FAITHFUL-ISH ALGEBRAIC MODELS FOR HOMOTOPY THEORY

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ABSTRACT. We present a algebraist-friendly framework which packages the structural aspects of "chromatic stable homotopy theory", together with an analysis of a few other simple examples, ultimately meant to spur curiosity about what might happen in general. This talk was delivered at a summer graduate student seminar at UC Berkeley.

1. INTRODUCTION

This talk amounts to a discussion of the broader landscape of stable homotopy theory, but because my intended audience is algebraic-leaning people who are comfortable with stacks and *not* a bunch of actively practicing stable homotopy theorists, I'd like to spend the first segment of the talk setting up the main ideas and players of the subject.

The basic object of study is the *spectrum*. Spectra come about by trying to reconcile the differences between homotopy and homology.

Definition 1. A *homology functor* is a functor $\widetilde{E}_*\colon h\mathsf{Spaces}_* \to \mathsf{AbelianGroups}^{\mathbb{Z}}$ that sends cone sequences to long exact sequences and (infinite) wedges to (infinite) direct sums.

For $A \le X$, there is a notion of "relative homotopy groups" $\pi_*(X,A)$ which belongs to a long exact sequence with $\pi_* X$ and $\pi_* A$, but the definition is *not* directly related to the "cone sequence" that appears in homology. It turns out to be possible to compare these in low degrees:

Theorem 2. *Let (X,A) be n-connected and A be m-connected. Then $(X,A) \to (X/A,*)$ is an $(n+m)$-equivalence.* □

Corollary 3. *This has the following consequences:*

1. *In the situation above, π_* converts the cone sequence to a long exact sequence through degree $n+m$.*
2. *The map $\pi_* X \to \pi_{*+1}\Sigma X$ is an equivalence through twice the connectivity of X.*
3. *For X n-connected and Y m-connected, π_* converts their wedge sum to a direct sum through degree $n+m$.* □

By suspending a space indefinitely and sending its connectivity towards infinity, these corollaries eliminate the axiomatic differences between homology and homotopy, and the resulting "stable homotopy groups" become a homology functor. It is then common to close up indefinitely suspended spaces under shifts and inductive colimits, and the resulting category is called Spectra. The first nontrivial success of the field is that Spectra is rich enough to house the objects involved for *any* homology functor:

Theorem 4 (Boardman, Brown, ...). *There are assignments Σ^∞ and B as in*

$$h\mathsf{Spaces}_* \xrightarrow{\Sigma^\infty} \mathsf{Spectra} \xleftarrow{B} \mathsf{HomologyTheories},$$

with Σ^∞ a functor and B not *a functor,*[1] *such that*

$$\widetilde{E}_*(X) = \pi_*(B(E) \otimes \Sigma^\infty X). \qquad \square$$

We have left one term here undefined: $\otimes$. The smash product of spaces can be promoted, with some cleverness, to a monoidal product on Spectra so that Σ^∞ is a monoidal functor and $\mathbb{S} = \Sigma^\infty S^0$ is its unit. The spectrum $\mathbb{S}$ acquires the structure of a *ring*, and generic spectra are modules for it, so that $\mathsf{Spectra} = \mathsf{Modules}_{\mathbb{S}}$ has something of the flavor of a derived category, where the spectra associated to finite cell complexes play the role of $\mathsf{Modules}_{\mathbb{S}}^{\mathrm{perf}}$. However, it is a bit dangerous to draw too much intuition from this, and it is especially dangerous to conflate $\mathsf{Modules}_{\mathbb{S}}^{\mathrm{perf}}$ with $D^{\mathrm{perf}}(\mathsf{Modules}_{\pi_*\mathbb{S}})$, because of the results like the following:

[1] Every map of homology theories lifts to a map of spectra, but perhaps not uniquely.

Lemma 5. *The homotopy groups $\pi_*\mathbb{S}/2$, with $\mathbb{S}/2 = \mathit{Cone}(2\colon \mathbb{S} \to \mathbb{S})$, are* not *all 2–torsion.* □

Nonetheless, because of the evident complexity of Spectra and difficulty of stable homotopy theory in general, we would like to codify some kind of algebraic model from which to draw intuition.

2. Contexts

Putting stable homotopy and homology on such even footing lets us directly compare them, and in particular we can try to use one to study the other. Applying B to ordinary homology $\widetilde{H}_*(-;R)$ yields a spectrum HR, which is a ring spectrum when R is a ring and which then receives a map $\eta\colon \mathbb{S} \to HR$ selecting the unit class

$$1 \in R = \widetilde{H}_*(S^0;R) = \pi_0 HR.$$

Tensoring η with a spectrum X and passing to homotopy gives the *Hurewicz map*

$$\pi_* X \to \pi_* \Sigma^\infty X \xrightarrow{\pi_*(\eta \otimes \Sigma^\infty X)} \pi_*(HR \otimes \Sigma^\infty X) = \widetilde{H}_*(X;R),$$

recognized from more algebro-geometric contexts as a sort of pullback along η.

Based on this, we might ask a more nuanced question: can we recover the $\mathbb{S}$–module $\Sigma^\infty X$ (or its homotopy) from $HR \otimes \Sigma^\infty X$ (or its homotopy) via descent along η? There is a rephrasing of descent that is particularly amenable to this situation: a descent datum for an S–module N along a map $f\colon R \to S$ is a choice of isomorphism $\varphi\colon (f \otimes 1)^* N \cong (1 \otimes f)^* N$, and an R–module M induces a descent datum on $N = f^* M = S \otimes_R M$ via the identity

$$(f \otimes 1)^* N = ((f \otimes 1) \circ f)^* M = ((1 \otimes f) \circ f)^* M = (1 \otimes f)^* N.$$

This is the 1–truncation of the homotopical situation, where the descent datum associated to an HR–module $Y = HR \otimes \Sigma^\infty X$ along the ring map η is given by the cosimplicial object

$$\mathscr{D}_{HR}(\Sigma^\infty X) := \left\{ \begin{array}{ccccccc} & & & \longrightarrow & HR & \longleftarrow & \\ & \xrightarrow{\eta\otimes 1} & HR & \longleftarrow & \otimes & \longrightarrow & \\ HR & \xleftarrow{\mu} & \otimes & \xrightarrow{1\otimes\eta\otimes 1} & HR & \longleftarrow & \\ \otimes & \xrightarrow{1\otimes\eta} & HR & \longleftarrow & \otimes & \longrightarrow & \cdots \\ \Sigma^\infty X & & \otimes & \longrightarrow & HR & \longleftarrow & \\ & & \Sigma^\infty X & & \otimes & \longrightarrow & \\ & & & & \Sigma^\infty X & & \end{array} \right\}.$$

Whereas the R–module M is classically recovered as an equalizer when f is of effective descent, the $\mathbb{S}$–module $\Sigma^\infty X$ is recovered as the homotopy limit of the diagram $\mathscr{D}_{HR}(\Sigma^\infty X)$ when η is of effective descent. The ability to formulate this *homotopy* limit is an important technical condition: the ring spectrum in question must be an "A_∞–ring spectrum"—which, luckily, HR always is. Passing to homotopy groups then yields the *HR–Adams spectral sequence*, whose E_1–page is assembled from the homology groups of X and which abuts to the stable homotopy of X. The basic effectivity result is as follows:

Theorem 6. *For $F \in \mathsf{Modules}_{\mathbb{S}}^{\mathrm{perf}}$, the homotopy limit of $\mathscr{D}_{H\mathbb{Z}}(F)$ (resp., of $\mathscr{D}_{H\mathbb{F}_p}(F)$) recovers F (resp., the p–completion of F), and the $H\mathbb{Z}$–based (resp., $H\mathbb{F}_p$–based) Adams spectral sequence is strongly convergent to $\pi_* F$ (resp., $\pi_* F^\wedge_p$).* □

A clear next step is to prove a bunch of theorems about the input of the Adams spectral sequence, since these directly translate through the spectral sequence to theorems about its output in homotopy groups. A more reserved first step is to assign a name to this input, where again we draw inspiration from algebraic geometry:

Definition 7. Let E be an A_∞–ring spectrum. The cosimplicial abelian group

$$\mathcal{M}_E(\Sigma^\infty X) := \left\{ \pi_*\begin{pmatrix} E \\ \otimes \\ \Sigma^\infty X \end{pmatrix} \begin{array}{c} \xrightarrow{\eta\otimes 1} \\ \xleftarrow{\mu} \\ \xrightarrow{1\otimes\eta} \end{array} \pi_*\begin{pmatrix} E \\ \otimes \\ E \\ \otimes \\ \Sigma^\infty X \end{pmatrix} \begin{array}{c} \longrightarrow \\ \longleftarrow \\ \xrightarrow{1\otimes\eta\otimes 1} \\ \longleftarrow \\ \longrightarrow \end{array} \pi_*\begin{pmatrix} E \\ \otimes \\ E \\ \otimes \\ E \\ \otimes \\ \Sigma^\infty X \end{pmatrix} \begin{array}{c} \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \end{array} \cdots \right\}$$

determines a quasicoherent simplicial sheaf over the simplicial scheme

$$\mathcal{M}_E := \left\{ \operatorname{Spec}\pi_*\begin{pmatrix} E \end{pmatrix} \begin{array}{c} \xleftarrow{\eta\otimes 1} \\ \xrightarrow{\mu} \\ \xleftarrow{1\otimes 1} \end{array} \operatorname{Spec}\pi_*\begin{pmatrix} E \\ \otimes \\ E \end{pmatrix} \begin{array}{c} \longleftarrow \\ \longrightarrow \\ \xleftarrow{1\otimes\eta\otimes 1} \\ \longrightarrow \\ \longleftarrow \end{array} \operatorname{Spec}\pi_*\begin{pmatrix} E \\ \otimes \\ E \\ \otimes \\ E \end{pmatrix} \begin{array}{c} \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \end{array} \cdots \right\},$$

called the *context* of E.[2] The input to the E–Adams spectral sequence is then computed by Čech cohomology:

$$E_2^{*,*} = \check{H}^*(\mathcal{M}_E; \mathcal{M}_E(\Sigma^* X)) \Rightarrow \pi_*\Sigma^\infty X.$$

In practice, this set-up is a lot less scary than it looks, which we illustrate for $E = H\mathbb{F}_2$. There is a Segal equivalence

$$E^{\otimes(j+1)} \xrightarrow{\simeq} (E\otimes E)^{\otimes_E j},$$

which, together with the Künneth isomorphism for homology with mod–2 coefficients, shows that $\mathcal{M}_{H\mathbb{F}_2}$ is 1–truncated—i.e., it is a groupoid-scheme. This object is manageable: the Čech cohomology of sheaves over groupoid-schemes is actually an invariant of the associated stack, which have a well-developed theory in algebraic geometry. In this example, this stack has been completely determined:

Theorem 8 (Milnor)**.** *The context* $\mathcal{M}_{H\mathbb{F}_2}$ *is given by the stacky quotient* $\operatorname{Spec}\mathbb{F}_2 // \underline{\operatorname{Aut}}(\widehat{\mathbb{G}}_a)$.[3] □

3. THE CHROMATIC PICTURE

Our daydream about contexts has some obnoxious realities.

- Suppose we work with the ring spectrum $E = H\mathbb{F}_p$ for an odd prime p. The "arrows" part of the groupoid-scheme, $\pi_*(H\mathbb{F}_p \otimes H\mathbb{F}_p)$, has odd-dimensional homotopy and is not a commutative ring. We therefore cannot apply "Spec" to it.
- Suppose we work with the ring spectrum $E = H\mathbb{Z}$. Not only is $\pi_*(H\mathbb{Z}\otimes H\mathbb{Z})$ not even, and it is also not a flat $\mathbb{Z}$–module, so the Künneth theorem for $H\mathbb{Z}$ does not apply. The resulting object is thus not 1–truncated.
- Suppose we work directly with the ring spectrum $\mathbb{S}$. Its structure maps are flat, but $\pi_*\mathbb{S}$ is also skew-commutative.

Amazingly, there is an A_∞–ring spectrum which is of effective descent for perfect $\mathbb{S}$–modules, whose homotopy is even, whose ring of cooperations is even, and whose context is 1–truncated.

Theorem 9 (Milnor)**.** *The complex bordism spectrum, MU, has these properties in* $\mathsf{Modules}_{\mathbb{S}}$. □

[2]It is important to note that this assignment from spectra to sheaves does *not* target the derived category and is *not* a triangulated functor. It can be shown that any such functor must be ordinary homology, eliminating most of the richness of stable homotopy theory.

[3]This group-scheme is quite large, so that actually the bulk of the theory of stacks is not so directly applicable. Alas.

This theorem alone is enough to produce a conservative functor

$$\mathcal{M}_{MU}(-): \mathsf{Modules}_{\mathbb{S}} \to \mathsf{QCoh}(\mathcal{M}_{MU}),$$

and the MU–Adams spectral sequence additionally controls to what extent the functor fails to be faithful. The algebraic target $\mathsf{QCoh}(\mathcal{M}_{MU})$ is not that useful, however, unless we can recognize it in plainer terms, as Milnor did for $H\mathbb{F}_2$. Amazingly, this can be done, and this amounts to the core of *chromatic homotopy theory*:

Theorem 10 (Quillen)**.** *The context associated to MU is the moduli stack of formal groups,* $\mathcal{M}_{\mathrm{fg}}$. □

This theorem is exceptionally useful: not only is $\mathcal{M}_{MU}$ recognizable, but $\mathcal{M}_{\mathrm{fg}}$ is an object that has received enormous attention in algebraic number theory, so that many of its geometric features are available in the literature.

This emboldened topologists so much that they began asking questions about the *fullness* and *surjectivity* of this functor as well. The most literal interpretation of these questions is too strong:

Theorem 11 (Adams)**.** *Consider the spectrum* $Cone(p: \mathbb{S} \to \mathbb{S})$*. At* $p = 2$*, there is no map* $v: \Sigma^2 Cone(2) \to Cone(2)$ *such that* $KU(v)$ *is multiplication by the Bott class* $\beta \in \pi_2 KU$*. By consequence,* $\mathcal{M}_{MU}(-)$ *is not full.* □

However, there are gentler interpretations, including the following version inspired by noncommutative geometry:

Definition 12. The *Balmer spectrum* of a monoidal ∞–category is the collection of thick prime tensor ideals, where a subcategory is thick when it is closed under cone sequences, retracts, and weak equivalences; it is a tensor ideal when it satisfies the ideal property against the monoidal structure; and it is prime when it satisfies the prime ideal property against the monoidal structure.[4]

The basic theorem about this definition is that the Balmer spectrum (when additionally equipped with a suitable structure sheaf) associated to the derived category of perfect R–modules recovers the Zariski spectrum of R. However, the definition is also well-adapted to homotopical situations like ours. Using this definition, one can prove the following "fullness" result:

Theorem 13 (Hopkins–Smith)**.** *The map* $\mathrm{Spec}(\mathsf{Modules}_{\mathbb{S}}^{\mathrm{perf}}) \to \mathrm{Spec}(\mathsf{Coh}(\mathcal{M}_{\mathrm{fg}}))$ *is a homeomorphism.* □

The main engine powering this result is the following:

Theorem 14 (Devinatz–Hopkins–Smith)**.** *The* E_∞*–page of the MU–Adams spectral sequence for* $X = \mathbb{S}$ *has an asymptotically flat vanishing curve.*[5] □

This ensures that primality results deduced for $\mathsf{QCoh}(\mathcal{M}_{\mathrm{fg}})$ are ultimately not too far from results about $\mathsf{Modules}_{\mathbb{S}}$.

In addition to the result about Balmer spectra, this theorem has all sorts of interesting consequences in stable homotopy theory. Here are two:

Remark 15. In Figure 1, we have included a picture of the $MU_{(2)}$–Adams spectral sequence through a small range. The dots in the picture correspond to various (higher-order) invariants of formal groups: for instance, α_1 for a group law $x +_\varphi y$ corresponds to the coefficient in the expansion

$$x +_\varphi x = 2x +_\varphi \alpha_1(\varphi) x^2 + \cdots \pmod 2,$$

or, alternatively, to the module of 1–jets associated to φ, and there are similar interpretations for the other glyphs. The differentials mediate the difference between the algebraic model and the actualities of stable homotopy theory, and so require genuine geometric input to deduce. That α_1 survives the spectral sequence means that it corresponds to an object in stable homotopy theory: in this case, the nontriviality of the cell decomposition for the space $\mathbb{C}\mathrm{P}^2$.

Remark 16. A central application of these ideas is that there exist a family of homology theories $\{K(n)\}_{n=0}^{\infty}$, each equipped with a kind of "generalized Bott class" $\beta(n)$, such that every p–local finite spectrum F admits an integer n and a map $v: \Sigma^m X \to X$ where $K(n)_* v$ induces multiplication by a power of $\beta(n)$ and $K(m)_* F = 0$ for $m < n$. The spectra $K(0)$ and $K(1)$ are familiar: they are $H\mathbb{Q}$ and KU/p respectively. The differing "wavelengths" of these elements $\beta(n)$, $|\beta(n)| = 2(p^n - 1)$, is where "chromatic homotopy theory" gets its poetic name.

[4]The Balmer spectrum can be enriched to carry a topology and a structure sheaf, either of rings or of ∞–categories of local objects.

[5]The curve is moreover conjectured to grow according to a square-root law.

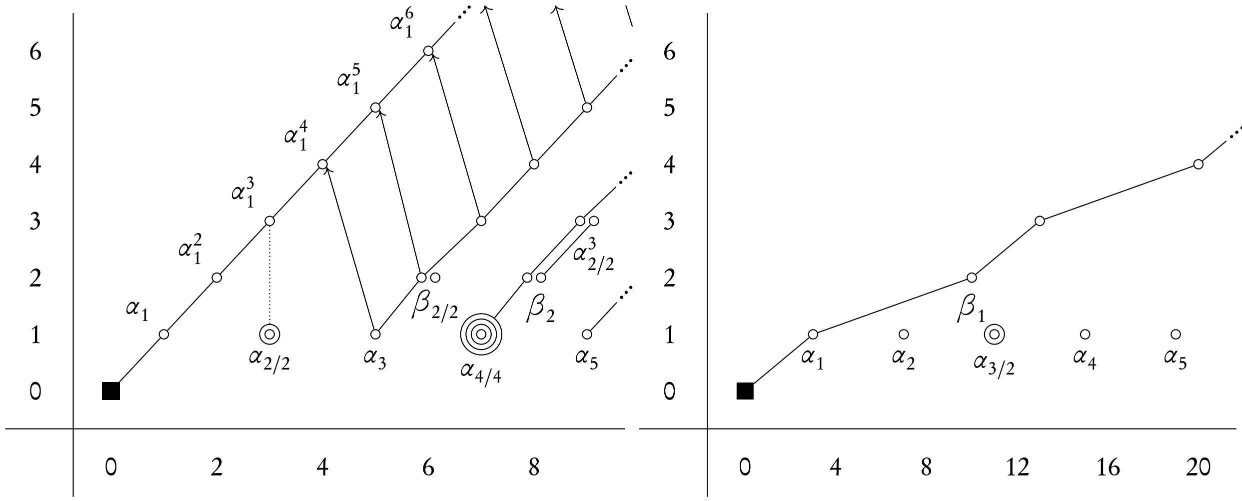

FIGURE 1. Pictures of the $MU_{(2)}$– and $MU_{(3)}$–Adams spectral sequences (left, right respectively) through a small range.

4. UNICORNS AND RESOLUTIONS

Many stunning results about the structure of Spectra have been deduced by topologists from this framework, but the techniques involved have not been transported to other settings. The core difficulty lies in our utilization of MU: whereas classical homotopy theorists were interested in MU because of its connection to complex manifolds, we do not have direct access to any such geometric link to guide us in a generic stable setting. In lieu of this, we instead simply enumerate the properties of MU which we have made use of:

Definition 17. An A_∞–ring object R in a monoidal stable ∞–category C is called a *unicorn* when its natural homotopy groups are even, its natural cooperation groups are even, its cooperation unit maps are flat, and when it induces a homeomorphism on the Balmer spectra of C and of $\mathsf{QCoh}(\mathscr{M}_R)$.

Having at least named the desirable properties of such objects, we then turn the question of where to find unicorns. It turns out that there is a classical theorem which characterizes a unicorn in Spectra and which does not make reference to complex geometry:

Theorem 18 (Priddy). *Let $X_0 = \mathbb{S}_{(p)}$ be the p–local sphere spectrum, and define X_n from X_{n-1} by attaching cells in degree $2n$ to minimally delete $\pi_{2n-1}X_{n-1}$. The spectrum X_∞ is then a ring and a minimal summand of $MU_{(p)}$.*[6] □

We propose to view this theorem not just as a recipe but as an all-or-nothing result: by trying to correct *one* of the failures of $\mathbb{S}_{(p)}$ to be a unicorn, we actually corrected *all* of its failures. This same spectrum also arises in another result which also has this flavor:

Theorem 19. *There is a map $X_\infty \to H\mathbb{F}_p$ which resolves away all odd-dimensional classes in $\pi_*(H\mathbb{F}_p \otimes H\mathbb{F}_p)$.* □

Again, correcting the failure of $H\mathbb{F}_p$ to have even cooperations produced X_∞, which somehow automatically has all the other desired properties.

However, although it is true that X_∞ carries the structure of an A_∞–ring, this is not clear from these constructions. A recent PhD thesis shows that it is also possible to carry out a similar construction in the setting of structured rings:

Theorem 20 (Beardsley). *Let $X(0) = \mathbb{S}_{(p)}$ be the p–local sphere spectrum, considered as an E_2–ring spectrum. Define the A_∞–$X(n-1)$–algebra $X(n)$ by attaching A_∞–$X(n-1)$–cells in degree $2n$ to delete $\pi_{2n-1}X(n-1)$. Then $X(n)$ is again an E_2–ring spectrum, and the spectrum $X(\infty)$ is $MU_{(p)}$.*[7] □

[6] For those in the know, this spectrum is usually called BP, the Brown–Peterson spectrum.

[7] These spectra $X(n)$ are those from the nilpotence and periodicity series: $X(n)$ is the Thom spectra of the tautological bundle on $\Omega SU(n+1)$.

Conjecture 21. If the monoidal unit of a monoidal stable ∞–category is connective, then a minimal A_∞–resolution of it which forms an even object gives a unicorn.

In the nonconnective case, one cannot rely on a Postnikov filtration to furnish us with an inductive procedure for removing the undesired odd-degree homotopy groups. Nonetheless, it seems like there is still a possibility for a result along these same lines, as illustrated in the following examples:

Example 22. The spectrum KO is not even. However, the complexification map $KO \to KU$ is a map of A_∞–KO–algebras which presents KU as a KO–unicorn. To see the evenness of the cooperations in KO–modules, use

$$KU \otimes_{KO} \Sigma KO \xrightarrow{1\otimes\eta} KU \otimes_{KO} KO \to KU \otimes_{KO} KU \to KU \otimes_{KO} \Sigma^2 KO.$$

This also entails effective descent.[8]

Example 23. Mildly more exotically, $KU_p^\wedge$ plays the role of a unicorn in the $K(1)$–local stable category. The unit of this local category, $L_{K(1)}\mathbb{S}$, has highly non-connective and non-even homotopy. Nevertheless, there is a fiber sequence

$$L_{K(1)}\mathbb{S} \xrightarrow{\eta} KU_p^\wedge \xrightarrow{\psi^\gamma - 1} KU_p^\wedge,$$

where ψ^γ is a certain *Adams operation*, presenting $\eta\colon L_{K(1)}\mathbb{S} \to KU_p^\wedge$ as being of effective descent. It can then be checked that p–adic K–theory has the other properties desired of a unicorn, provided one does not exit the p–complete setting.[9]

[8] The general result for fixed-points of $K(n)$–local spectra may follow from work of Mathew–Meier on Tate vanishing.

[9] Deducing what modifications, if any, need to be made to count the Goerss–Henn–Mahowald–Rezk resolution as a solution to the unicorn problem for $L_{K(2)}\mathbb{S}$ might be an interesting place to get started.