Bordism: An introduction and sampler

Graduate Student Topology & Geometry Conference

Eric Peterson

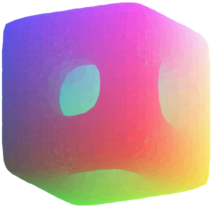
Thanks, Neil Strickland, for your talk notes.

http://math.harvard.edu/~ecp/latex/talks/bordism.pdf

April 2, 2016

Definition

A *manifold* ( ) is a compact space that locally looks like $\mathbb{R}^n$ ( ).



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Vocabulary: Manifolds

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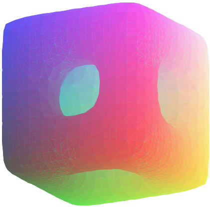

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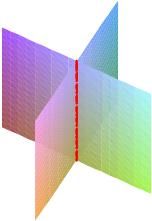

not a manifold

Vocabulary: Manifolds

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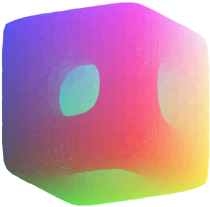
A *manifold* (with boundary, or "∂") is a compact space that locally looks like $\mathbb{R}^n$ (or $\mathbb{R}^{n-1} \times [0, \infty)$).



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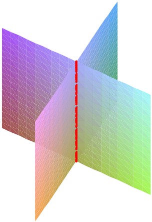

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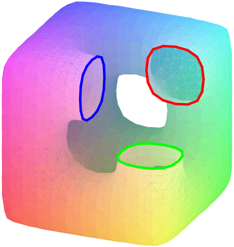

manifold with ∂

Vocabulary: Bordisms

Good examples

- "Most" of most naturally occuring spaces.
- Curves in dimension 1, surfaces in dimension 2.
- All finite CW-complexes are h'topy equivalent to a manifold.
- Physical spacetimes.
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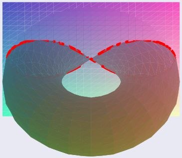

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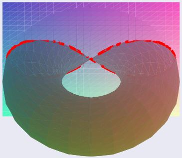
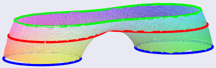


perturb $\rightsquigarrow$



a "bordism"

Using manifolds to probe spaces

Definition

$$C_n(X) := \mathbb{Z}\left\{\Delta^n \xrightarrow{\sigma} X \,\middle|\, \begin{array}{c} \Delta^n \text{ the standard } n\text{–simplex}, \\ \sigma \text{ continuous} \end{array}\right\},$$

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$C_n^{\mathrm{geom}}(X)$ is *already* an abelian monoid with cancellation under $\sqcup$, and ∂ gives a map of monoids.

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- It is valued in abelian groups.
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- It converts disjoint unions to direct sums.
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- "Dimension axiom": $H_*^{\mathrm{geom}}(pt)$?

The bordism ring

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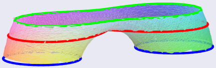

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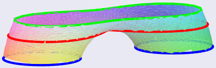

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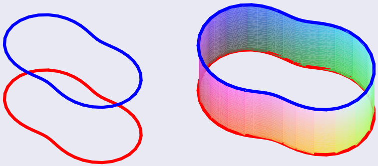
Definition

The resulting ring $H_*^{\mathrm{geom}}(pt)$ is called the *bordism ring*.
(Remember that this is a natural place for "intersection theory".)

Features of the bordism ring

It is 2–torsion, since every closed manifold has a cylinder:



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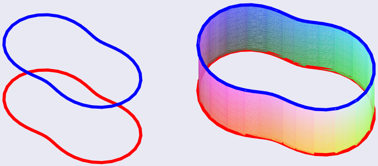

It is empty in dimension 1:

$$\left.\begin{aligned} [S^1]+[S^1]+[S^1] &= 0, \\ 2\cdot[S^1] &= 0 \end{aligned}\right\} \quad \Rightarrow \quad 0=[S^1].$$

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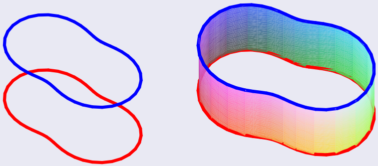

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Surprise!

Using homotopy theory, we can completely compute this ring.

Designer homotopy types

There are *Eilenberg–Mac Lane* spaces $K(A, n)$ with

$$\pi_* K(A, n) = \begin{cases} A & \text{if } * = n, \\ 0 & \text{otherwise.} \end{cases}$$

They have two important properties:

$$H^n(X; A) = \mathrm{HoSpaces}(X, K(A, n)),$$
$$H_n(X; A) = \operatorname*{colim}_m \pi_{n+m}(X \wedge K(A, m)).$$

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Claim

There are analogous spaces for H_*^{geom}.

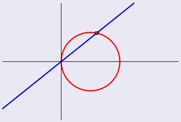
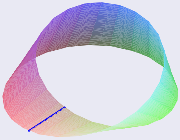
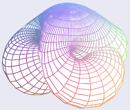
This gives us a program for computing $H_*^{\mathrm{geom}}(pt)$: find these special spaces and compute their homotopy.

Grassmannians

Definition

- $B(m,k)$ is the space of k–planes in $\mathbb{R}^{m+k}$.
- $E(m,k)$ is the space of pairs $(V, v \in V)$, $V \in B(m,k)$.
- $M(m,k) = E(m,k) \cup \{\infty\}$.

Example: $m = 1$, $k = 1$

$B(1,1) \simeq \mathbb{R}\mathrm{P}^1$	$E(1,1)$	$M(1,1) \simeq \mathbb{R}\mathrm{P}^2$
		

Main claim

geometric cornerstone	Δ^n	manifolds
homology functor	$H_n(-; A)$	$H_n^{\text{geom}}(-)$
system of spaces	$\{K(A, n)\}_n$	$\{M(m, k)\}_{m,k}$

As justification, we need assignments

$$\pi_{n+k} M(m, k) \leftrightarrows H_n^{\text{geom}}(pt).$$

The Pontryagin–Thom construction

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The rightward direction: intersection theory

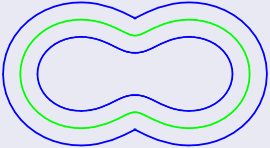
$$\begin{tikzcd} \begin{array}{c} S^{n+k} \cap B(m,k) \\ + \text{ perturbation} \end{array} \arrow[r] \arrow[d] & B(m,k) \arrow[d] \\ S^{n+k} \arrow[r] & M(m,k) \end{tikzcd}$$

The Pontryagin–Thom construction

The leftward direction

Start with an m–dimensional manifold M, as well as...

- ...an embedding $M \subseteq \mathbb{R}^{m+k}$.
- ...an embedding of its normal bundle $M \subseteq \nu \subseteq \mathbb{R}^{m+k}$.



$$M \longrightarrow \nu \longrightarrow \mathbb{R}^{m+k} \cup \{\infty\}$$

$$B(m,k) \to E(m,k) \longrightarrow M(m,k).$$

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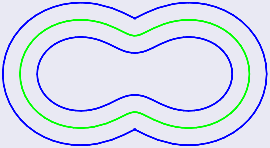

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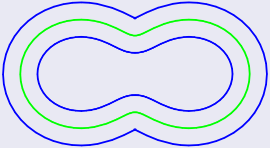

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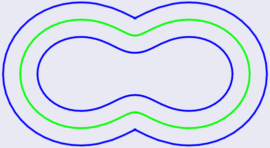

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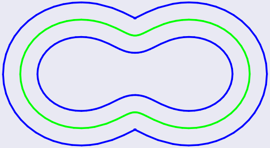

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Main claim

These constructions are inverses, so

$$H_n^{\mathrm{geom}}(pt) = \operatorname*{colim}_{m,k} \pi_{n+k} M(m,k).$$

Using stable homotopy to reduce the problem

Systems like $S = \{M(m,k)\}_{m,k}$ belong to *stable homotopy*.

The *E–Adams spectral sequence* attempts to recover $\pi_* S$ from $E_* S$ + structure. (Not always possible: S can be “invisible” to E.)

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Plan

Let's use this machine to compute

$$\{H_*(M(m,k);\mathbb{F}_2)\}_{m,k} \rightsquigarrow \operatorname*{colim}_{m,k} \pi_{n+k} M(m,k) = H_n^{\mathrm{geom}}(pt).$$

Definition

The *ring of (all) functions* on G is the ring k^G.

It is a commutative Hopf algebra with diagonal

$$\begin{aligned} \Delta: \quad k^G &\to k^{G \times G}, \\ \Delta: \quad f &\mapsto ((x, y) \mapsto f(xy)). \end{aligned}$$

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$$\begin{array}{ccc} M \text{ a } G\text{–module} & \leftrightarrows & M \text{ a } k[G]\text{–module} \\ & \overset{\text{finiteness}}{\leftrightarrows} & M^* \text{ a } k^G\text{–comodule} \end{array}$$

Interlude: Algebraic geometry and group cohomology

Cohomology using sheaves over spaces

$$Y \xleftarrow[f]{} X \qquad\qquad \mathrm{Sheaves}_Y \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathrm{Sheaves}_X$$

Cohomology: set $f\colon X \to pt$ and use $H^*(X; A) = \pi_* Rf_* \underline{A}$.

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(This is uninteresting without more algebraic geometry.)

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Comodules over Hopf algebras

$$H \xleftarrow{f} G \qquad \begin{array}{ccc} M & \longmapsto & M \\ \mathrm{Comodules}_{kH} & \underset{f_*}{\overset{f^*}{\rightleftarrows}} & \mathrm{Comodules}_{kG} \\ N\square_{kG} k^H & \longleftarrow\!\!\shortmid & N, \end{array}$$

where

$$N_1 \square_C N_2 = \ker\left(N_1 \otimes N_2 \xrightarrow{\psi_1\otimes 1 - 1\otimes\psi_2} N_1 \otimes_k C \otimes_k N_2\right).$$

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$$H \xleftarrow{f} G \qquad \begin{array}{ccc} M & \longmapsto & M \\ \mathrm{Comodules}_{kH} & \underset{f_*}{\overset{f^*}{\rightleftarrows}} & \mathrm{Comodules}_{kG} \\ N \square_{kG} k^H & \longleftarrow\!\shortmid & N, \end{array}$$

where

$$N_1 \square_C N_2 = \ker\left(N_1 \otimes N_2 \xrightarrow{\psi_1 \otimes 1 - 1 \otimes \psi_2} N_1 \otimes_k C \otimes_k N_2\right).$$

Cohomology: use $f\colon G \to 1$ and $H^*(G; N) = \pi_* Rf_* N$.

Definition

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Back to the $H\mathbb{F}_2$–Adams spectral sequence

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Theorem

$H_*(X; \mathbb{F}_2)$ is a comodule for $\mathcal{A}$, i.e., a *generalized representation*.

The $H\mathbb{F}_2$–Adams SS starts w/ *generalized group cohomology*:

$$H^*(\mathcal{A}; H_*(X; \mathbb{F}_2)) \Rightarrow \pi_* X^{\widehat{}}_2.$$

The Adams spectral sequence applied to $\{M(n,k)\}_{n,k}$

Calculations

$$\begin{aligned} H_*(\operatorname*{colim}_{m,k} B(m,k)) &\cong H_* \operatorname*{colim}_{k} BO(k) \cong H_* BO \\ &\cong \mathbb{F}_2[b_0, b_1, b_2, \ldots] \end{aligned}$$

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Any $f \in M(T)$ can be written uniquely as $f = g \circ h$ for $g \in A(T)$, $h(x) = x + \sum_{j \neq 2^k} a_{j-1} x^j$.

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Setting $Q(T) := \{h \in T[\![x]\!] \mid h = x + \sum_{j \neq 2^k - 1} a_j x^{j+1}\}$, then $H^*(\mathcal{A}; \mathcal{M}) = \mathcal{Q}$ is the cofixed points

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Continuations

Punchline

$$H_*^{\mathrm{geom}}(pt) = \mathcal{Q} = \mathbb{F}_2[c_j \mid j \geq 1, j \neq 2^k - 1]$$
$$= \mathbb{F}_2[c_2, c_4, c_5, c_6, c_8, c_9, c_{10}, c_{11}, c_{12}, c_{13}, c_{14}, c_{16}, \ldots].$$

We figured this out using stable homotopy and algebraic geometry.

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Different sorts of bordism:

- Oriented, spin, symplectic, ...
- Complex, complex-orientable, . . .

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- Oriented, spin, symplectic, ...
- Complex, complex-orientable, ...
    - $\rightsquigarrow$ chromatic homotopy.

The moral story: descent

Algebra

$$\begin{tikzcd} \mathrm{Modules}_R \arrow[r, shift left] \arrow[rr, bend left, "f^*"] & \mathrm{Comodules}_{S \otimes_R S} \arrow[l, shift left, "H^0"] \arrow[r] & \mathrm{Modules}_S \end{tikzcd}$$

In excellent cases, the left-hand pair is an equivalence.

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It's possible to assign objects to concepts in stable homotopy:

$$\text{stable homotopy} \rightsquigarrow \mathbb{S}, \quad \text{mod–2 homology} \rightsquigarrow H\mathbb{F}_2.$$

There is a ring map $f\colon \mathbb{S} \to H\mathbb{F}_2$, and its descent is controlled by a coring $H\mathbb{F}_2 \wedge H\mathbb{F}_2$ with homotopy groups $\mathcal{A}$.

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A bridge

The Adams spectral sequence mediates between homotopy (potentially very hard) and homology (pure algebra).

Thank you!!

http://math.harvard.edu/~ecp/latex/talks/bordism.pdf