Algebraic topology and algebraic number theory

Graduate Student Topology & Geometry Conference

Eric Peterson

http://math.berkeley.edu/~ericp/latex/talks/austin-2014.pdf

April 5, 2014

Formal groups

In this talk, p is an *odd* prime and k is a finite field, char $k = p$.

Definition

A *formal group law* is a power series $x +_F y \in R[\![x, y]\!]$ satisfying

$$x +_F 0 = x, \quad x +_F y = y +_F x, \quad x +_F (y +_F z) = (x +_F y) +_F z.$$

Idea

Complex geometry: these come from charts on Lie groups.
Arithmetic geometry: take a more exotic R than $\mathbb{C}$, like $\mathbb{F}_p$.

Example: $\mathbb{G}_m$

The formal multiplicative group $\mathbb{G}_m$ is presented by

$$x +_{\mathbb{G}_m} y = 1 - (1 - x)(1 - y) = x + y - xy.$$

Some local class field theory

Most number theoretic questions can be interpreted through $\bar{K}$ and $\mathrm{Gal}(\bar{K}/K)$ for K some (local) number field.

Some local class field theory

Most number theoretic questions can be interpreted through $\bar{K}$ and $\mathsf{Gal}(\bar{K}/K)$ for K some (local) number field.

Theorem (Lubin–Tate)

For a local number field K, there is a *maximal abelian extension*:

$$\mathsf{Gal}(K^{\mathrm{ab}}/K) = \mathsf{Gal}(\bar{K}/K)^{\mathrm{ab}}.$$

One can construct it explicitly by studying the torsion points of a certain formal group Γ_K over $\mathcal{O}_K$.

Example: $K = \mathbb{Q}_p$

$\Gamma_{\mathbb{Q}_p}$ is given by $\mathbb{G}_m$. "Torsion points" of $\mathbb{G}_m$ are unipotent elements, and in fact $\mathbb{Q}_p^{\mathrm{ab}} = \mathbb{Q}_p(\zeta_n : n > 0)$.

Fields in stable homotopy theory

A *field spectrum* is a ring spectrum with Künneth isomorphisms.
E.g.:

$$\left.\begin{aligned} H^{dR} = H\mathbb{R}, \\ H\mathbb{F}_p, \end{aligned}\right\} \text{ ordinary}$$

A *field spectrum* is a ring spectrum with Künneth isomorphisms. E.g.:

$$\left.\begin{aligned} H^{dR} = H\mathbb{R}, \\ H\mathbb{F}_p, \end{aligned}\right\} \text{ ordinary} \qquad \left.\begin{aligned} KU/p, \\ \ldots? \end{aligned}\right\} \text{ extraordinary}$$

Fields in stable homotopy theory

A *field spectrum* is a ring spectrum with Künneth isomorphisms. E.g.:

$$\left.\begin{array}{r} H^{dR} = H\mathbb{R}, \\ H\mathbb{F}_p, \end{array}\right\} \text{ ordinary} \qquad \left.\begin{array}{r} KU/p, \\ \ldots? \end{array}\right\} \text{ extraordinary}$$

Theorem (Devinatz–Hopkins–Smith)

There is a bijection

$$\left\{ \begin{array}{c} \text{formal groups} \\ \text{over } k \end{array} \right\} \xrightarrow{K} \left\{ \begin{array}{c} \text{2-periodic} \\ \text{field spectra} \\ \text{with } \pi_0 = k \end{array} \right\}.$$

The spectrum $K(\Gamma)$ is called *the Morava K-theory for* Γ.

Example: $\Gamma = \mathbb{G}_m$

KU/p is a model for $K(\mathbb{G}_m)$.

Our three examples come with natural "deformations":

$$H\mathbb{R} \to H\mathbb{R}, \qquad H\mathbb{Z}_p \to H\mathbb{F}_p, \qquad KU_p^\wedge \to KU/p.$$

All of these are governed by Bockstein operations.

Homology operations and Morava E-theories

Our three examples come with natural "deformations":

$$H\mathbb{R} \to H\mathbb{R}, \qquad H\mathbb{Z}_p \to H\mathbb{F}_p, \qquad KU_p^\wedge \to KU/p.$$

All of these are governed by Bockstein operations.

Theorem (Morava et al.)

For $d = \mathrm{ht}(\Gamma)$ finite, $K(\Gamma)$ has d Bocksteins, giving a spectrum

$$E(\Gamma) \to K(\Gamma)$$

called *the Morava E-theory for* Γ. It takes values in modules over $\mathrm{Def}(\Gamma) = \mathbb{W}(k)[\![u_1, \ldots, u_{d-1}]\!]$ with an $\mathrm{Aut}\,\Gamma$ action.

Example: $\Gamma = \mathbb{G}_m$

$KU_p^\wedge \to KU/p$ is a model for $E(\mathbb{G}_m) \to K(\mathbb{G}_m)$.

Our three examples come with natural "deformations":

$$H\mathbb{R} \to H\mathbb{R}, \qquad H\mathbb{Z}_p \to H\mathbb{F}_p, \qquad KU_p^\wedge \to KU/p.$$

All of these are governed by Bockstein operations.

Theorem (Morava et al.)

For $d = \mathrm{ht}(\Gamma)$ finite, $K(\Gamma)$ has d Bocksteins, giving a spectrum

$$E(\Gamma) \to K(\Gamma)$$

called *the Morava E-theory for* Γ. It takes values in modules over $\mathrm{Def}(\Gamma) = \mathbb{W}(k)[[u_1, \ldots, u_{d-1}]]$ with an $\mathrm{Aut}\,\Gamma$ action.

Example: $\Gamma = \mathbb{G}_m$

$KU_p^\wedge \to KU/p$ is a model for $E(\mathbb{G}_m) \to K(\mathbb{G}_m)$.

Homotopy groups via L-functions

$E(\Gamma)$ is valued in modules over $\mathrm{Def}(\Gamma) = \mathbb{W}(k)[\![u_1, \ldots, u_{d-1}]\!]$ with an $\mathrm{Aut}\,\Gamma$ action.

Theorem (Harris–Taylor et al., "local Langlands correspondence")

There is a correspondence among certain representations of:

$$\mathrm{Aut}\,\Gamma, \quad \mathrm{GL}_d(K), \quad W_K \subset \mathrm{Gal}(\bar{K}/K).$$

Homotopy groups via L-functions

$E(\Gamma)$ is valued in modules over $\mathrm{Def}(\Gamma) = \mathbb{W}(k)[[u_1, \ldots, u_{d-1}]]$ with an $\mathrm{Aut}\,\Gamma$ action.

Theorem (Harris–Taylor et al., "local Langlands correspondence")

There is a correspondence among certain representations of:

$$\mathrm{Aut}\,\Gamma, \quad \mathrm{GL}_d(K), \quad W_K \subset \mathrm{Gal}(\bar{K}/K).$$

Theorem (Salch(–Morava), special case of $\Gamma = \mathbb{G}_m$)

For nice enough finite cell complexes X, there is an L-function

$$L(E(\mathbb{G}_m)_*(X); s).$$

It is analytic to the right of a pole at $s = \dim X$, and its special values at $s > \dim X$ have denominators encoding the ranks of the $E(\mathbb{G}_m)$-local homotopy groups of X away from 2.

Homotopy groups via L-functions

Example: $L_{E(\mathbb{G}_m)}S^0$ (Adams–Hopkins–Ravenel)

The L-function associated to S^0 is the Riemann ζ-function.

n	1	2	3	4	5	6	7
$\lvert\pi_{2n+1}L_{E(\mathbb{G}_m)}S^0\rvert$	$2^3 3^1$	1	$2^4 3^1 5^1$	2^1	$2^3 3^2 7^1$	1	$2^5 3^1 5^1$
$\mathrm{denom}(\zeta(-n))$	$2^2 3^1$	1	$2^3 3^1 5^1$	1	$2^2 3^2 7^1$	1	$2^4 3^1 5^1$

Example: $L_{E(\mathbb{G}_m)} S^0$ (Adams–Hopkins–Ravenel)

The L-function associated to S^0 is the Riemann ζ-function.

n	1	2	3	4	5	6	7
$\lvert\pi_{2n+1} L_{E(\mathbb{G}_m)} S^0\rvert$	$2^3 3^1$	1	$2^4 3^1 5^1$	2^1	$2^3 3^2 7^1$	1	$2^5 3^1 5^1$
$\mathrm{denom}(\zeta(-n))$	$2^2 3^1$	1	$2^3 3^1 5^1$	1	$2^2 3^2 7^1$	1	$2^4 3^1 5^1$

Example: $L_{E(\mathbb{G}_m)}S^0$ (Adams–Hopkins–Ravenel)

The L-function associated to S^0 is the Riemann ζ-function.

n	1	2	3	4	5	6	7
$\lvert\pi_{2n+1}L_{E(\mathbb{G}_m)}S^0\rvert$	$2^3 3^1$	1	$2^4 3^1 5^1$	2^1	$2^3 3^2 7^1$	1	$2^5 3^1 5^1$
$\mathrm{denom}(\zeta(-n))$	$2^2 3^1$	1	$2^3 3^1 5^1$	1	$2^2 3^2 7^1$	1	$2^4 3^1 5^1$

Idea

This set-up encourages us to work one prime at a time and invoke Euler factorizations and p-adic L-functions.

What might $n \in \mathbb{Z}_p$ mean on the homotopy groups side?

Lines in the $K(\Gamma)$-local category

What's a sphere, anyway?

The $\wedge$-invertible spectra are exactly the stable spheres.

What's a sphere, anyway?

The $\wedge$-invertible spectra are exactly the stable spheres.

$$\begin{tikzcd} \text{Spectra} \arrow[rr, bend left, "E(\Gamma)_*"] & & \mathrm{Mod}_{\mathrm{Def}(\Gamma), \mathrm{Aut}\,\Gamma} \end{tikzcd}$$

What's a sphere, anyway?

The $\wedge$-invertible spectra are exactly the stable spheres.

$$\begin{tikzcd} \text{Spectra} \arrow[r, "L_{K(\Gamma)}"'] \arrow[rr, bend left, "E(\Gamma)_*"] & \text{Spectra}_{K(\Gamma)} \arrow[r, "E(\Gamma)_*", "\text{conservative}"'] & \text{Mod}_{\text{Def}(\Gamma), \text{Aut}\,\Gamma} \end{tikzcd}$$

What's a sphere, anyway?

The $\wedge$-invertible spectra are exactly the stable spheres.

$$\begin{tikzcd}
\mathrm{Spectra} \arrow[r, "L_{K(\Gamma)}"'] \arrow[rr, bend left, "E(\Gamma)_*"] & \mathrm{Spectra}_{K(\Gamma)} \arrow[r, "E(\Gamma)_*", "\text{conservative}"'] & \mathrm{Mod}_{\mathrm{Def}(\Gamma), \mathrm{Aut}\,\Gamma}
\end{tikzcd}$$

Question

What are the $\wedge$-invertible objects in $\mathrm{Spectra}_{K(\Gamma)}$?

Lines in the $K(\Gamma)$-local category

Theorems (Hopkins–Mahowald–Sadofsky; Hopkins–Strickland)

For $\Gamma = \mathbb{G}_m$ and $p \geq 3$, $\mathrm{Pic} = \mathbb{Z}_p^\times \rtimes \mathbb{Z}/2$ (even spheres $\rtimes$ S^1). The homotopy groups of $L_{K(\mathbb{G}_m)} S^{-1}$ indexed on the $\mathbb{Z}_p^\times$-factor in Pic agree with the p-adic interpolation of the ζ-function.

Lines in the $K(\Gamma)$-local category

Theorems (Hopkins–Mahowald–Sadofsky; Hopkins–Strickland)

For $\Gamma = \mathbb{G}_m$ and $p \geq 3$, $\mathrm{Pic} = \mathbb{Z}_p^\times \rtimes \mathbb{Z}/2$ (even spheres $\rtimes$ S^1). The homotopy groups of $L_{K(\mathbb{G}_m)} S^{-1}$ indexed on the $\mathbb{Z}_p^\times$-factor in Pic agree with the p-adic interpolation of the ζ-function.

Theorems (Goerss, Hopkins, Mahowald, Rezk, Sadofsky)

- $\Gamma = \mathbb{G}_m$, $p = 2$: $\mathbb{Z}/2 \times (\mathbb{Z}_2^\times \rtimes \mathbb{Z}/2)$.
- $\Gamma = C_{ss}$, $p \geq 5$: $(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}/(p^2 - 1)) \rtimes \mathbb{Z}/2$.
- $\Gamma = C_{ss}$, $p = 3$: $\mathbb{Z}/3 \times \mathbb{Z}/3 \times ((\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}/(3^2 - 1)) \rtimes \mathbb{Z}/2)$.

All other values unknown. How can we compute them? What can these say about Salch's L-functions?

Bonus slide: Lines in the $K(\Gamma)$-local category

Theorem (Hopkins–Mahowald–Sadofsky)

A spectrum X is $K(\Gamma)$-locally invertible if and only if $K(\Gamma)_*X$ is a $K(\Gamma)_*$-line (i.e., $\dim K(\Gamma)_*X = 1$).

Bonus slide: Lines in the $K(\Gamma)$-local category

Theorem (Hopkins–Mahowald–Sadofsky)

A spectrum X is $K(\Gamma)$-locally invertible if and only if $K(\Gamma)_* X$ is a $K(\Gamma)_*$-line (i.e., $\dim K(\Gamma)_* X = 1$).

Theorem (P.)

If X is a space with $K(\Gamma)^* X$ a power series ring, there is a map

$$T_+ L_{K(\Gamma)} \Sigma^\infty X \to L_{K(\Gamma)} \Sigma^\infty X$$

selecting its algebro-geometric tangent space on cohomology.

E.g.:

- $T_+ \mathbb{C}\mathrm{P}^\infty \simeq \mathbb{C}\mathrm{P}^1 \simeq S^2$.

Bonus slide: Lines in the $K(\Gamma)$-local category

Theorem (Hopkins–Mahowald–Sadofsky)

A spectrum X is $K(\Gamma)$-locally invertible if and only if $K(\Gamma)_*X$ is a $K(\Gamma)_*$-line (i.e., $\dim K(\Gamma)_*X = 1$).

Theorem (P.)

If X is a space with $K(\Gamma)^*X$ a power series ring, there is a map

$$T_+ L_{K(\Gamma)}\Sigma^\infty X \to L_{K(\Gamma)}\Sigma^\infty X$$

selecting its algebro-geometric tangent space on cohomology.

E.g.:

- $T_+\mathbb{CP}^\infty \simeq \mathbb{CP}^1 \simeq S^2$.
- $T_+K(\mathbb{Q}_p/\mathbb{Z}_p, d) \simeq S^0[\det]$ for $p \gg d$.