# BRST QUANTIZATION OF A RELATIVISTIC POINT-PARTICLE

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ABSTRACT. In this note, we record the computation of the physical states of the BRST quantization of a relativistic point-particle. It was written in preparation for a short presentation at the end of a course on QFTs taught by Reshetikhin.

## 1. THE BRST CHARGE

In this note, we analyze the BRST quantization of the local behavior of a relativistic point-particle system in a $D$-dimensional spacetime. In the classical Lagrangian formalism, such a system in $n$-dimensional space is described by a path generalized coordinate fields $X^\mu$ and an auxiliary field $e = \sqrt{g_{\tau\tau}}$, called the *einbein*. A path through this phase space describes a hypothetical world-line of a particle; we define the Lagrangian action on such a worldline by the integral

$$S = \int d\tau \frac{1}{2}\left(\frac{1}{e}\dot{X}_\mu \dot{X}^\mu - m^2 e\right).$$

The field $e$ has the effect of coupling the point-particle to 1-dimensional gravity. We further define the Lagrangian $L$ itself to be the integrand in this expression. Classical mechanics says that physically realizable worldlines are those for which this action is minimized, and we'll spend the remainder of this note analyzing this problem.

We can transform the $D+1$ second-order Euler-Lagrange equations into $2(D+1)$ first-order equations by replacing the target manifold with its cotangent bundle. In the cotangent bundle, the velocities $\dot{X}^\mu$ and $\dot{e}$ essentially become first-order objects, corresponding to generalized momenta defined by $p_\mu = \frac{\delta S}{\delta \dot{X}^\mu}$ and $p_e = \frac{\delta S}{\delta \dot{e}}$; hence the cotangent bundle has coordinates $\{X^\mu, p_\mu, e, p_e\}$. To produce the analogue of the Lagrangian action, one performs the Legendre transformation to produce the Hamiltonian $H = \frac{\delta S}{\delta \dot{X}^\mu}\dot{X}^\mu + \frac{\delta S}{\delta \dot{e}}\dot{e} - L$. As ingredients to this formula, we compute

$$\frac{\delta S}{\delta \dot{X}^\mu} = e^{-1}\dot{X}_\mu, \qquad \dot{X}_\mu = e p_\mu, \qquad \frac{\delta S}{\delta \dot{e}} = 0,$$

which begets the following formula for the Hamiltonian:

$$H = p_\mu \dot{X}^\mu - L = p_\mu \dot{X}^\mu - \frac{1}{2}\left(\frac{1}{e}\dot{X}_\mu \dot{X}^\mu - m^2 e\right) = e p_\mu p^\mu - \frac{1}{2}\left(e p_\mu p^\mu - m^2 e\right) = \frac{1}{2}e\left(p_\mu p^\mu + m^2\right).$$

The cotangent bundle is an example of a symplectic manifold, so comes equipped with a Poisson bracket

$$\{A, B\} = \frac{\partial A}{\partial p_\mu}\frac{\partial B}{\partial X^\mu} - \frac{\partial A}{\partial X^\mu}\frac{\partial B}{\partial p_\mu}.$$

The bracket can be calculated on these fields to be $\{X^\mu, p_\nu\} = \delta^\mu_\nu$ and $\{e, p_e\} = 1$, with all others vanishing. In turn, this gives us the means to compute $\frac{dA}{d\tau} = \{A, H\}$ for any physical quantity $A(q^\mu, p_\nu, e, p_e)$.

From the form of the Hamiltonian, we can identify the role of the einbein: it is a Lagrange multiplier, enforcing the constraining condition $\frac{dG}{d\tau} = 0$ for $G = \frac{1}{2}(p_\mu p^\mu + m^2)$. Additionally, there is a group of reparametrizations of the phase space which act on this system, detected by the einbein. The goal of the BRST method is two-fold: first, we may set the einbein to $e = 1$, which we accomplish by introducing a field $B$ selecting this parametrization via a $\delta$-functional; second, with the einbein eliminated we must promote the constraint $G$ to another first-order object, which we accomplish by extending the target space $F$ by a Grassmann algebra of "ghost-antighost pairs," while maintaining its structure as a symplectic manifold. That is, we're going to promote the cotangent bundle to a super-Poisson manifold whose odd coordinates are given by the ghost-antighost pairs. In our situation, to counterbalance the single constraint $G$ we introduce a single ghost-antighost pair $c$ and $\bar{c}$, over which we extend the Poisson bracket by the general formula

$$\{A,B\} = \frac{\partial A}{\partial p_\mu}\frac{\partial B}{\partial X^\mu} - \frac{\partial A}{\partial X^\mu}\frac{\partial B}{\partial p_\mu} + i(-1)^{|A|}\left(\frac{\partial A}{\partial c^\alpha}\frac{\partial B}{\partial \bar{c}_\alpha} + \frac{\partial A}{\partial \bar{c}_\alpha}\frac{\partial B}{\partial c^\alpha}\right).$$

The second bicovector field is rotated by the imaginary unit to prevent it from interfering with the original Poisson bracket. In particular, we can compute $\{c^\alpha, \bar{c}_\beta\} = -i\delta^\alpha_\beta$.

The purpose of introducing these new fields is to produce a vector field of infinitesimal transformations, the BRST charge $\Omega$, which encodes the constraint equation and against which the action is invariant. The charge is defined in general by the equation

$$\Omega = c_\alpha(G_\alpha + M_\alpha),$$

where $M_\alpha$ is some quantity $M_\alpha = \sum_{n=1}^\infty \frac{i^n}{2n!} c^{\alpha_1}\cdots c^{\alpha_n} M^{\beta_1\cdots\beta_n}_{\alpha\alpha_1\cdots\alpha_n}\bar{c}_{\beta_1}\cdots\bar{c}_{\beta_n}$, determined recursively in $n$. Specifically, $M_\alpha$ is chosen so that the action of $\Omega$ is nilpotent, i.e., $\{\Omega,\Omega\} = 0$. In our case, because $\{G,G\} = 0$, we may take $M = 0$ and hence $\Omega = cG$.

## 2. The BRST-extended action

The BRST construction outlined so far has involved extending the phase space by some additional fields "meant to encode the constraints $G_\alpha$." It would be nice to extend the action $S$ on the classical phase space to an action $S_{BRST}$ on the BRST phase space, to study its properties as a physical system. In particular, we would request two things: first, that the extended action reduces to the classical one on the physical sector, and second, that the BRST charge $\Omega$ constructed above is conserved.

Faddeev-Popov techniques instruct us to extend the classical action by a coordinate-momentum term relating the ghost and antighost fields to find $S_{BRST}$:

$$S_{BRST} = \int d\tau \left(p_\mu \dot{X}^\mu + i\bar{c}\dot{c} - \frac{1}{2}(p_\mu p^\mu + m^2)\right).$$

Note, here we have integrated out the $B$ field, which fixes $e = 1$. We now want to study the invariance properties of $S_{BRST}$ under $\delta_\Omega(-) = \{\Omega, -\}$, and then from that deduce the conserved charge using Noether's theorem. First, we compute the action of $\delta_\Omega$ on the individual fields:

$$\{\Omega, X^\mu\} = \frac{\partial \Omega}{\partial p_\mu} = cp^\mu, \qquad \{\Omega, p_\mu\} = -\frac{\partial\Omega}{\partial X^\mu} = 0,$$

$$\{\Omega, c\} = i\frac{\partial\Omega}{\partial \bar{c}} = 0, \qquad \{\Omega, \bar{c}\} = i\frac{\partial \Omega}{\partial c} = \frac{i}{2}(p_\mu p^\mu + m^2).$$

Collectively, using integration by parts to swap derivatives around this yields

$$\delta_\Omega S_{BRST} = \int d\tau \frac{d}{d\tau}\left(p_\mu c p^\mu + ici\frac{1}{2}(p_\mu p^\mu + m^2) + 0\right) = \int d\tau \frac{d}{d\tau}\left[\frac{1}{2}c(p^2 - m^2)\right].$$

This shows that $S_{BRST}$ is invariant under the action of $\Omega$, up to a total time derivative. It's further worth noting that this calculation shows our choice of $S_{BRST}$ is not uniquely determined by the above two goals. Rather, we may add any BRST-exact term to the BRST-extended Hamiltonian and produce a system which, while still the same classical system on the physical sector, may differ on the non-physical sector. In particular, this allows us to introduce BRST-exact terms which make subsequent computations simpler — a common technique.

As a consistency check, we now apply Noether's theorem, which states that if $S$ is invariant under an infinitesimal transformation $q^\mu \mapsto (q')^\mu = q^\mu + \varepsilon f^\mu(q)$, then the charge $\Omega = p_\mu f^\mu(q)$ is conserved. In our situation, this gives

$$\Omega = p\delta_\Omega X^\mu + i\bar{c}\delta_\Omega c - \delta_\Omega S_{BRST} = \frac{1}{2}c(p^2 + m^2).$$

This is exactly the BRST charge we began with.

## 3. Physical states

Having introduced these nonphysical fields, we are now challenged to produce the physical states for the original system from ours. Invariance under the BRST charge operator exactly corresponds to satisfying the original constraint equations — but there are more states invariant under the BRST charge than there are physical states. Namely, if $|\Psi\rangle$ satisfies $\Omega|\Psi\rangle = 0$, then so does $|\Psi\rangle + \Omega|\chi\rangle$ for any state $\chi$, since $\Omega^2 = 0$. It turns out this is the only thing that goes wrong: the cohomology group $\ker\Omega/\operatorname{im}\Omega$ encodes exactly the distinguishable physical states of the system.

Now we have an operator algebra for which we'd like to study the irreducible representations, i.e., the building blocks of state spaces on which these fields may act. Since the classical fields and ghost fields commute with one another, we may treat them separately. The classical states are labeled by $D$-momenta $k$, and the ghost states are labeled by $\uparrow$ and $\downarrow$, corresponding to the unique 2-dimensional irreducible representation of the ghost algebra. The complete set of states in total is labeled by $|\uparrow, k\rangle$ and $|\downarrow, k\rangle$, which are determined by the operators

$$p^\mu|\uparrow, k\rangle = k^\mu|\uparrow, k\rangle, \qquad p^\mu|\downarrow, k\rangle = k^\mu,$$
$$\bar{c}|\uparrow, k\rangle = |\uparrow, k\rangle, \qquad \bar{c}|\downarrow, k\rangle = 0,$$
$$c|\uparrow, k\rangle = 0, \qquad c|\downarrow, k\rangle = |\downarrow, k\rangle.$$

Using the definition of $\Omega$, then, we can compute

$$\Omega|\downarrow, k\rangle = \frac{1}{2}(k^2 + m^2)|\uparrow, k\rangle, \qquad \Omega|\uparrow, k\rangle = 0.$$

So, the physical states are $|\uparrow, k\rangle$ for any $k$ and $|\downarrow, k\rangle$ for $k$ satisfying $k^2+m^2 = 0$. However, because $\Omega\left(\frac{2}{k^2+m^2}|\downarrow, k\rangle\right) = |\uparrow, k\rangle$, the states $|\uparrow, k\rangle$ with $k^2+m^2 \neq 0$ are not physically distinguishable from $0$ (i.e., they are BRST-coboundaries), so we are left only with $|\uparrow, k\rangle$ and $|\downarrow, k\rangle$ for $k^2 + m^2 = 0$.[1]

## 4. Propagators

If a physical theory is to be taken seriously by physicists then it should be able to predict various observations made in nature - one should be able to compute physically meaningful observables of the theory. In quantum theory the observables correspond to vacuum expectation values of products of the operators defined in the theory,

$$\langle 0|\mathscr{O}_1(x_1)\cdots\mathscr{O}_n(x_n)|0\rangle.$$

The importance of such expectation values stems from the fact that they are related, via the reduction formula, to scattering amplitudes. Note, we are ignoring subtleties such as the time-ordering of operators.

In terms of path integrals, the vacuum expectation values of such operators are computed as

$$\langle 0|\mathscr{O}_1(x_1)\cdots\mathscr{O}_n(x_n)|0\rangle = \frac{\int D[\Psi]D[\Psi^\dagger]\mathscr{O}_1(x_1)\cdots\mathscr{O}_n(x_n)e^{iS[\Psi,\Psi^\dagger]}}{\int D[\Psi]D[\Psi^\dagger]e^{iS[\Psi,\Psi^\dagger]}},$$

where $\Psi$ corresponds to all fields appearing in the theory. If our theory is free - the fields do not interact - then the action is Gaussian and these integrals can easily be evaluated. However, free theories are (usually) very boring, so we need a way of interpreting expressions like above when the integrals are not Gaussian. As we will now demonstrate, a way around this is to expand the action in powers of the coupling constant.

To begin, note that every action can be decomposed into a free part, corresponding to the quadratic term, and an interaction part, correspoding to the coupled fields,

$$S[\Psi, \Psi^\dagger] = S_0[\Psi, \Psi^\dagger] + S_I[\Psi, \Psi^\dagger].$$

If we now expand $\exp(iS_I)$ out in a power series then we have a sum of path integrals which all look like

$$\frac{\int D[\Psi]D[\Psi^\dagger]\mathscr{O}_1(x_1)\cdots\mathscr{O}_n(x_n)g^nP_n(\Psi,\Psi^\dagger)e^{iS_0[\Psi,\Psi^\dagger]}}{\int D[\Psi]D[\Psi^\dagger]e^{iS_0[\Psi,\Psi^\dagger]}},$$

[1] Bram then additionally says that $|\uparrow, k\rangle$ is disallowed, because its BRST-momentum is nonvanishing, and its pairing with a genuine physical state results in a delta function. This, for whatever reason, "violates kinematic law." I don't understand this.

where $g$ is the coupling constant and $P_n(\Psi, \Psi^\dagger)$ are monomials in the fields. Thus, we now have a sum of Gaussian integrals which we can compute. Thanks to Feynman, we have a diagrammatic way of telling which part of the sum we are computing and also the contribution due to this integral.

A Feynman diagram is nothing more than a graph where each edge corresponds to the propagation of particles (of the same type) and each vertex corresponds to places where these particles can meet and interact. So, we can see that the two most important things needed to compute a Feynman diagram are the propagators and the interaction terms. The propagators correspond to the free part of the action and the interaction terms correspond to the interaction part of the action, both in the momentum representation. More precisely, the propagator - which is related to the amplitude for a state to evolve into another - is given by the two-point function

$$\langle 0|\mathcal{O}(x_1)\mathcal{O}(x_2)|0\rangle \equiv G(x_2, x_1).$$

For our example this becomes

$$G(x_2^\mu, x_1^\mu) = \int_0^\infty dT \int \mathscr{D}x\mathscr{D}p\mathscr{D}B\mathscr{D}e\mathscr{D}c\mathscr{D}\bar{c}\ e^{iS_{gf}[x,p,B,e,c,\bar{c}]},$$

with the gauge-fixed action $S_{gf}$ from before, namely

$$S_{gf} = \int_0^T d\tau \left( p_\mu \dot{x}^\mu + B(e-1) + i\bar{c}\dot{c} - \frac{1}{2}(p_\mu p^\mu + m^2) \right).$$

Now, as the fields $B$, $e$, $c$ and $\bar{c}$ are linear in the action we can simply integrate them out using various delta functions. This leaves us with

$$G(x_2^\mu, x_1^\mu) = \int_0^\infty dT \int \mathscr{D}x\mathscr{D}p\ e^{i\int d\tau \left[p_\mu \dot{x}^\mu - \frac{1}{2}(p_\mu p^\mu + m^2)\right]}.$$

This can be evaluated in the usual method of taking time slices giving

$$G(x_2^\mu, x_1^\mu) = \int_0^\infty dT\ T^{-D/2} \exp\left[\frac{1}{2} i \left( \frac{(x_2^\mu - x_1^\mu)^2}{T} - (m^2 - i\varepsilon)T \right)\right],$$

where we've mulitplied the integrand by $e^{-\varepsilon T/2}$ to regulate the integral (a common practice in QFT). Note, the propagator only depends on the difference between the two coordinates $G(x_2^\mu - x_1^\mu)$, as one would expect for two-point functions. To get the usual expression for the propagator we perform a Fourier transform to momentum space,

$$G(x_2^\mu - x_1^\mu) = \int \frac{d^D p}{(2\pi)^D} \frac{e^{ip\cdot(x_2 - x_1)}}{p^2 + m^2 - i\varepsilon}.$$

## 5. References

In preparing this document, we made use of Polchinski's and Green-Schwarz-Witten's books on string theory, Bram Wouter's M.Sc. thesis from the University of Amsterdam, and various incredibly helpful papers by van Holten on the subject of BRST quantization.