}_{(2)}, && H_0(\mathbb{R}\mathrm{P}^3; \mathbb{Q}) = \mathbb{Q}.
\end{aligned}$$

The descent spectral sequence for this cover of the site has

$$\begin{aligned}
E^1_{p,q} &= \pi_p \operatorname{hofib}(\operatorname{Tot}^q \mathcal{F}(\check{\mathrm{C}}\mathrm{ech}\, U_*) \to \operatorname{Tot}^{q-1} \mathcal{F}(\check{\mathrm{C}}\mathrm{ech}\, U_*)) \\
&= \pi_p \Omega^q \bigvee_J \mathcal{F}(U_{J_0} \cap \cdots \cap U_{J_q}) \\
&= \bigoplus_{|J|=q} H_{p+q}(\mathbb{R}\mathrm{P}^3; \mathcal{O}_{U_{J_0}} \otimes \cdots \otimes \mathcal{O}_{U_{J_q}}),
\end{aligned}$$

and so we have the groups

$$\begin{aligned}
&E^1_{-1,0} = 2^{-1}\mathbb{Z} \oplus \mathbb{Z}_{(2)}, && E^1_{0,0} = \mathbb{Z}/2, && E^1_{2,0} = 2^{-1}\mathbb{Z} \oplus \mathbb{Z}_{(2)}, \\
&E^1_{-2,2} = \mathbb{Q}, && E^1_{1,2} = \mathbb{Q}.
\end{aligned}$$

There are two locations for a d^1-differential, both of type signature $2^{-1}\mathbb{Z} \oplus \mathbb{Z}_{(2)} \to \mathbb{Q}$, and both given by the formula $d^1(a/2^b, n/m) = \frac{am+2^b n}{2^b m}$. This has kernel exactly $(a/1, -a/1)$, which is isomorphic to $\mathbb{Z}$. Hence, we compute

$$\begin{aligned} H_3(\mathbb{R}\mathrm{P}^3;\mathbb{Z}) &= \mathbb{Z}, \\ H_2(\mathbb{R}\mathrm{P}^3;\mathbb{Z}) &= 0, \\ H_1(\mathbb{R}\mathrm{P}^3;\mathbb{Z}) &= \mathbb{Z}/2, \\ H_0(\mathbb{R}\mathrm{P}^3;\mathbb{Z}) &= \mathbb{Z}. \end{aligned}$$

5. Miscellania

I will almost certainly be out of time at this point, but there are various points beyond this that are worth mentioning.

- **Sheaf cohomology:** The construction of the descent spectral sequence is entirely natural in our choice of cover — cover refinements induce maps of spectral sequences, converging to isomorphisms on the E^∞ pages. Just as Čech cohomology refines in the limit to sheaf cohomology, the E^2-page of the descent spectral sequence refines under this construction to the sheaf cohomology $E^2_{p,q} = H^{p+q}(X;\pi_p F)$.
- **The Eilenberg-Moore spectral sequence:** There is a homotopy theoretic version of the two-sided bar construction, which for an input space X produces a simplicial object that in good cases is weakly equivalent to ΩX. The skeletal filtration produces a simple example of the Eilenberg-Moore spectral sequence, but for a generalized cohomology theory rather than ordinary cohomology over a field. This spectral sequence suffers from interesting convergence problems for unbounded cohomology theories.
- **Other homotopy sheaves of spectra:** A huge goal of derived algebraic geometry is to provide a source for these homotopy sheaves. For instance, the extremely interesting cohomology theory tmf appears as the global sections of a sheaf of (structured) ring spectra over the moduli stack of elliptic curves. There is also a theorem producing spectra related to the Morava E-theories by taking the global sections of some sheaf of spectra over a stack supporting a a p-divisible group.
- **The Adams spectral sequence:** The Adams spectral sequence is also formed by building a cosimplicial object in the stable category, effectively covering a spectrum by "free" spectra over a fixed ground ring spectrum.