Cotangent spectra and the determinantal sphere

by

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Abstract

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We explore the generalization of cellular decomposition in chromatically localized stable categories suggested by Picard–graded homotopy groups. In particular, for $K(d)$ a Morava K–theory, we show that the Eilenberg–Mac Lane space $K(\mathbb{Z}, d+1)$ has a $K(d)$–local cellular decomposition tightly analogous to the usual decomposition of infinite–dimensional complex projective space (alias $K(\mathbb{Z}, 2)$) into affine complex cells. Additionally, we identify these generalized cells in terms of classical invariants — i.e., we show that their associated line bundles over the Lubin–Tate stack are tensor powers of the determinant bundle. (In particular, these methods give the first choice–free construction of the determinantal sphere $\mathbb{S}[\det]$.) Finally, we investigate the bottom attaching map in this exotic cellular decomposition, and we justify the sense in which it selects a particular $K(d)$–local homotopy class

$$\Sigma^{-1}\mathbb{S}[\det]^2 \to \mathbb{S}[\det]$$

generalizing the classical four "Hopf invariant 1" classes h in the cofiber sequences

$$\Sigma^{-1}(\mathbb{S}^n)^{\wedge 2} \xrightarrow{h} \mathbb{P}^1_k \to \mathbb{P}^2_k$$

associated to the four normed real division algebras:

k	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
n	1	2	4	8
h	2	η	ν	σ

We also include a lengthy introduction to the subject of chromatic homotopy theory, outlining all of the tools relevant to the statements of our original results.

Contents

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Chapter 0

Introduction

Before proceeding to the body of the thesis, we give an overview of what to expect in more detail than provided in the abstract above.

In Chapter 1, we will introduce a sequence of homology theories K_Γ and E_Γ, the study of which is generally referred to as "chromatic homotopy theory". Morava first constructed these homology theories by employing a fundamental connection, uncovered by Quillen, between stable homotopy theory and the theory of formal Lie groups. Because of the strength of this connection, the Morava K–theories exert a remarkable amount of control over the stable category, and it is often profitable to study a given problem in stable homotopy theory by considering its analogues in the K_Γ–local stable category for every K_Γ.

Our primary goal in this thesis is to compute an involved example of the following maxim:

After K_Γ–localization, spectra can acquire unusually efficient cellular decompositions.

There is an uninteresting sense in which the maxim above is often true: any kind of localization has the potential to lose information, and hence it may render a complicated object simple if those complications are locally invisible. A more interesting interpretation of the maxim comes from the observation that the notion of "cell" in the K_Γ–local stable category is dramatically enlarged from the usual one. In the global stable category, an $(n+1)$–cell is attached to a spectrum X along a homotopy class $\mathbb{S}^n \to X$ — i.e., using an element in $\pi_n X$. The sphere spectra $\mathbb{S}^n$ are uniquely characterized in the global stable category by the property that $\mathbb{S}^n$ has a mate $\mathbb{S}^{-n}$ such that $\mathbb{S}^n \wedge \mathbb{S}^{-n}$ is the monoidal unit $\mathbb{S}^0$. In the K_Γ–local stable category, however, there are many more such objects, and it has been previously observed that it is profitable to grade homotopy groups using the entire collection. If we take these as selecting the loci along which we are permitted to attach new cells, then we can build considerably more intricate decompositions. (Precisely how much more intricate is unknown, as there are very few K_Γ for which we can exhaustively name the invertible K_Γ–local spectra.)

However, the K_Γ–local category has bad properties which prohibit the easy manufacture of cellular decompositions. First consider, again, the global situation. For an n–connective spectrum X, the Hurewicz isomorphism $H_n X \cong \pi_n X$ shows that any generating set of $H_n X$ can be lifted to a sequence of homotopy classes of X which, in turn, can be used to model the bottom cells of X. Continuing inductively shows that the integral homology groups $H\mathbb{Z}_*X$ can be used to describe a (non-functorial) cellular structure on X which is *minimal* in the sense that it uses the minimum number of cells in each graded dimension. However, in the K_Γ–local case, it is a theorem of Hovey and Strickland that the K_Γ–local category has no localizing or colocalizing subcategories, and hence no natural notion of cellular or Postnikov decomposition (since a t–structure would, in particular, have such subcategories). Hence, for a K_Γ–local spectrum, there is generally no natural way to select a "bottom-most class" on K_Γ–homology. This makes extracting a K_Γ–local cellular decomposition from K_Γ–homology difficult.

As a concrete example, we consider the Eilenberg–Mac Lane spaces $K(\mathbb{Z}, t)$. Ravenel and Wilson computed the K_Γ–cohomology ring

$$K_\Gamma^*(K(\mathbb{Z}, t))$$

to be a finite–dimensional power series ring, and there is a dual description of the K_Γ–homology. In the motivational case $t = 2$, the space $K(\mathbb{Z}, 2) \simeq \mathbb{C}\mathrm{P}^\infty$ has an efficient cellular decomposition in the global stable category. In particular there is a map

$$\beta : S^2 \simeq \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^\infty$$

which on K_Γ–homology selects a "bottom-most" class. The image of this map on homology selects the subspace dual to the linear polynomials in the power series ring. For $t > 2$, however, the global cellular decomposition described by $H\mathbb{Z}_*K(\mathbb{Z}, t)$ is known, but quite complicated. Moreover, we compute in Chapter 2 the richer invariant

$$E_\Gamma(K(\mathbb{Z}, t)),$$

and that calculation shows that there cannot exist a map

$$\mathbb{S}^k \to L_\Gamma \Sigma^\infty K(\mathbb{Z}, t)$$

which is nonzero on K_Γ–homology, no matter what k is allowed to be. This completely obstructs the existence of a K_Γ–local decomposition by classical cells which strips off one K_Γ–homology class at a time.

In spite of this negative result, we show in Chapter 3 that this defect is in part reparable by passing to the larger context of generalized K_Γ–local cellular decompositions described above. Associated to any space X we give a construction of a map

$$T_+^* L_\Gamma \Sigma_+^\infty X \to L_\Gamma \Sigma_+^\infty X$$

which, in the case of $X = K(\mathbb{Z}, t)$ above, is injective on K_Γ–homology with image exactly the subspace dual to the linear polynomials in the power series ring. Ravenel and Wilson also show that there is a particular value $t = T > 2$ (dependent upon the choice of K_Γ) for which the K_Γ–cohomology ring is 1–dimensional. In this case, this construction gives a map of spectra which on

K_Γ–homology selects a single "bottom-most" class, as desired. Moreover, a theorem of Hopkins, Mahowald, and Sadofsky shows that the spectrum

$$T_+^* L_\Gamma \Sigma_+^\infty K(\mathbb{Z}, T)$$

is a K_Γ–local generalized sphere. We also show that this process is iterable, giving a complete decomposition of $K(\mathbb{Z}, T)$ into K_Γ–local generalized cells.

The formal methods used to extract this exotic cellular decomposition of $K(\mathbb{Z}, T)$ also apply to $K(\mathbb{Z}, 2)$ to give the classical cellular decomposition. One can then ask what other theorems can be lifted from the classical context of $\mathbb{C}P^\infty$ to this new spectrum. As an example of such a transported theorem, $\mathbb{C}P^\infty \simeq BU(1)$ arises as the classifying space for a $U(1)$–bundles. Additionally, a classical theorem of Adams states that $U(1) \simeq S^1$ is one of only four H–spaces (along with $O(1)$, $Sp(1)$, and the unit–length vectors in the octonians) with the homotopy type of a sphere. In analogy, we prove that there exists a K_Γ–local generalized sphere spectrum $G(1)$ with an A_∞–multiplication so that

$$BG(1) \simeq L_\Gamma \Sigma_+^\infty K(\mathbb{Z}, n+1).$$

This shows that the K_Γ–local category has exotic examples of spheres of "Hopf invariant 1", despite Adams's theorem on the global stable category.

In addition to a strong tolerance for algebro-geometric language, the main tool of this thesis is coalgebraic algebra. In Chapter 1, we introduce a presentation of suitably nice formal schemes by coalgebras and we recall the role in stacky geometry of comodules for a Hopf algebroid, showing their utility in algebraic geometry. In Chapter 2, the main object of study are ring objects in a category of coalgebras (sometimes referred to as "Hopf rings"), which we use to encode and wrangle certain unstable cooperations arising in chromatic homotopy theory. Finally, in Chapter 3 we use a homotopical version of coalgebras to construct the functor T_+^* described above. The behavior of the tools used in that construction are themselves controlled by spectral sequences involving the homological algebra of inverse systems of comodules (over a Hopf algebroid), as described in Appendix A. In each case, we take some care to introduce the material gently, mindful of the general unpopularity of coalgebraic algebra.

Finally, because chromatic homotopy theory is such a forbiddingly intricate subject, we provide a lengthy introduction in Chapter 1 which outlines all of the material from algebraic geometry and algebraic topology used later in the thesis.

Chapter 1

Background on E– and K–theory

In this first chapter, we introduce many of the basic players from chromatic homotopy theory needed for our later discussion, espousing a relentlessly algebro-geometric point of view.

- In Section 1.1, we give a very abbreviated sequence of definitions in algebraic geometry, included so as to emphasize the "functor of points" perspective, as this is somewhat idiosyncratic but used to exclusion in this document and elsewhere in the stable homotopy theory literature.

- In Section 1.2, we recount parts of the theory of formal Lie groups, in preparation for an immediate application to stable homotopy theory in the following Section 1.3. This includes the definition of formal Lie groups (as built upon the functor of points language in Section 1.1), their classification (due to Lazard), and various features of their moduli stack.

- In Section 1.3, we describe two important constructions which yield algebro-geometric objects from homotopical input. We explore this in some classical examples, and we leverage it (using theorems from Section 1.2) to produce homology theories E_Γ and K_Γ tied to a formal Lie group Γ. These homology theories will be our main focus for the rest of the document.

- In Section 1.4, we describe some salient features of stable homotopy theory after localizing at K_Γ, i.e., after forcing $K_\Gamma(X) = 0$ to imply $X \simeq \mathrm{pt}$. In particular, we describe the "continuous" version of E_Γ, for which there is a good duality theory between homology and cohomology, and we describe the example of the K_Γ–local homotopy groups of the sphere for $\Gamma = \widehat{\mathbb{G}}_m$.

We write $\mathsf{C}(X, Y)$ for the set of arrows in a locally small category C with source X and target Y.

1.1 Schemes and formal schemes

Throughout this document, we will make use of algebraic geometry and also of coalgebras and coalgebraic geometry. Motivation for this will be made in Section 1.3 and Section 1.4, but it's useful first to record the basic tools involved.

Essentials of algebraic geometry

The essential idea of scheme theory is to make commutative algebra look formally categorically similar to the study of moduli objects in topology and geometry. For example, just as cohomology classes in $H^n(X; G)$ correspond naturally to homotopy classes of maps

$$X \to K(G, n),$$

by fixing an algebra A and studying it by maps $A \to T$ for test rings T we will often see that A has a fruitful interpretation as a moduli–theoretic object. To this end, we record our first definition:

Definition 1.1.1. An affine scheme X over a ring R is a representable functor

$$\begin{aligned} X \colon \mathsf{Algebras}_R &\to \mathsf{Sets}, \\ X(T) &\cong \mathsf{Algebras}_R(A, T) \end{aligned}$$

for some R–algebra A.[1,2] The functor $\mathsf{Algebras}_R(A, -)$ is written $\operatorname{Spec}(A)$.

Example 1.1.2. The R–algebra $R[x]$ determines a functor $\operatorname{Spec}(R[x]) =: \mathbb{A}^1_R$, referred to as the affine line (over R). This functor has the property that $\mathbb{A}^1_R(T) = T$ and hence that[3]

$$\mathsf{AffineSchemes}_R(\operatorname{Spec} T, \mathbb{A}^1_R) \cong T.$$

Example 1.1.3. Generally, we define $\mathbb{A}^n_R$ by

$$\mathbb{A}^n_R := \operatorname{Spec} R[x_1, \ldots, x_n].$$

It represents the functor $T \mapsto T^{\times n}$.

We now define a certain interesting class of subfunctors of affine schemes: the closed subschemes.

Definition 1.1.4. A map of schemes $Y \to X$ is called a subscheme when the induced map $Y(T) \to X(T)$ is injective for all T. Moreover, a subscheme is said to be closed when for any of the following pullbacks along a map $\operatorname{Spec} T \to F$ there is an ideal I of T and an isomorphism completing the left-hand triangle:

---

[1] A (not necessarily affine) scheme is a functor which is "locally" isomorphic to such a representable functor, where the representing object A is allowed to change in a controlled way.

[2] This isomorphism is not necessarily canonical; a choice of such an isomorphism is called a chart.

[3] This evaluation is sometimes also written $\mathscr{O}(\operatorname{Spec} T) := \mathsf{AffineSchemes}_R(\operatorname{Spec} T, \mathbb{A}^1_R)$.

$$\begin{tikzcd}
\operatorname{Spec} T/I \arrow[r, "\simeq"] \arrow[dr] & \operatorname{Spec} T \times_F G \arrow[r] \arrow[d] & G \arrow[d] \\
& \operatorname{Spec} T \arrow[r] & F.
\end{tikzcd}$$

Remark 1.1.5. The "closed" nomenclature is motivated by the calculus of ideals, which shows that, e.g., the finite union of closed subschemes is itself a closed subscheme.

Example 1.1.6. The only affine schemes with no proper closed subschemes are of the form $\operatorname{Spec} k$ for k a field. For this reason, a map $\operatorname{Spec} k \to X$ is sometimes called a closed point of X. More generally, a scheme with at most one proper closed subscheme must have the form $\operatorname{Spec} R$ for R a local ring, and maps $\operatorname{Spec} R \to X$ are called points of X.

With closed subschemes in hand, it is natural to wonder about open subschemes. These have a more complicated definition, because the complement of a closed subscheme may not be an affine scheme. For instance, $\widehat{\mathbb{A}}^2 \setminus \{(0,0)\}$ is not an affine scheme — but it is covered jointly by the affine schemes $\operatorname{Spec} R[x,y][x^{-1}]$ and $\operatorname{Spec} R[x,y][y^{-1}]$. This behavior turns out to be generic, and one winds up at the following definition:

Definition 1.1.7. A subscheme $Y \to X$ of X is called an open subscheme when for any pullback along a map $\operatorname{Spec} T \to F$ there is a collection of elements $s_k \in T$ such that for any prime p in T with a lift to the pullback there exists a further lift:

$$\begin{tikzcd}
\operatorname{Spec} T_{(p)} \arrow[d, dotted] \arrow[dr] \arrow[ddr] & & \\
\check{C}(\{\operatorname{Spec} T[s_k^{-1}]\}_k) \arrow[r] \arrow[dr] & \operatorname{Spec} T \times_F G \arrow[r] \arrow[d] & G \arrow[d] \\
& \operatorname{Spec} T \arrow[r] & F.
\end{tikzcd}$$

Here $\check{C}(\{\operatorname{Spec} T[s_k^{-1}]\}_k)$ denotes the simplicial object

$$\check{C}(\{\operatorname{Spec} T[s_k^{-1}]\}_k) := \left\{ \coprod_{k_1} \operatorname{Spec} T[s_{k_1}^{-1}] \begin{matrix} \longleftarrow \\ \longrightarrow \\ \longleftarrow \end{matrix} \coprod_{k_1,k_2} \operatorname{Spec} T \begin{bmatrix} s_{k_1}^{-1} \\ , \\ s_{k_2}^{-1} \end{bmatrix} \begin{matrix} \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \end{matrix} \cdots \right\},$$

where the reader can take "$\coprod$" to be a formal symbol expressing the many commutative diagrams to check.

Example 1.1.8. The R–algebra $R[x,y]/(xy-1)$ determines a functor called the "multiplicative group":

$$\operatorname{Spec} R[x,y]/(xy-1) =: \mathbb{G}_m.$$

This presentation of this functor comes with a natural embedding

$$\frac{R[x,y]}{(xy-1)} \leftarrow R[x,y]$$
$$\mathbb{G}_m \to \mathbb{A}^2$$

which on functors of points induces a closed inclusion $\mathbb{G}_m(T) \subseteq \mathbb{A}^2(T)$. The further projection

$$\mathbb{G}_m \to \mathbb{A}^2 \to \mathbb{A}^1$$

onto either coordinate x or y of $\mathbb{A}^2$ also gives an open inclusion by

$$\operatorname{Spec}((R[x])[x^{-1}]) \to \operatorname{Spec} R[x].$$

Its effect on points $\mathbb{G}_m(T) \subseteq \mathbb{A}^1(T) \cong T$ is to select exactly the multiplicative group of unit elements in T.

Remark 1.1.9. Both Example 1.1.2 and Example 1.1.8 are examples of schemes with further algebraic structures. The scheme $\mathbb{G}_m$ is naturally valued in abelian groups, and hence (by the Yoneda lemma, as it is a representable functor) receives the following maps:

$$\begin{array}{rcc}
\mathbb{G}_m \times \mathbb{G}_m \xrightarrow{\mu} \mathbb{G}_m & \qquad & x_1 \otimes x_2 \mapsfrom x \\
 & & y_1 \otimes y_2 \mapsfrom y, \\
\mathbb{G}_m \xrightarrow{\chi} \mathbb{G}_m & & (y,x) \mapsfrom (x,y), \\
\operatorname{Spec} R \xrightarrow{\eta} \mathbb{G}_m & & 1 \mapsfrom x,y.
\end{array}$$

These assemble to make $\mathbb{G}_m$ into a *group scheme*, i.e., an abelian group object in the category of affine schemes. Similarly, the functor $\mathbb{A}^1$ is valued in commutative algebras, so it is a *ring scheme*, i.e., it has addition and subtraction maps which intertwine with the multiplication map described above.

Finite schemes and formal schemes

The most basic and well–behaved class of affine schemes is that of *finite* affine schemes X defined over a field k, where "finite" means that X has a chart $X = \operatorname{Spec} A$ where A is a k–algebra which is finitely generated as a k–module.

Example 1.1.10. The truncation $\mathbb{A}^{1,(n)} := \operatorname{Spec} k[x]/x^{n+1}$ is such a finite affine scheme. It represents the subfunctor of $\mathbb{A}^1$ which selects those elements which are nilpotent of order at most n.

In this nice setting, there is a second presentation of X: writing cX for the dual of the chart

$$cX := \mathsf{Modules}_k(\mathbb{A}^1_k(X), k),$$

the functor X can be expressed as

$$X(T) \cong \mathrm{Sch}(cX)(T) := \left\{ u \in cX \otimes_k T \,\middle|\, \begin{array}{c} \Delta u = u \otimes u \in (cX \otimes_k T) \otimes_T (cX \otimes_k T), \\ \varepsilon u = 1 \in T \end{array} \right\},$$

where Δ is dual to the multiplication on A and ε is dual to the unit of A. Given such a k-coalgebra, we can define such a functor Sch generally; diagrammatically, we have the following functors:

$$\begin{tikzcd}
\mathsf{FiniteCoalgebras} \arrow[r, shift right, "(-)^\vee"'] \arrow[rr, bend right, "\mathrm{Sch}"'] & \mathsf{FiniteAlgebras} \arrow[l, shift right, "(-)^\vee"'] \arrow[r, shift right, "\mathrm{Spec}"'] & \mathsf{FiniteSchemes}. \arrow[l, shift right, "\mathbb{A}^1"'] \arrow[ll, bend right, "c"']
\end{tikzcd}$$

Additionally, each pair of functors gives an equivalence of categories.

Our immediate goal is to explore the behavior of this equivalence as we loosen the hypotheses of being finite and of being a module over a field k. It is instructive to handle these adjectives separately and to relax finiteness first. In the case that A is an infinite-dimensional k-module there is an inequivalence $(A \otimes_k A)^\vee \not\cong A^\vee \otimes_k A^\vee$ and hence no natural composite diagonal map

$$A^\vee \xrightarrow{\mu^\vee} (A \otimes_k A)^\vee \leftarrow A^\vee \otimes_k A^\vee.$$

It follows already that linear duality fails to provide an equivalence of categories between algebras and coalgebras. However, coalgebras over a field k enjoy the following structure theorem:

Lemma 1.1.11 (Demazure's Lemma, [17, pg. 12], [52, Appendix 5.3]). *If C is a k-coalgebra and E is a finite-dimensional vector subspace of C, then there exists a finite-dimensional vector subspace F of C with $E \subseteq F \subseteq C$ and F a k-subcoalgebra.* □

Corollary 1.1.12. *Every k-coalgebra is "ind-finite", i.e., there is a natural equivalence between (possibly infinite) k-coalgebras and their lattices of finite k-subcoalgebras.* □

Applying linear-algebraic duality to this lattice of finite k-subcoalgebras naturally takes values in profinite algebras, and indeed there is the following commuting network of equivalences of categories:

$$\begin{tikzcd}
\mathsf{FiniteCoalgebras} \arrow[r, shift right, "(-)^\vee"'] \arrow[d] & \mathsf{FiniteAlgebras} \arrow[l, shift right, "(-)^\vee"'] \arrow[r, shift right, "\mathrm{Spec}"'] \arrow[d] & \mathsf{FiniteSchemes} \arrow[l, shift right, "\mathbb{A}^1"'] \arrow[d] \\
\mathsf{Coalgebras} \arrow[r, shift right, "(-)^\vee"'] & \mathsf{ProfiniteAlgebras} \arrow[l, shift right, "(-)^\vee"'] \arrow[r, shift right, "\mathrm{Spf}"'] & \mathsf{FormalSchemes}, \arrow[l, shift right, "\mathbb{A}^1"']
\end{tikzcd}$$

where we have made the following implicit definition:

Definition 1.1.13. A formal k-scheme is an ind-system of finite k-schemes.

Example 1.1.14. Our favorite example of a formal affine scheme will be affine n-space:

$$\widehat{\mathbb{A}}^n := \operatorname{Spf} k[\![x_1, \ldots, x_n]\!] := \operatorname{colim}_{(i_1,\ldots,i_n)} \operatorname{Spec} \frac{k[x_1, \ldots, x_n]}{(x_1^{i_1}, \ldots, x_n^{i_n})}.$$

The dual system of coalgebras is given by the modules

$$C_{n,(i_1,\ldots,i_n)} := k\{\beta_{(j_1,\ldots,j_n)} \mid j_m \le i_m \text{ for all } m\}$$

with diagonal

$$\Delta\beta_{(j_1,\ldots,j_n)} = \sum_{k_1 \le j_1} \cdots \sum_{k_n \le j_n} \beta_{k_1,\ldots,k_n} \otimes \beta_{j_1-k_1,\ldots,j_n-k_n}.$$

Lemma 1.1.15 ([74, pg. 32]). *There is an isomorphism between maps $\widehat{\mathbb{A}}^n \to \widehat{\mathbb{A}}^m$ and m-tuples of n-variate power series with vanishing constant term.* ☐

Lemma 1.1.15 allows us to reinterpret various theorems from the analytic geometry of power series in this "algebraic" context of formal schemes. For example, there is the following version of the inverse function theorem:

Lemma 1.1.16 (Inverse function theorem). *If $f : \widehat{\mathbb{A}}^n \to \widehat{\mathbb{A}}^n$ is a map of formal varieties, then f has an inverse if and only if $T_0 f : T_0\widehat{\mathbb{A}}^n \to T_0\widehat{\mathbb{A}}^n$ is an invertible linear transformation.* ☐

Having this identification of Hom-sets is so useful that we define an interesting class of formal schemes for which these lemmas can be used.

Definition 1.1.17. A formal variety V is a formal scheme which is (noncanonically) isomorphic to formal affine n-space for some n. For such an isomorphism $\varphi : V \to \widehat{\mathbb{A}}^n$, φ is called a coordinate (for V) and the inverse isomorphism $\varphi^{-1} : \widehat{\mathbb{A}}^n \to V$ is called a parameter (for V).

We now turn to the case of a general ground ring k, where Demazure's lemma fails [52, Appendix 5.3]. Instead, it is standard practice to incorporate his lemma into a definition.

Definition 1.1.18 ([74, Definition 4.58]). Suppose that C is a k-coalgebra (where k is not necessarily taken to be a field), and further suppose that as a k-module C is free. If there exists a basis $C \cong R\{e_i \mid i \in I\}$ such that each finitely generated submodule M of C can be enlarged to a subcoalgebra of the form $R\{e_{i_j} \mid j \in J\} \subseteq M$ on a finite indexing set J, then C is said to be a "good coalgebra" and the basis $\{e_i\}_{i\in I}$ is said to be a "good basis".

Remark 1.1.19 ([52, Appendix 5.3]). If C is an R-coalgebra, free as an R-module, where is R a principal ideal domain with no zerodivisors, then C is automatically good with that basis.

Theorem 1.1.20 ([74, Proposition 4.64]). *Good k-coalgebras form a full subcategory of all k-coalgebras, and* Sch *is a fully faithful functor onto its image in* $\mathsf{FormalSchemes}_k$ *(called "coalgebraic formal schemes"). If $\mathscr{F} : I \to \mathsf{GoodCoalgebras}_k$ is a diagram of coalgebras with colimit F, and additionally both $\mathscr{F}$ and its colimit F factor through the subcategory of good coalgebras, then* $\operatorname{Sch} F$ *is the colimit of* $\operatorname{Sch} \circ \mathscr{F}$ *in the category of formal schemes.* ☐

Example 1.1.21. Formal affine n–space $\widehat{\mathbb{A}}^n$ is a coalgebraic formal scheme.

Lemma 1.1.22. *The following analogue of Lemma 1.1.15 holds in the generality of Example 1.1.21: there is an isomorphism between maps $\widehat{\mathbb{A}}^n \to \widehat{\mathbb{A}}^m$ and m–tuples of n–variate power series with nilpotent constant term.* ☐

Hence, in the presence of Demazure's lemma, we can accomplish most anything for general coalgebraic formal schemes that we could have accomplished for coalgebraic formal schemes over a field. For example, given a closed inclusion

$$\operatorname{Spf} S \xrightarrow{s} \operatorname{Spf} A =: X$$

of a coalgebraic formal R–scheme $\operatorname{Spf} A$, we can form the algebraic tangent space $T_s X$ of s in X in three different ways:

1. Extensions of the shape

$$\begin{tikzcd}
\operatorname{Spf} S \arrow[rr, "s"] \arrow[dr] & & \operatorname{Spf} A \\
& \operatorname{Spf} S[\varepsilon]/(\varepsilon^2), \arrow[ur] &
\end{tikzcd}$$

   where the map $\operatorname{Spf} S \to \operatorname{Spf} S[\varepsilon]/(\varepsilon^2)$ is given by the dual map $\varepsilon \mapsto 0$.

2. Because s is a closed inclusion, it follows that $S = A/I_s$ for I_s the *ideal of definition*. There is also a natural isomorphism $T_s X \cong (I_s/I_s^{\otimes_R 2})^*$ with the tangent space as defined above.

3. Because the formal schemes are assumed to be coalgebraic, we also have that s is represented by a map

$$cs : D \hookrightarrow C$$

   whose quotient $M = C/D$ gives a C–comodule. Dualizing the defining coequalizer diagram for $M \otimes_R N$ for a k–algebra R, right R–module M, and left R–module N:

$$\begin{tikzcd}
M \otimes_R N & M \otimes_k N \arrow[l] & M \otimes_k R \otimes_k N, \arrow[l, shift right, "\alpha_M \otimes 1"'] \arrow[l, shift left, "1 \otimes \alpha_N"]
\end{tikzcd}$$

   one produces a defining equalizer diagram for $M \square_C N$ for a k–coalgebra C, a right C–comodule M, and a left C–comodule N:

$$\begin{tikzcd}
M \square_C N \arrow[r] & M \otimes_k N \arrow[r, shift left, "\psi_M \otimes 1"] \arrow[r, shift right, "1 \otimes \psi_N"'] & M \otimes_k C \otimes_k N.
\end{tikzcd}$$

In particular, for M the coideal of definition of a closed inclusion of coalgebraic formal schemes as above, we have

$$T_s^* X := \ker\left(M \xrightarrow{\Delta} M \square_C M\right).$$

Remark 1.1.23. The scheme $\operatorname{Spf} S[\varepsilon]/(\varepsilon^2)$ used in the first construction could also be called $\widehat{\mathbb{A}}_S^{1,(1)}$, as it participates in the ind–system defining the formal affine line over S. Thus, one can more generally consider the collection of maps to X from $\widehat{\mathbb{A}}_S^{1,(j)} = \operatorname{Spf} S[x]/(x^{j+1})$ participating in a similar commuting triangle, altogether called the j–jets at s (cf. Example 1.1.10).

Sheaves of modules

Finally, we note that the theory of A–modules is visible to the affine scheme $\operatorname{Spec} A$:

Definition 1.1.24 ([74, Definition 2.42, Proposition 2.46]). Given an A–module M and an affine scheme $\operatorname{Spec} B$ over $\operatorname{Spec} A$, we define the value of the sheaf $\mathscr{M}$ at $\operatorname{Spec} B$ by

$$\mathscr{M}(\operatorname{Spec} B \to \operatorname{Spec} A) := M \otimes_A B.$$

Sheaves on the site of affines over $\operatorname{Spec} A$ which are isomorphic to sheaves of this form are said to be "quasicoherent", and there is an equivalence of categories between A–modules and quasicoherent sheaves on $\operatorname{Spec} A$.

Remark 1.1.25 ([74, Proposition 4.47]). This definition immediately extends to the setting of a formal scheme X by considering collections of quasicoherent sheaves, one over each finite scheme in the defining ind–system for X, together with compatible isomorphisms among the restrictions.

Definition 1.1.26. For a map $\varphi : \operatorname{Spec} A \to \operatorname{Spec} B$ of affine schemes, there is an induced adjunction on categories of quasicoherent sheaves:

$$\begin{tikzcd} \mathsf{QCoh}(\operatorname{Spec} A) \arrow[r, shift left, "\varphi_*"] & \mathsf{QCoh}(\operatorname{Spec} B). \arrow[l, shift left, "\varphi^*"] \end{tikzcd}$$

On the level of modules, these are described by the assignments

$$\begin{aligned} \varphi_*(M_A) &= M_A \text{ (considered as a } B\text{-module)}, \\ \varphi^*(M_B) &= M_B \otimes_B A. \end{aligned}$$

Example 1.1.27. An ideal I of a ring A is an A–submodule of A and so by Definition 1.1.24 begets a quasicoherent sheaf on $\operatorname{Spec} A$ called the ideal sheaf. A point $p : \operatorname{Spec} P \to \operatorname{Spec} A$ is said to lie in the support of I when $p^* I \neq 0$, and p lifts to a point of the associated closed subscheme $\operatorname{Spec} A/I \to \operatorname{Spec} A$ precisely when it is not in the support of I.

Example 1.1.28. Dually, one can define comodules for a coalgebra and the quasicoherent sheaves that they determine. In particular, an inclusion $C \to C'$ of coalgebras induces a quotient C'–comodule C'/C, and generally we define a C'–coideal to be a C'–comodule quotient of C' itself.

1.2 Formal Lie groups

One of the applications of Lemma 1.1.22 is the insertion of some Lie theory into algebraic geometry. In a coordinate chart centered at the identity element of a n–dimensional Lie group, the multiplication law of the Lie group expands into a n–tuple of $(2n)$–variate power series. Interpreting this as a morphism

$$\widehat{\mathbb{A}}^{2n} \cong \widehat{\mathbb{A}}^n \times \widehat{\mathbb{A}}^n \to \widehat{\mathbb{A}}^n,$$

this inspires the following defintion:

Definition 1.2.1. A (commutative) formal group law over a ring R is map $\widehat{\mathbb{A}}^n \times \widehat{\mathbb{A}}^n \to \widehat{\mathbb{A}}^n$ (i.e., an n–tuple of $(2n)$–variate power series), written $x +_F y$ with x the first n coordinates and y the second n, satisfying the following identities:

$$\begin{aligned} x +_F 0 &= x && \text{(identity)}, \\ x +_F y &= y +_F x && \text{(symmetry)}, \\ (x +_F y) +_F z &= x +_F (y +_F z) && \text{(associativity)}. \end{aligned}$$

(In particular, R is no longer required to be $\mathbb{R}$ or $\mathbb{C}$. These equalities make sense over any ring.)

We will primarily be concerned with the case of formal groups of dimension 1. Throughout the rest of this document, unless otherwise specified the dimension will implicitly taken to be 1.

Remark 1.2.2 ([43, Theorem I]). In the 1–dimensional setting, the extra symmetry condition is inoffensive: every 1–dimensional formal group law is automatically symmetric if and only if the ground ring contains no elements which are simultaneously nilpotent and torsion.

Formal group laws are reasonably well–behaved objects, and many basic theorems from Lie groups (especially those whose proofs crucially use the chart at the identity!) carry over immediately. For example, there is the following lemma:

Lemma 1.2.3. *Every formal group law has an inverse law.*

Proof. Consider the shearing map $\widehat{\mathbb{A}}^{2n} \to \widehat{\mathbb{A}}^{2n}$ defined by $\sigma : (x, x') \mapsto (x, x +_F x')$. Its action on tangent spaces is given by

$$T_0(\sigma) = \left(\begin{array}{c|c} I_n & T_0(- +_F 1_{\widehat{\mathbb{A}}^n}) \\ \hline 0 & I_n \end{array} \right).$$

Since this matrix is upper-triangular with unit entries on the diagonal, it is invertible. It follows by Lemma 1.1.16 that an inverse to σ exists, and by restricting to $x = 0$ one extracts the desired inverse map. □

Inspired by these uses of geometry, we define a "deeper" geometric object from which formal group laws arise.

Definition 1.2.4. A formal group $\widehat{\mathbb{G}}$ is a formal variety equipped with a multiplication.[4]

Remark 1.2.5. A formal group law thus arises from selecting a coordinate on a formal group and transporting the multiplication across the isomorphism. In the motivating situation of a Lie group, we might draw the following nonsensical diagram:

$$\begin{tikzcd}
G \times G \arrow[d] & G_0^\wedge \times G_0^\wedge \arrow[l] \arrow[d] & \widehat{\mathbb{A}}^n \times \widehat{\mathbb{A}}^n \arrow[l, "\cong"'] \arrow[d] \\
G & G_0^\wedge \arrow[l] & \widehat{\mathbb{A}}^n, \arrow[l, "\cong"']
\end{tikzcd}$$

where $G_0^\wedge$ denotes an "infinitesimal neighborhood of 0" without an explicit choice of chart. While it is not actually possible to draw such a diagram in the usual category of manifolds, formal groups give a means by which this can be studied. The operation "$(G, 0) \mapsto G_0^\wedge$" is meaningful in formal schemes: for a Noetherian affine group scheme $G = \operatorname{Spec} A$ with zero-locus detected by the closed subscheme $\operatorname{Spec}(A/I) \to G$, we can associate the formal scheme

$$\left\{\operatorname{Spec}(A/I) \to \operatorname{Spec}(A/I^2) \to \cdots \to \operatorname{Spec}(A/I^n) \to \cdots\right\} =: G_0^\wedge \to G.$$

The middle vertical arrow in the above diagram corresponds to the restriction of the multiplication map to this formal geometric object, and the choice of horizontal arrows (i.e., a chart) presents the multiplication as a power series (i.e., a Taylor expansion).

Example 1.2.6. We will explore the diagram in Remark 1.2.5 in the case that $G = \mathbb{A}^1$, which as we saw in Example 1.1.2 is naturally valued in abelian groups. Associated to the zero-section $\operatorname{Spec} R \to \mathbb{A}^1$, we have the formal scheme $(\mathbb{A}^1)_0^\wedge$ given by

$$\operatorname{Spf} R[\![x]\!] := \left(\operatorname{Spec} R[x]/(x) \to \operatorname{Spec} R[x]/(x)^2 \to \cdots \to \operatorname{Spec} R[x]/(x)^n \to \cdots\right).$$

Using the formula

$$(\operatorname{Spf} R[\![x]\!])(T) = \operatorname{colim}_n \{(\operatorname{Spec} R[x]/(x^n))(T)\}_n,$$

we see again that $\widehat{\mathbb{A}}^1$ selects the ideal of nilpotent elements in T. This carries the natural structure of an abelian group restricting the structure of the abelian group on the whole ring — just as described in Remark 1.2.5. Writing $\widehat{\mathbb{G}}_a$ for this formal variety understood with this group structure and chart, the structure maps are specified by

$$\begin{aligned}
\widehat{\mathbb{G}}_a \times \widehat{\mathbb{G}}_a \xrightarrow{+} \widehat{\mathbb{G}}_a, &\qquad x' + x'' \leftarrow\!\shortmid x, \\
\widehat{\mathbb{G}}_a \xrightarrow{\chi} \widehat{\mathbb{G}}_a, &\qquad -x \leftarrow\!\shortmid x,
\end{aligned}$$

where we've written x' for $x \otimes 1$ and x'' for $1 \otimes x$.

---

[4] Some authors use "formal group" to signify a group object in the category of formal schemes, and they called a formal group "smooth" or "of Lie type" when the underlying formal scheme is a formal variety. Because our applications are so narrow, we will skip the extra adjectives.

Example 1.2.7. Similarly, we can complete the scheme $\mathbb{G}_m$ of Example 1.1.8 at the unit section $\operatorname{Spec} R \to \mathbb{G}_m$ to produce a formal group scheme $\widehat{\mathbb{G}}_m$. Since the unit section is a smooth point of $\mathbb{G}_m$, it follows that

$$\widehat{\mathbb{G}}_m \to \widehat{\mathbb{A}}^1, \qquad\qquad 1-x \leftarrow\!\shortmid x$$

is a coordinate. We can then calculate the structure maps in terms of this coordinate:

$$\begin{aligned}
\widehat{\mathbb{G}}_m \times \widehat{\mathbb{G}}_m \xrightarrow{\mu} \widehat{\mathbb{G}}_m & \qquad\qquad 1-(1-x')(1-x'') \leftarrow\!\shortmid x \\
& \qquad\qquad x' + x'' - x' \cdot x'' = \\
\widehat{\mathbb{G}}_m \xrightarrow{\chi} \widehat{\mathbb{G}}_m & \qquad\qquad 1-(1-x)^{-1} \leftarrow\!\shortmid x \\
& \qquad\qquad -x - x^2 - x^3 - \cdots - x^n - \cdots =
\end{aligned}$$

Remark 1.2.8. The procedure of completing a scheme X at a closed subscheme Y is generally very useful. It sometimes goes by the name of building the "infinitesimal deformation space" of the subscheme, as it has the property that if $Y \to Y' \to X$ is any nilpotent thickening of Y equipped with a map to X prolonging the inclusion of Y, then there is a factorization

$$\begin{tikzcd}
Y \arrow[r] \arrow[d] & X^\wedge_Y \arrow[d] \\
Y' \arrow[r] \arrow[ur, dotted, "\exists!"] & X.
\end{tikzcd}$$

This construction is of special interest when X presents a moduli problem and Y selects a certain solution. As $X^\wedge_Y$ captures the local geometry of X infinitesimally closed to Y, it follows that $X^\wedge_Y$ describes solutions of the moduli problem which are infinitesimally close to the given solution Y. This is often considerably easier to fully analyze than X itself and still gives important partial information about the broader behavior of X.

Example 1.2.9. The completion of the inclusion of the closed point $\operatorname{Spec} \mathbb{F}_p \to \operatorname{Spec} \mathbb{Z}$ gives the p-adic integers

$$\operatorname{Spec} \mathbb{F}_p \to \operatorname{Spf} \mathbb{Z}_p \to \operatorname{Spec} \mathbb{Z}.$$

The assertion of Remark 1.2.8 in this context reads that if A is any complete local ring with residue field $\mathbb{F}_p$, then A is automatically a $\mathbb{Z}_p$-algebra. More generally, if k is a perfect field of positive characteristic, there is an analogous object $\mathbb{W}(k)$, called the p-typical Witt vectors over k, so that if A is a complete local ring with residue field k, then A is automatically a $\mathbb{W}(k)$-algebra. For example, for ζ_{p^d-1} a primitive $(p^d-1)^{\text{th}}$ root of unity:

$$\mathbb{W}(\mathbb{F}_{p^d}) \cong \mathbb{Z}_p(\zeta_{p^d-1}), \qquad\qquad \mathbb{W}(\mathbb{F}_p) \cong \mathbb{Z}_p.$$

The moduli of formal groups

Let us return to considering formal groups by trying to understand their moduli. We have a fairly thorough classification both of formal groups and formal group laws, due to various arithmetic geometers.

Theorem 1.2.10 ([43]). *The functor assigning a ring R to the set of (commutative, 1-dimensional) formal group laws over R is an affine scheme. It is corepresented by the Lazard ring, which has a noncanonical isomorphism to an infinite-dimensional polynomial ring:*

$$L \cong \mathbb{Z}[c_1, c_2, \ldots, c_n, \ldots]. \quad \square$$

Corollary 1.2.11. *Let F be any formal group law over any ring R. For any surjective ring map $S \to R$ (which, e.g., can be taken to be torsion-free), there always exists a lift of F to S.* □

Before proceeding to describe the classification of formal groups, we construct a corresponding geometric moduli object in terms of which we will phrase the results.

Definition 1.2.12. Set $\mathcal{M}_{\mathbf{PS}}$ to be the moduli of power series with no constant term:

$$\mathcal{M}_{\mathbf{PS}} = \operatorname{Spec} \mathbb{Z}[a_1, a_2, \ldots, a_n, \ldots],$$

where the universal such power series classified by the identity map is given by $\sum_{i=1}^{\infty} a_i x^i$. This is a monoid scheme under composition, and the submoduli $\mathcal{M}_{\mathbf{PS}}^{\mathrm{gpd}}$ of invertible power series is given by

$$\mathcal{M}_{\mathbf{PS}}^{\mathrm{gpd}} = \operatorname{Spec} \mathbb{Z}[a_1^{\pm}, a_2, \ldots, a_n, \ldots].$$

Now, a map

$$\operatorname{Spec} R \to \operatorname{Spec} L \times \mathcal{M}_{\mathbf{PS}}^{\mathrm{gpd}} =: X_1$$

classifies a formal group law F along with a change-of-coordinate power series φ. By conjugating F with φ to get a new law

$$x +_{F'} y := \varphi^{-1}(\varphi x +_F \varphi y),$$

we produce a second map $\operatorname{Spec} R \to \operatorname{Spec} L$. In the universal case, this is a map

$$\operatorname{Spec} L \times \mathcal{M}_{\mathbf{PS}}^{\mathrm{gpd}} \xrightarrow{\text{target}} \operatorname{Spec} L =: X_0.$$

Continuing in this fashion, we also construct the following maps:

$$\begin{aligned}
X_1 &\xrightarrow{\text{source}} X_0 && \text{(selects the FGL } F\text{)},\\
X_1 &\xrightarrow{\text{target}} X_0 && \text{(selects the } \varphi\text{-conjugate of } F\text{)},\\
X_1 {}^{s}\!\times^{t}_{X_0} X_1 &\xrightarrow{\text{compose}} X_1 && \text{(composes } \varphi_1 \text{ and } \varphi_2 \text{ to } \varphi_1 \circ \varphi_2\text{)},\\
X_0 &\xrightarrow{\text{identity}} X_1 && \text{(augments } F \text{ with } \varphi(x) = x\text{)},\\
X_1 &\xrightarrow{\text{invert}} X_1 && \text{(replaces } F \text{ by its } \varphi\text{-conjugate, replaces } \varphi \text{ by } \varphi^{-1}\text{)}.
\end{aligned}$$

Altogether, this makes the pair $(\operatorname{Spec} L, \operatorname{Spec} L \times \mathcal{M}_{\mathbf{PS}}^{\mathrm{gpd}})$ into a groupoid scheme. We define the moduli of formal Lie groups to be the following groupoid–valued functor, written as a "stacky quotient" or "homotopy quotient":

$$\operatorname{Spec} L \xrightarrow{C} \mathcal{M}_{\mathbf{fg}} := (\operatorname{Spec} L) // (\operatorname{Spec} L \times \mathcal{M}_{\mathbf{PS}}^{\mathrm{gpd}}).$$

Definition 1.2.13. It will also be useful to define an auxiliary moduli problem: the moduli of formal groups equipped with a specified unit tangent vector. To form this, set $\mathcal{M}_{\mathbf{SPS}}$ to be the further submoduli of $\mathcal{M}_{\mathbf{PS}}^{\mathrm{gpd}}$ of those power series with leading term x (i.e., $a_1 = 1$). Then, we make the definition

$$\mathcal{M}_{\mathbf{fg}}^{(1)} := \operatorname{Spec} L // (\operatorname{Spec} L \times \mathcal{M}_{\mathbf{SPS}}).$$

This moduli has a $\mathbb{G}_m$–action by rescaling the tangent vector, and it follows that

$$\mathcal{M}_{\mathbf{fg}}^{(1)} // \mathbb{G}_m \simeq \mathcal{M}_{\mathbf{fg}}.$$

The rational moduli of formal groups

The behavior of $\mathcal{M}_{\mathbf{fg}}$ organizes substantially after localizing at an arithmetic prime, which we investigate now. At the generic point, its behavior is very simple:

Lemma 1.2.14 ([43, Proposition 4]). *Rationally, every formal group law F admits a unique "strict logarithm", $\log_F$:*

$$\log_F(x +_F y) = x + y = x +_{\widehat{\mathbb{G}}_a} y.$$

(That is, $\mathcal{M}_{\mathbf{fg}} \times \operatorname{Spec} \mathbb{Q}$ is valued in contractible groupoids.)

Proof. We say that a 1–form $f(x)dx$ is (left–)invariant under F if the following holds:

$$f(x)dx = f(y +_F x)d(y +_F x) = f(y +_F x)\frac{\partial(y +_F x)}{\partial y}dx.$$

Restricting to the origin by setting $x = 0$, we deduce the condition

$$f(0) = f(y) \cdot \left(\frac{\partial(y +_F x)}{\partial y}\right)\bigg|_{x=0}.$$

Setting the boundary condition $f(0) = 1$ gives the "strict invariant differential ω_F", and integrating against y yields

$$\log_F(y) = \int \left(\frac{\partial(y +_F x)}{\partial x}\right)\bigg|_{x=0} dy.$$

To see that the series $\log_F$ has the claimed homomorphism property, note that

$$\frac{\partial \log_F(y +_F x)}{\partial x} = f(y +_F x)d(y +_F x) = f(x)dx = \frac{\partial \log_F(x)}{\partial x},$$

and hence that $\log_F(y +_F x)$ and $\log_F(x)$ differ by a constant. Checking at $x = 0$ shows that the constant is $\log_F(y)$. $\square$

Remark 1.2.15. It is worth remarking that we only need to be able to divide by integers in order to define additive logarithms. This contrasts with the positive characteristic case, where we also need to be able to take roots to put formal group laws into a canonical form; see Theorem 2.3.1.

The p-local moduli of formal groups

At finite primes, the classification is considerably more involved. In the rest of the section, $\mathscr{M}_{\mathrm{fg}}$ implicitly refers to the localization $\mathscr{M}_{\mathrm{fg}} \times \operatorname{Spec} \mathbb{Z}_{(p)}$.

Lemma 1.2.16 ([15]). *Each formal group law over a torsion-free p-local ring is naturally isomorphic to one which is "p-typical", meaning its rational logarithm has the form*[5]

$$\log_F x = x + \sum_{n=1}^{\infty} m_n x^{p^n}. \quad \square$$

However, the M-chart for $\mathscr{M}_{\mathrm{fg}} \times \operatorname{Spec} \mathbb{Q}$ does not obviously extend to a chart of $\mathscr{M}_{\mathrm{fg}} \times \operatorname{Spec} \mathbb{Z}_{(p)}$, as not all formal group laws have logarithms. One can get around this by trying to use the p-typical logarithm property to describe other invariants of formal group laws which do not require extra assumptions on the ground ring, such as the following:

Theorem 1.2.17 ([6, Theorem 3.6]). *Define the n-series of F to be*

$$[n]_F(x) = \overbrace{x +_F \cdots +_F x}^{n \text{ copies of } x}.$$

For a p-typical formal group law F over a ring R, the following holds:

$$[p]_F(x) = px +_F v_1 x^p +_F v_2 x^{p^2} +_F \cdots +_F v_d x^{p^d} +_F \cdots,$$

for some coefficients $v_d \in R$. $\square$

Remark 1.2.18. In fact, the rational logarithm coefficients can be recursively recovered from the coefficients v_d, using the following manipulation:

$$\begin{aligned} p \log_F(x) &= \log_F([p]_F(x)) \\ p \sum_{n=0}^{\infty} m_n x^{p^n} &= \log_F \left({\sum_{d=0}^{\infty}}{}_F \, v_d x^{p^d} \right) = \sum_{d=0}^{\infty} \log_F \left(v_d x^{p^d} \right) \\ \sum_{n=0}^{\infty} p m_n x^{p^n} &= \sum_{d=0}^{\infty} \sum_{j=0}^{\infty} m_j v_d^{p^j} x^{p^{d+j}} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} m_k v_{n-k}^{p^k} \right) x^{p^n}, \end{aligned}$$

implicitly taking $m_0 = 1$ and $v_0 = p$.

---

[5] An equivalent concentration condition can be specified on the rational exponential. For a second discussion of p-typical curves see Remark 2.3.3.

In any event, as the definition of $[p]_F(x)$ requires no conditions on the ground ring R, we find a new chart:

Theorem 1.2.19. *The ring* $\mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_d, \ldots]$ *provides a* p*-local chart:*

$$\operatorname{Spec}\mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_d, \ldots] \xrightarrow{V} \mathcal{M}_{\mathrm{fg}} \times \operatorname{Spec}\mathbb{Z}_{(p)}.$$

Proof. Starting with a formal group law F over any ground ring R, pick a torsion-free p-local ring R' surjecting onto R by a map $f : R' \to R$. By Corollary 1.2.11, there exists a lift F' of F to R', so that $f^*F' = F$. Applying Lemma 1.2.16, one finds an invertible power series φ' over R' so that the φ'-conjugate of F' is p-typical, and hence by Theorem 1.2.17 we have

$$[p]_{(\varphi')^{-1}F'\varphi'}(x) = px +_{(\varphi')^{-1}F'\varphi'} w_1' x^p +_{(\varphi')^{-1}F'\varphi'} w_2 x^{p^2} + \cdots$$

for some coefficients $w_*' \in R'$. Translating all of this information through f, we have produced an invertible power series $\varphi = f^*\varphi'$ such that

$$[p]_{\varphi^{-1}F\varphi}(x) = px +_{\varphi^{-1}F\varphi} w_1 x^p +_{\varphi^{-1}F\varphi} w_2 x^{p^2} + \cdots$$

for $w_* = f(w_*')$. □

All this begs a geometric interpretation. The following definition captures the most prominent feature of the V-chart.

Definition 1.2.20. Let F be a formal group law defined over a complete local ring R for which p lies in the maximal ideal. Then, the height of F is any one of the following equivalent invariants:

1. Recall from Lemma 1.2.14 that the logarithm of a torsion-free lift F' of F is determined by the integral equation

$$\log_{F'}(x) = \int \omega_{F'}(x) = \int \left(\frac{\partial(x +_{F'} y)}{\partial y}\right)\Bigg|_{y=0} dx.$$

   The first coefficient for which this power series integral may fail to be p-local must be of the form x^{p^d}. In fact, d is independent of choice of lift, and it is called the height of F.

2. It follows from Theorem 1.2.17 that the lowest-order nonvanishing coefficient of $[p]_F(x)$ over the residue field of R must be in degree p^d, and the integer d is called the height of F.

3. Geometrically, the subscheme of p-torsion points $\widehat{\mathbb{G}}[p]$ of the formal group $\widehat{\mathbb{G}}$ associated to F is finite and free of rank p^d for some d, called the height of F. (This follows from a form of the Weierstrass Preparation Theorem [74, Lemma 5.14].) In particular, it is clear from this definition that height is actually an invariant of the formal group rather than the formal group law.

Remark 1.2.21. It is common to say that the additive formal group $\widehat{\mathbb{G}}_a$ defined over a ring R as above has height ∞, which is an obvious extension of each of the definitions of height given above. It is also common to say that a formal group defined over a rational ring has height 0, which — though rational rings are not complete and local against a maximal ideal $\mathfrak{m}$ containing p — is an extension of the geometric definition. After all, the rational additive group has no nontrivial p-torsion points, so $\widehat{\mathbb{G}}_a[p]$ is of rank $1 = p^0$.

The stratification of p-local formal group laws by height is *the* way to break up their moduli into pieces, as captured by the following theorems.

Theorem 1.2.22 ([41]). *There is a* unique *closed substack $\mathcal{M}_{\mathbf{fg}}^{\geq d} \subseteq \mathcal{M}_{\mathbf{fg}}$ for each positive codimension n. It selects the formal groups of height at least d, and thus corresponds to the ideal $(p, v_1, \ldots, v_{d-1})$ of the cover V.* □

Moreover, the content of each stratum is well-understood:

Theorem 1.2.23 ([43, Théorème IV]). *There is a unique geometric point of $\mathcal{M}_{\mathbf{fg}}$ in each $\mathcal{M}_{\mathbf{fg}}^{\geq d}$ which is not in $\mathcal{M}_{\mathbf{fg}}^{\geq (d+1)}$. It can be modeled by a formal group law over $\mathbb{F}_p$ with p-series $[p]_{F_d}(x) = x^{p^d}$, called the Honda formal group law. The only geometric point of $\mathcal{M}_{\mathbf{fg}}$ not captured by this sequence is $\widehat{\mathbb{G}}_a$, considered over $\mathbb{F}_p$.* □

Theorem 1.2.24 ([45, Proposition 1.1]). *The geometric point F_d in Theorem 1.2.23 has deformation space in $\mathcal{M}_{\mathbf{fg}}^{(1)}$ equivalent to $(d-1)$-dimensional formal affine space over $\mathbb{Z}_p$. That is, the completion (as in Example 1.2.6) of the map $F_d : \operatorname{Spec} \mathbb{F}_p \to \mathcal{M}_{\mathbf{fg}}^{(1)}$ is given by*

$$\begin{tikzcd}
\operatorname{Spec} \mathbb{F}_p \arrow[r] \arrow[dd, equal] \arrow[drr, "F_d"] & LT_{F_d} \arrow[r] \arrow[d] & \operatorname{Spec} \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_d, \ldots] \arrow[d] & \textit{(covering schemes)} \\
& \operatorname{Def}(F_d) \arrow[r] \arrow[d] & \mathcal{M}_{\mathbf{fg}}^{(1)} \arrow[d] & \textit{(stacks)} \\
\operatorname{Spec} \mathbb{F}_p \arrow[r] & \operatorname{Spf} \mathbb{Z}_p \arrow[r] & \operatorname{Spec} \mathbb{Z}_{(p)}, & \textit{(ground rings)}
\end{tikzcd}$$

where LT_{F_d} is noncanonically isomorphic to $\widehat{\mathbb{A}}^{d-1}_{\mathbb{Z}_p}$. □

Definition 1.2.25. The formal scheme LT_{F_d} is often referred to as "Lubin–Tate space", and its ring of functions as the "Lubin–Tate ring".

Because Lubin–Tate space is smooth, the stacky deformation space $\operatorname{Def}(F_d)$ is given by the stacky quotient of Lubin–Tate space by the automorphisms of F_d as a formal group:

$$\operatorname{Def}(F_d) \simeq LT_{F_d} // \mathbb{S}_d.$$

This automorphism group $\mathbb{S}_d$ is also known explicitly.

Theorem 1.2.26 ([19, 46])**.** *Suppose that F_d is defined over a perfect field k. Then its endomorphism algebra takes the following form*

$$\operatorname{End} F_d \cong \mathbb{W}(k)\langle S\rangle \Bigg/ \left(\begin{array}{c} Sa = a^{\varphi} S, \\ S^d = p \end{array} \right),$$

where φ is a lift of the Frobenius from k to the ring of Witt vectors $\mathbb{W}(k)$. In the case that the instantiation of F_d with $[p]_{F_d}(x) = x^{p^d}$ is chosen, S represents the geometric Frobenius on $\widehat{\mathbb{A}}^1$: $S(x) = x^p$. The automorphism group

$$\mathbb{S}_d := \operatorname{Aut} F_d = (\operatorname{End} F_d)^{\times}$$

is referred to as the d^{th} (Morava) stabilizer group.[6] ☐

Remark 1.2.27 ([73, Section 24]). Since it is perhaps not immediately apparent, we indicate how $\mathbb{S}_d$ acts on LT_{F_d}. Given a $\gamma \in \mathbb{S}_d$, we can construct the diagram

$$\begin{tikzcd}
 & \widetilde{F}_d \arrow[rr, dotted, "\widetilde{\gamma}"] \arrow[dd] & & \widetilde{F}_d \arrow[dd] \\
F_d \arrow[ur] \arrow[rr, "\gamma"] \arrow[dd] & & F_d \arrow[ur] \arrow[dd] & \\
 & LT_{F_d} \arrow[rr, dotted, "\exists \gamma_{LT}"] & & LT_{F_d} \\
\operatorname{Spec} k \arrow[ur] \arrow[rr, equal, "1_{LT_{F_d}}"] & & \operatorname{Spec} k. \arrow[ur] &
\end{tikzcd}$$

The dotted arrows exist because the left-most $\widetilde{F}_d$ is a versal deformation of F_d and the right-most $\widetilde{F}_d$ is *some* deformation of F_d, hence there is a map γ_{LT} selecting it. This gives the desired map $\mathbb{S}_d \to \operatorname{Aut} LT_{F_d}$.

Remark 1.2.28. Let K be a local number field with residue field k and let D be the division K-algebra with Hasse invariant $1/d$. Arithmetic geometers may then recognize $\operatorname{End} F_d$ as a maximal order in D. Algebraic topologists who were wondering where our choice of "d" to denote height came from (rather than their usual "n") now know.

1.3 Basic applications to topology

These elements of algebraic geometry make contact with homotopy theory via cohomology functors. For a ring spectrum E and space X, the homotopy groups $\pi_* E = E_*$ and cohomology

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[6] See Example 2.3.7 for a sketch of a proof of this fact.

groups E^*X both form rings, and so we can employ the language of affine algebraic geometry to study them. For the coefficient ring, consider the following construction:

Definition 1.3.1. Suppose that E is a ring spectrum with $\pi_*E^{\wedge j}$ commutative for each j (for instance, it suffices for each $\pi_*E^{\wedge j}$ to be even). We define the simplicial scheme associated to E to be

$$\mathscr{M}_E := \left\{ \begin{tikzcd} \operatorname{Spec}\pi_*E \arrow[r, shift left=2, leftarrow] \arrow[r] \arrow[r, shift right=2, leftarrow] & \operatorname{Spec}\pi_*\begin{pmatrix}E\\ \wedge\\ E\end{pmatrix} \arrow[r, shift left=4, leftarrow] \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] \arrow[r, shift right=4, leftarrow] & \operatorname{Spec}\pi_*\begin{pmatrix}E\\ \wedge\\ E\\ \wedge\\ E\end{pmatrix} \arrow[r, shift left=6, leftarrow] \arrow[r, shift left=4] \arrow[r, shift left=2, leftarrow] \arrow[r] \arrow[r, shift right=2, leftarrow] \arrow[r, shift right=4] \arrow[r, shift right=6, leftarrow] & \cdots \end{tikzcd} \right\}.$$

Remark 1.3.2. Since we've espoused the functor–of–points viewpoint in Definition 1.1.1, it is worth remarking on a similar viewpoint here. Namely, for a ring T, levelwise evaluation of the simplicial functor in $\mathscr{M}_E$ at T yields a simplicial set. This "levelwise evaluation" construction actually takes place internally to the category of simplicial schemes: set $\Delta^j \otimes \operatorname{Spec} T$ to be the semisimplicial functor with $\binom{j}{k}$ disjoint copies of $\operatorname{Spec} T$ in dimension k, and face and degeneracy maps given by list insertion and deletion. A T–point of $\operatorname{Spec}\pi_*E^{\wedge(j+1)}$ is then equivalent to a $(\Delta^j \otimes \operatorname{Spec} T)$–point of $\mathscr{M}_E$, so we are studying the function space from a constant simplicial scheme.

Since E–homology is valued in E_*–modules, which by Definition 1.1.24 are also known as quasicoherent sheaves over $\operatorname{Spec} E_*$, we are prompted to recall the following definitions:

Definition 1.3.3. Let X be a simplicial object in affine schemes and let $\mathscr{F}[n]$ be a quasicoherent sheaf on $X[n]$.[7]

1. $\mathscr{F}$ is collectively called a sheaf [67, Tag 09VK] on X when each map $\varphi : [m] \to [n]$ inducing a map $X(\varphi) : X[n] \to X[m]$ naturally induces a sheaf map
$$\mathscr{F}(\varphi)_* \colon \mathscr{F}[m] \to X(\varphi)_*\mathscr{F}[n].$$
2. $\mathscr{F}$ is called quasicoherent [67, Tag 07TF] when it is comprised levelwise of quasicoherent sheaves.
3. $\mathscr{F}$ is called Cartesian quasicoherent [67, Tag 07TF] when it is quasicoherent and the adjoint map
$$\mathscr{F}(\varphi)^* \colon X(\varphi)^*\mathscr{F}[m] \to \mathscr{F}[n]$$
is an isomorphism.

---

[7] This is a slight abuse of notation: $\mathscr{F}[n]$ is not the n^{th} level of a simplicial object.

In our specific example, we construct such a sheaf in the following way:

Definition 1.3.4. For a spectrum E as in Definition 1.3.1 and input spectrum X, we define the following diagram of abelian groups:

$$\begin{tikzcd}
\mathcal{E}(X) := \Bigg\{ \pi_*\begin{pmatrix} E \\ \wedge \\ X \end{pmatrix} \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] & \pi_*\begin{pmatrix} E \\ \wedge \\ E \\ \wedge \\ X \end{pmatrix} \arrow[r, shift left=4] \arrow[r, leftarrow, shift left=2] \arrow[r] \arrow[r, leftarrow, shift right=2] \arrow[r, shift right=4] & \pi_*\begin{pmatrix} E \\ \wedge \\ E \\ \wedge \\ E \\ \wedge \\ X \end{pmatrix} \arrow[r, shift left=6] \arrow[r, leftarrow, shift left=4] \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] \arrow[r, leftarrow, shift right=4] \arrow[r, shift right=6] & \cdots \Bigg\},
\end{tikzcd}$$

where all of the coface and codegeneracy maps are induced by the unit map $\mathbb{S} \to E$ and the multiplication map $E \wedge E \to E$. (In particular, X is not involved.) The j^{th} object is a module for $\mathcal{O}(\mathcal{M}_E[j])$, and hence determines a quasicoherent sheaf over the scheme $(\mathcal{M}_E[j])$. Suitably interpreted, the maps of abelian groups determine maps of pushforwards so that $\mathcal{E}(X)$ is a quasicoherent sheaf over the simplicial scheme $\mathcal{M}_E$.

In many nice cases, these simplicial constructions are highly redundant and can actually be expressed very simply through equivariant algebraic geometry:

Lemma 1.3.5 (cf. [76, Theorems 13.75 and 17.8] and [67, Tag 07TP])**.** *Consider E_*E as an E_*-module via the structure map induced by $\mathbb{S} \wedge E \to E \wedge E$. If the other structure map $E \wedge \mathbb{S} \to E \wedge E$ is a flat map of E_*-modules, then $\mathcal{M}_E$ is naturally weakly equivalent to*

$$\mathcal{M}_E \simeq (\operatorname{Spec} E_*)//(\operatorname{Spec} E_*E),$$

and the sheaves constructed in Definition 1.3.4 are Cartesian *quasicoherent.*

Proof sketch. The hypothesis implies $\pi_* E^{\wedge j} \cong (E_*E)^{\otimes_{E_*}(j-1)}$. This isomorphism prohibits nondegenerate higher simplices and the result follows immediately. Moreover, the associated sheaves $\mathcal{E}(X)$ are determined by the E_*-module E_*X and its coaction map $\psi : E_*X \to E_*E \otimes E_*X$. □

Example 1.3.6. The above flatness and commutativity hypotheses are satisfied in the case that $E = H\mathbb{F}_2$ is ordinary mod–2 homology. In this case, $\pi_* H\mathbb{F}_2 \cong \mathbb{F}_2$ gives a one-point scheme, acted on by the Hopf algebra

$$\pi_*(H\mathbb{F}_2 \wedge H\mathbb{F}_2) = \mathcal{A}_* \cong \mathbb{F}_2[\xi_1, \xi_2, \ldots, \xi_n, \ldots]$$

with algebra generators ξ_n in degrees $|\xi_n| = 2^n - 1$. The dual Steenrod algebra $\mathcal{A}_*$ has the diagonal structure map [53, Theorem 3]

$$\Delta\colon \mathcal{A}_* \to \mathcal{A}_* \otimes_{\mathbb{F}_2} \mathcal{A}_*, \qquad \Delta(\xi_n) = \sum_{j=0}^{n} \xi_j \otimes \xi_{n-j}^{2^j}.$$

This can be interpreted through algebraic geometry as follows: a generic power series $f(z)$ is an automorphism of the additive formal group law $x' +_{\widehat{\mathbb{G}}_a} x'' = x' + x''$ exactly when it satisfies

$$f(x') + f(x'') = f(x') +_{\widehat{\mathbb{G}}_a} f(x'') = f\left(x' +_{\widehat{\mathbb{G}}_a} x''\right) = f(x' + x''),$$

and such an f is further called a "strict" automorphism when $f'(0) = 1$ (cf. Definition 1.2.13). In characteristic 2, such strict automorphisms are precisely the power series of the form

$$f(x) = x + \sum_{n=1}^{\infty} \xi_n x^{2^n}$$

in indeterminates ξ_n. The functor $\underline{\mathrm{Aut}}(\widehat{\mathbb{G}}_a)$ on $\mathbb{F}_2$–algebras selecting such power series is exactly corepresented by $\mathscr{A}_*$. More than this, both $\underline{\mathrm{Aut}}(\widehat{\mathbb{G}}_a)$ and $\operatorname{Spec} \mathscr{A}_*$ are group schemes in an obvious way, and the isomorphism indicated above respects these structure maps. For example, two such series $f(x)$ and $g(x) = x + \sum_{m=0}^{\infty} \zeta_m x^{2^m}$ compose to give

$$f(g(x)) = \sum_{n=0}^{\infty} \xi_n \left(\sum_{m=0}^{\infty} \zeta_m x^{2^m} \right)^{2^n} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \xi_n \zeta_m^{2^m} x^{2^{m+n}} = \sum_{\ell=0}^{\infty} \left(\sum_{n=0}^{\ell} \xi_n \cdot \zeta_{\ell-n}^{2^n} \right) x^{2^\ell},$$

where $\xi_0 = \zeta_0 = 1$. It follows that

$$\mathscr{M}_{H\mathbb{F}_2} = * // \underline{\mathrm{Aut}}(\widehat{\mathbb{G}}_a).$$

Remark 1.3.7. A similar statement can be made for $H\mathbb{F}_p$, $p \geq 3$, but care (in the guise of formal supergeometry) is required to encode the odd–degree, noncommutative Bockstein τ_* generators. See work of Inoue for details [37].

There are much more instructive examples of the utility of this stacky construction, but in order to properly appreciate them we should introduce the second contact point with algebraic geometry.

Definition 1.3.8. Suppose that X is a space, E is a ring spectrum, and among the compact subspaces $X_\alpha \subseteq X$ of X there is a cofinal subsystem α' for which $E^* X_{\alpha'}$ is even–concentrated.[8] Then, we define X_E to be the formal scheme

$$X_E := \operatorname{Spf} E^* X := \{\operatorname{Spec} E^* X_{\alpha'}\}_{\alpha'}.$$

Example 1.3.9. For example, consider the space $X = \mathbb{C}\mathrm{P}^\infty$ and cohomology theory HR for a commutative ring R. Then X admits an exhaustive filtration by the compact subspaces $X_n = \mathbb{C}\mathrm{P}^n$, hence

$$\mathbb{C}\mathrm{P}^\infty_{HR} = \operatorname{colim}_n \operatorname{Spec} HR^*(\mathbb{C}\mathrm{P}^n) \cong \operatorname{colim}_n \operatorname{Spec} R[x]/x^{n+1} \cong \widehat{\mathbb{A}}^1_R.$$

---

[8] This is the sort of caveat Strickland's definitions are meant to compensate for [74, Definition 8.15].

Furthermore, because $\mathbb{C}\mathrm{P}^\infty$ is an H–space there is an induced associative, symmetric, and unital map

$$\mathbb{C}\mathrm{P}^\infty_{HR} \times_{\operatorname{Spec} R} \mathbb{C}\mathrm{P}^\infty_{HR} \xrightarrow{\mu} \mathbb{C}\mathrm{P}^\infty_{HR}.$$

Because there is no positive–degree homotopy in HR_*, this must be given on coordinates by

$$\begin{aligned} R[\![x]\!]\widehat{\otimes}_R R[\![x]\!] &\xleftarrow{\mathscr{O}(\mu)} R[\![x]\!] \\ x \otimes 1 + 1 \otimes x &\hookleftarrow x, \end{aligned}$$

and hence as a group scheme $\mathbb{C}\mathrm{P}^\infty_{HR}$ is isomorphic to $\widehat{\mathbb{G}}_a$.

Example 1.3.10. Again consider $X = \mathbb{C}\mathrm{P}^\infty$ but now consider its complex K–theory. The ring $KU^*(\mathbb{C}\mathrm{P}^\infty)$ also takes the form

$$KU^*(\mathbb{C}\mathrm{P}^\infty) \cong KU_*[\![x]\!], \quad KU_* \cong \mathbb{Z}[\beta^{\pm}].$$

where x is the degree–zero class representing the virtual bundle $\beta^{-1}\mathscr{L}$—1. The H–space structure on $\mathbb{C}\mathrm{P}^\infty$ classifies the tensor product of line bundles, and hence we compute

$$\begin{aligned} \beta^{-1}(\mathscr{L}\cdot\mathscr{L}')-1 &= \beta^{-1}(\mathscr{L}\cdot\mathscr{L}')-\beta^{-1}\mathscr{L}-\beta^{-1}\mathscr{L}'+1+\beta^{-1}\mathscr{L}-1+\beta^{-1}\mathscr{L}'-1 \\ &= \beta(\beta^{-1}\mathscr{L}-1)(\beta^{-1}\mathscr{L}'-1)+(\beta^{-1}\mathscr{L}-1)+(\beta^{-1}\mathscr{L}'-1), \end{aligned}$$

corresponding to the group law $\beta\cdot(x'\cdot x'')+x'+x''$. It follows that the formal group $\mathbb{C}\mathrm{P}^\infty_{KU}$ is

$$\mathbb{C}\mathrm{P}^\infty_{KU} \cong \widehat{\mathbb{G}}_m.$$

This pair of examples is inspiring: varying the cohomology theory and fixing the space, a formal scheme of the "same shape" (albeit with different group structures and over different bases) appeared from this construction. However, not all cohomology theories E have the property that $\mathbb{C}\mathrm{P}^\infty_E$ is a formal line — for instance, the analogous isomorphism is false for real K-theory KO [83, Corollary 2.13]. This motivates the following definition:

Definition 1.3.11. A multiplicative cohomology theory E is said to be complex orientable when $\mathbb{C}\mathrm{P}^\infty_E$ is a one–dimensional formal variety over $\operatorname{Spec}\pi_*E$. In this case, a choice of coordinate on $\mathbb{C}\mathrm{P}^\infty_E$ is called a complex orientation of E.

Remark 1.3.12. When a complex orientable cohomology theory is represented by a spectrum, complex orientations are in bijective correspondence with factorizations of the unit map:

$$\begin{tikzcd} \mathrm{Thom}(\mathscr{L}-1\downarrow\mathbb{C}\mathrm{P}^0) \arrow[r, equal] \arrow[d] & \mathbb{S} \arrow[r, "\eta"] \arrow[d] & E \\ \mathrm{Thom}(\mathscr{L}-1\downarrow\mathbb{C}\mathrm{P}^\infty) \arrow[r, equal] & \Sigma^{-2+\infty}\mathbb{C}\mathrm{P}^\infty, \arrow[ur, dashed] & \end{tikzcd}$$

i.e., E–cohomology classes of $\mathbb{C}\mathrm{P}^\infty$ of degree 2 which map to the unit under the suspension isomorphism [2, Lemma I.4.6]. Much more deeply, the theory of Thom spectra supplies us with a multiplicative spectrum MU with the property that complex orientations of E are in bijective correspondence with homotopy–multiplicative maps $MU \to E$.

The following theorem of Quillen is the first profound demonstration of the degree to which the geometry of formal groups embeds into the study of multiplicative cohomology theories:

Theorem 1.3.13 (Quillen [60, Theorem 6.5], [58], [56, Appendix 1]). *There is an identification of MU_* with the Lazard ring (see Theorem 1.2.10), so that* $\operatorname{Spec} MU_*$ *represents the scheme of formal group laws and $\mathbb{C}\mathrm{P}^\infty_{MU}$, which is a formal affine line with a canonical coordinate, carries the universal formal group law with rational logarithm*

$$\log(x) = \sum_{n=1}^{\infty} \frac{[\mathbb{C}\mathrm{P}^{n-1}]}{n} \cdot x^n.$$

Moreover, the ring spectrum MU satisfies the hypotheses of Lemma 1.3.5 and $\operatorname{Spec} MU_* MU$ *represents the scheme of formal group laws and "strict isomorphisms" (cf. Definition 1.2.13), so that*

$$\mathcal{M}_{MU} \simeq \mathcal{M}_{\mathbf{fg}}^{(1)}.$$

Remark 1.3.14. The complex bordism ring MU_* comes with a natural grading by dimension, and this is reflected on Lazard's ring by the following $\mathbb{G}_m$–action: for a unit λ and formal group law F, we can produce a new formal group law by the formula

$$x +_{\lambda \cdot F} y := \frac{(\lambda x) +_F (\lambda y)}{\lambda}.$$

This grading is also what inhibits Theorem 1.3.13 from being about $\mathcal{M}_{\mathbf{fg}}$ properly, and it's repaired by forgetting the grading on MU. Namely, let

$$MUP := MU[u^\pm] = \bigvee_{n=-\infty}^{\infty} \Sigma^{2n} MU,$$

so that the following hold:

$$MUP_0 \cong L, \quad MUP_0 MUP \cong L[b_0^\pm, b_1, b_2, \ldots], \quad \operatorname{Spec} MUP_0 // \operatorname{Spec} MUP_0 MUP \cong \mathcal{M}_{\mathbf{fg}}.$$

Throughout this document, we will prefer to ignore gradings, working in the periodified setting instead.

Remark 1.3.15. We take the opportunity to relate Theorem 1.3.13 to the ordinary integral and rational homologies of MU. The map $MU \to H\mathbb{Z}$ given by

$$MU \to \operatorname{colim}_n MU/(c_1, c_2, \ldots, c_n)$$

selects the integral additive formal group on homotopy. The induced map $MU \wedge MU \to H\mathbb{Z} \wedge MU$ presents

$$H\mathbb{Z}_* MU = \operatorname{colim}_n MU[b_1, b_2, \ldots]/(c_1, c_2, \ldots, c_n) = \mathbb{Z}[b_1, b_2, \ldots]$$

as the universal ring with an integrally-defined strict exponential map $\exp_b(x) = \sum_j b_j x^{j+1}$ selecting another integral formal group law

$$x +_b y = \exp_b(\exp_b^{-1} x + \exp_b^{-1} y).$$

By Corollary 1.2.11, every formal group law admits a lift to a torsion-free ring. Furthermore, by Lemma 1.2.14 every formal group law over a rational ring has a logarithm. It thus follows that the maps

$$\pi_* MU \to H\mathbb{Z}_* MU \to H\mathbb{Q}_* MU$$

are both injections.

With Theorem 1.3.13 in hand, the following observation of Landweber furnishes us with a great deal of cohomology theories:

Theorem 1.3.16 ([42]). *When $i\colon \operatorname{Spec} R \to \mathcal{M}_{\mathrm{fg}}$ is a flat map, the restriction $i^* \mathcal{M}_{MU[u^\pm]}(X)$ determines a complex orientable homology theory. If $j\colon \operatorname{Spec} R \to \operatorname{Spec} L$ is a lift of such an i across Lazard's C-cover, then j determines an even-periodic complex oriented homology theory by the formula*

$$X \mapsto MUP_*(X) \otimes^j_{MUP_*} R.$$

Remark 1.3.17. It is worth emphasizing that Landweber's theorem has two serious limitations: it only gives a homology theory rather than a spectrum, and it only applies to flat *affine* maps. In particular, this makes it very hard to arrange to use Landweber's theorem to approach nonaffine flat maps, e.g., $\mathcal{M}_{\mathrm{ell}} \to \mathcal{M}_{\mathrm{fg}}$. Much of the work surrounding the construction of topological modular forms (alias *TMF*) grapples with this issue.

Example 1.3.18 (Brown–Peterson theories). The inclusion $\mathcal{M}_{\mathrm{fg}} \times \operatorname{Spec} \mathbb{Z}_{(p)} \to \mathcal{M}_{\mathrm{fg}}$ is flat since the localization morphism $\operatorname{Spec} \mathbb{Z}_{(p)} \to \operatorname{Spec} \mathbb{Z}$ is flat. It follows from Theorem 1.3.16 that there is a homology theory BPP with coefficients given by the V-covering ring $\mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_d, \ldots]$ of Theorem 1.2.19.

Example 1.3.19 (Johnson–Wilson theories). The inclusion of open substacks is flat. It follows from Theorem 1.2.22 and Theorem 1.3.16 that there is a cohomology theory $EP(d)$ associated to the open submoduli $\mathcal{M}_{\mathrm{fg}}^{\leq d}$ of the p-local moduli $\mathcal{M}_{\mathrm{fg}} \times \operatorname{Spec} \mathbb{Z}_{(p)}$. In the V-cover, this is naively given by the coefficient ring $\mathbb{Z}_{(p)}[v_1, \ldots, v_{d-1}, v_d, v_d^{-1}, v_{d+1}, \ldots]$, but in fact the ring $\mathbb{Z}_{(p)}[v_1, \ldots, v_d][v_d^{-1}]$ can be used.

Example 1.3.20 (Morava E-theories). The closed points of Theorem 1.2.23 are generally not selected by flat maps. However, completion is meant to correct this problem: the completion of a closed substack of a Noetherian stack prolongs the inclusion to a flat map [50]:

$$\begin{tikzcd}
\operatorname{Spec} k \arrow[rr, "\Gamma"] \arrow[dr] & & \mathcal{M}_{\mathrm{fg}}^{\leq d} \arrow[r, "\text{open}"] & \mathcal{M}_{\mathrm{fg}} \\
& \operatorname{Def}(\Gamma). \arrow[ur, "\text{flat}"] \arrow[urr, "\text{flat}"'] & &
\end{tikzcd}$$

It follows from Theorem 1.3.16 that there is a cohomology theory E_Γ with coefficients given by the Lubin–Tate ring for Γ, a formal group of finite height. This homology theory was famously first considered by Morava [54].

Example 1.3.21 (Finite height Morava K–theories). Having constructed E_Γ for a finite height formal group Γ, we can then use Theorem 1.2.24 to construct a theory associated to the actual classifying map

$$\Gamma\colon \operatorname{Spec} k \to \mathcal{M}_{\mathrm{fg}}.$$

Namely, that theorem guarantees the existence of a regular sequence $(p, u_1, \dots, u_{d-1})$ given by a coordinatization $(u_1, \dots, u_{d-1})$ of $\operatorname{Def}(\Gamma) \cong \widehat{\mathbb{A}}^{d-1}_{\mathbb{W}(k)}$. In turn, we can define K_Γ to be the quotient $E_\Gamma/(p, u_1, \dots, u_{d-1})$, where "quotient" is taken to mean the iterated cofiber of multiplication by these homotopy elements.

Example 1.3.22 (Exceptional Morava K–theories). In accordance with Remark 1.2.21, we define the Morava K–theory associated to an additive group over a field to be the Eilenberg–Mac Lane spectrum for that field.

Remark 1.3.23. Theorem 1.3.16 bestows all these cohomology theories with 2–periodic gradings, and this is often not minimal. Indeed, recall from Remark 1.3.14 that the "P" in our notation is meant to denote "Periodic".

1. MUP was formed by setting $MUP = MU[u^\pm]$ with $|u| = 2$, and indeed this gives a minimal multiplicative decomposition of MUP into wedge summands.

2. BPP similarly decomposes multiplicatively as $BPP = BP[u^\pm]$ for some connective ring spectrum BP and $|u| = 2$. Moreover, $L_p MU$ decomposes as an infinite wedge sum of shifts of BP.

3. $EP(d)$ splits into a wedge of $(p^d - 1)$ summands of identical $2(p^d - 1)$–periodic spectra called $E(d)$.

4. K_{F_d}, for F_d the formal group law described in Theorem 1.2.23, splits into a wedge of $(p^d - 1)$ summands of identical $2(p^d - 1)$–periodic spectra called $K(d)$.

Remark 1.3.24. As all of the preceding examples are based on base-change, it is prudent to mention that algebraic geometers have long favored p–divisible groups over formal groups in similar situations. A p–divisible group is an inductive sequential system $\{G_k\}$ of finite, flat group schemes, subject to the condition that $G_{k+1}[p^k] \cong G_k$ with quotient $G_{k+1}/G_k \cong G_1$. Over a p–complete ground scheme, connected p–divisible groups are equivalent to formal groups of finite height [78, Proposition 1]. However, they behave quite differently under base change: as a finite flat scheme, the rank of G_1 is constant under base-change (though the system $\{G_k\}$ may not be sent to a *connected* p-divisible group!), whereas the height of a formal group can vary after base-change. The theory of p-divisible groups has made some impact in algebraic topology; the reader should turn to Hopkins–Kuhn–Ravenel character theory as a prime example [29], as well as its "transchromatic" extension by Stapleton [68].

Remark 1.3.25. Having now constructed an ample supply of important cohomology theories, it is now also worth remarking that it is similarly valuable to vary X as well as E in the construction of X_E. This is especially true if X is taken to be a space "spiritually near" to $\mathbb{C}\mathrm{P}^\infty$. For example, for a complex-oriented cohomology theory E, $BU(n)_E$ can be identified with a certain scheme of effective divisors of weight n on $\mathbb{C}\mathrm{P}^\infty_E$ [74, Section 8.3]. When E is complex-orientable and its homotopy ring is complete against n, a choice of isomorphism $\mathbb{Z}/n \cong U(1)[n]$ identifies the scheme $B\mathbb{Z}/n_E$ with the subscheme $\mathbb{C}\mathrm{P}^\infty_E[n]$ of n-torsion of the formal group $\mathbb{C}\mathrm{P}^\infty_E$. See Ravenel and Wilson [64, Theorem 5.7] for an early version of this second theorem[9] or Hopkins–Kuhn–Ravenel [29, Section 5] and Stapleton [68, Theorem 2.1 and Proposition 2.3] for more elaborate versions.

Remark 1.3.26. The reader should also be warned of a small inconsistency in the additive case: the formal group on which $\mathrm{Aut}\,\widehat{\mathbb{G}}_a$ acts in Example 1.3.6 is $\mathbb{R}\mathrm{P}^\infty_{H\mathbb{F}_2}$ rather than $\mathbb{C}\mathrm{P}^\infty_{H\mathbb{F}_2}$, and similarly the cohomological object in question in Remark 1.3.7 is $H\mathbb{F}_p^*(B\mathbb{Z}/p)$.

Remark 1.3.27. The E_2-page of the E-based Adams spectral sequence for $\pi_* X_E^\wedge$ can be interpreted as the stack cohomology of $\mathcal{M}_E(X)$ over $\mathcal{M}_E$. Because cohomology is involved, this is a place where the full force of stacky technology can be profitably brought to bear on the problem. For our purposes, however, it will typically suffice to think of "stack" as shorthand for "equivariant algebraic geometry" or "simplicial algebraic geometry", without worrying about descent or gluing.

Remark 1.3.28. It is tempting to think of the pro-spectrum $\{F(X_\alpha, \mathbb{S})\}$ associated to a space X as the primal object being studied, and that

$$X_E = \{\operatorname{Spec} E^* X_\alpha\} = \{\operatorname{Spec} \pi_*(F(X_\alpha, \mathbb{S}) \wedge_{\mathbb{S}} E)\}$$

arises from some kind of base-change construction. This is sometimes useful for intuition, and Mike Mandell has proven results along these lines for ordinary cohomology [49, 48] extending the rational results of Quillen [59]. However, the author does not know how to make this thought satisfyingly precise for the periodic cohomology theories E_Γ relevant to chromatic homotopy theory.

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[9] Beware that the subscripts in the proof of Ravenel and Wilson's theorem have typos: β_{nj+i} and β_i should be $\beta_{(nj+i)}$ and $\beta_{(i)}$ respectively.

Fields in stable homotopy theory

We conclude this section with some remarks on “field cohomology theories” in stable homotopy theory. A field k in commutative algebra is a commutative ring characterized by the property that its category of modules is free under direct sum on the single generator k itself. Analogously, we make the following definition:

Definition 1.3.29. A field spectrum K is a ring spectrum so that any K–module (in the homotopy category) splits as a wedge of suspensions of K.

The classification of these objects is due to Devinatz, Hopkins, and Smith [32].

Theorem 1.3.30 ([32, Proposition 1.9])**.** *The following statements are true:*

1. *If K is a field spectrum, then there is a d such that K decomposes as a wedge of suspensions of the spectrum $K(d)$ of Remark 1.3.23 for the formal group F_d of height d.*
2. *The construction K_Γ determines a bijection*

$$\begin{tikzcd}
\left\{ \begin{array}{c} \textit{formal Lie groups} \\ \textit{over } k \end{array} \right\} \arrow[r, bend left, "K_{(-)}"] & \left\{ \begin{array}{c} \textit{2–periodic field spectra} \\ \textit{with } \pi_0 = k \end{array} \right\} . \arrow[l, bend left, "\mathbb{CP}^{\infty}_{(-)}"]
\end{tikzcd}$$

Remark 1.3.31. The reader may like to enrich Theorem 1.3.30 to a statement about categories rather than a bijection of sets of isomorphism classes, but we warn that this turns out to be quite tricky. A smattering of results in this direction can be found in work of Goerss, Hopkins, and Miller [21, Corollary 7.6], of Ando [4, Theorems 1 and 5], and of Strickland [71].

Remark 1.3.32. The reader may also recall the primacy of formal groups in Lubin and Tate’s explicit local class field theory. In that context, for a local number field K with ring of integers $\mathcal{O}_K$ and residue field k, the points of the maximal totally ramified abelian extension K^{trab} and the Artin reciprocity morphism can both be described in terms of a certain formal group Γ_K, defined over $\mathcal{O}_K$ and with height prescribed by properties of k. That is, some significant chunk of the arithmetic of K is controlled by a background formal group. We would like to make the vague assertion that something similar is happening here: the behavior of the field spectrum K_Γ is again controlled by a background formal group $\Gamma \simeq \mathbb{CP}^{\infty}_{K_\Gamma}$.

Remark 1.3.33. Paul Balmer has given a procedure for associating a space $\operatorname{Spec} \mathsf{C}$ to a tensor triangulated category C, the points of which are specified by the thick tensor ideals of C [8]. For a ring R, the Balmer spectrum $\operatorname{Spec} D^{\mathrm{fin}}(\mathsf{Modules}_R)$ associated to the derived category of R–modules agrees with the Zariski spectrum, and the points are in natural bijection with the closed subschemes of the functor $\operatorname{Spec} R$ as defined in Definition 1.1.4. One of the consequences [32, Theorem 7] of Theorem 1.3.30 is that the points in the Balmer spectrum of $\mathsf{Spectra}^{\mathrm{fin}}$ are selected by the K_Γ–acyclics for various formal groups Γ [8, Corollary 9.5].

1.4 The Γ–local stable category

To fully employ the arithmetic geometry attached to Morava's cohomology theories, it is common to move to the associated "local" category — i.e., to declare that spectra which are K_Γ–acyclic are in fact contractible, or equivalently to declare that maps which are K_Γ–homology isomorphisms are in fact weak homotopy equivalences. This eliminates the "pathology" of topological phenomena which are invisible to K_Γ, and so more tightly binds the behavior of $\mathsf{Spectra}$ to K_Γ. For reasons to be discussed in this section, we will refer to this simply as the "Γ–local category" with localization functor L_Γ, suppressing the letter "K".

When Γ has finite height d this category has fascinating properties, many of which will be the subject of the remainder of this paper.

From here on, we will always consider d to be finite and positive unless otherwise stated.

Continuous E–theory

To begin our study of the Γ–local category, we cite some bulk results which eludicate features of the localization functor:

Theorem 1.4.1. *Let d be the height of Γ.*

1. *([63, Theorem 7.5.6]) In keeping with the above geometrically-centric notation, let $L_{\mathcal{M}_{\mathrm{fg}}^{\le d}}$ denote the Bousfield localization functor for $E(d)$. This functor is smashing, i.e.,*

$$L_{\mathcal{M}_{\mathrm{fg}}^{\le d}} X \simeq \left(L_{\mathcal{M}_{\mathrm{fg}}^{\le d}} \mathbb{S}\right) \wedge X.$$

2. *([33, Lemma 2.3]) There is the following weak equivalence, natural in X:*

$$L_\Gamma(X) \simeq \lim_I \left(L_{\mathcal{M}_{\mathrm{fg}}^{\le d}} \mathbb{S}^0 \wedge M_0(v^I) \wedge X\right),$$

where $\{M_0(v^I)\}_I$ is an inverse system of finite spectra which have bottom cell in dimension zero, which have maps $M_0(v^I) \to M_0(v^{I'})$ for $I \ge I'$, and which have

$$BP_* M_0(v^I) \cong BP_*/(p^{I_0}, v_1^{I_1}, \dots, v_{n-1}^{I_{n-1}}).$$

(Such a system is guaranteed to exist for large enough I by Hopkins–Smith [32, Proposition 5.14].)

3. *(Theorem 1.3.30.1) The localization functor L_Γ is an invariant of d and of the characteristic p of the ground field.*
4. *([61, Theorem 2.1.d, Lemma 2.3]) There is a natural pullback square*

$$\begin{tikzcd}
L_{\mathcal{M}_{\mathrm{fg}}^{\leq d}} X \arrow[r] \arrow[d] & L_\Gamma X \arrow[d] \\
L_{\mathcal{M}_{\mathrm{fg}}^{\leq (d-1)}} X \arrow[r] & L_{\mathcal{M}_{\mathrm{fg}}^{\leq (d-1)}} L_\Gamma X.
\end{tikzcd}$$

5. *([63, Theorem 7.5.7]) For X a finite spectrum, there is a natural equivalence*

$$X_{(p)} \simeq \lim_d L_{\mathcal{M}_{\mathrm{fg}}^{\leq d}} X. \quad \square$$

Before continuing, we must address one important caveat:

Definition 1.4.2. The homological Bousfield class of a spectrum E is the collection of all spectra X which have $E_* X \neq 0$. Similarly, one can define a cohomological Bousfield class for E to be the collection of all spectra X which have $E^* X \neq 0$.

Lemma 1.4.3 (Theorem 1.4.4, Corollary 1.4.6)**.** *The cohomological Bousfield classes of K_Γ and E_Γ agree. The homological Bousfield class of E_Γ is strictly larger than the homological Bousfield class of K_Γ.* ☐

This lemma has some important philosophy behind it. Example 1.3.20 and Example 1.3.21 define E–theory as associated to the infinitesimal deformation space of K–theory, which suggests the presence of a cohomological Bockstein spectral sequence

$$K_\Gamma(X) \otimes A \Rightarrow E_\Gamma(X).$$

Here A consists of Bocksteins arising from square–zero deformation fiber sequences of the flavor

$$\begin{tikzcd}
\cdots \arrow[r] & E_\Gamma/\mathfrak{m}^{j+1} \arrow[r] \arrow[d] & E_\Gamma/\mathfrak{m}^{j} \arrow[r] \arrow[d] & \cdots \\
& E_\Gamma \otimes \mathfrak{m}^{j+1}/\mathfrak{m}^{j+2} \arrow[d, equal] & E_\Gamma \otimes \mathfrak{m}^{j}/\mathfrak{m}^{j+1} \arrow[l, "{[+1]}"] \arrow[ul, "{\partial,[+1]}"] \arrow[d, equal] & \\
& \displaystyle\bigvee_{x_i \text{ basis of } \mathfrak{m}^{j+1}/\mathfrak{m}^{j+2}} \Sigma^{|x_i|} K_\Gamma & \displaystyle\bigvee_{y_j \text{ basis of } \mathfrak{m}^{j}/\mathfrak{m}^{j+1}} \Sigma^{|y_j|} K_\Gamma, \arrow[l, "{\text{Bocksteins},[+1]}"] &
\end{tikzcd}$$

where $\mathfrak{m}$ is the maximal ideal of the Lubin–Tate ring. Such a spectral sequence indeed turns out to exist for the cohomology theories K_Γ and E_Γ. In particular, there is the following theorem:

Theorem 1.4.4 ([35, Proposition 2.5]). *If X is a spectrum so that $K_\Gamma^* X$ is even–concentrated, then $E_\Gamma^* X$ is pro-free and even–concentrated, so that $K_\Gamma^* X = E_\Gamma^* X \otimes_{E_\Gamma^*} E_\Gamma^* / \mathfrak{m}$.*

Proof sketch. The differentials in the Bockstein spectral sequence take even–degree elements to odd–degree elements. By assumption there are no odd–degree elements, and hence the spectral sequence collapses at E_1. □

However, such a spectral sequence for homology (with reasonable convergence properties) is prohibited by Lemma 1.4.3: there are spectra with vanishing K_Γ–homology for which the E_Γ–homology is nonzero. This is "corrected" by remaining inside the Γ–local category: we define the covariant functor E_Γ by the formula

$$E_\Gamma(X) := \pi_* L_\Gamma(E_\Gamma \wedge X) \cong \pi_* L_\Gamma(E_\Gamma \wedge L_\Gamma X).$$

The bifunctor $(X, Y) \mapsto L_\Gamma(X \wedge Y)$ on Γ–local spectra determines a monoidal structure for which the localization map L_Γ is a map of monoidal categories. This means that the above definition of E_Γ is the natural one for considerations internal to the Γ–local category — and it has the pleasant extra effect of forcefully correcting the "homological Bousfield class" of E_Γ. In turn, Theorem 1.4.1 gives the result we sought:

Lemma 1.4.5 ([35, Propositions 7.10 and 8.4]). *There is a Milnor exact sequence*

$$0 \to \lim_j{}^1 (E_\Gamma/\mathfrak{m}^j(\Sigma^{-1} X)) \to E_\Gamma(X) \to \lim_j (E_\Gamma/\mathfrak{m}^j(X)) \to 0,$$

where the derived inverse limit is taken in abelian groups.

Proof. This is a direct corollary of Theorem 1.4.1 and the Milnor sequence for homotopy inverse limits (cf. Remark A.4). □

Corollary 1.4.6 ([35, Proposition 8.4]). *If $E_\Gamma(X)$ is pro-free then $K_\Gamma(X) \cong E_\Gamma(X)/\mathfrak{m}_\Gamma$. In the other direction, if $K_\Gamma(X)$ is concentrated in even dimensions, then $E_\Gamma(X)$ is pro-free.*

Proof sketch. Recall from Theorem 1.4.1 and Lemma 1.4.5 the defining sequence

$$\pi_* L_\Gamma(E_\Gamma \wedge X) \cong \pi_* \lim_I \left(M_0(v^I) \wedge E_\Gamma \wedge X\right).$$

(We have used that E_Γ is $E(d)$–local.) The inverse system is a cofinal subsystem of the system $\pi_* \lim_I \left(E_\Gamma/(v^I) \wedge X\right)$, and in the situation that $I' < 2I$ (i.e., the ideal (v^I) is square–zero in $\pi_* E_\Gamma/(v^{I'})$) then the induced map $E_\Gamma/(v^{I'}) \to E_\Gamma/(v^I)$ has fiber a wedge of even–degree suspensions of K_Γ. It follows that the long exact sequence of homotopy determining $E_\Gamma/(v^{I'})$ from $E_\Gamma/(v^I)$ degenerates into easily studied short exact sequences. □

Remark 1.4.7. The covariant functor E_Γ does *not* satisfy the axioms of a homology functor, because it does not commute with infinite colimits. (In particular, there is a spectral sequence arising from Hovey's lemma [33, Lemma 2.3] comparing the behavior of infinite colimits to a kind of derived completion [34].) Another way of viewing the difference between the two notions of E_Γ-homology is that E_Γ is naturally expressed as an inverse limit, but the following equivalence fails, sometimes wildly:

$$\pi_*\left(\left(\lim_j E_\Gamma/\mathfrak{m}^j\right)\wedge X\right)\not\cong \pi_*\lim_j\left((E_\Gamma/\mathfrak{m}^j)\wedge X\right).$$

This observation directly connects this difference to a topic in our Appendix A and in the MIT E-theory seminar notes [57, Section 14]. Nonetheless, Corollary 1.4.6 is reason enough to call the covariant E_Γ the "correct" notion of Morava E-homology, and we will unabashedly refer to it as a "homology functor" in the rest of this document.

Remark 1.4.8. In the case that $K_\Gamma(X)$ is even-concentrated for a space X, the compact subspaces of X can be used to topologize $E_\Gamma(X)$ and $E^*_\Gamma(X)$ so that they become continuously $(E_\Gamma)_*$-linearly dual to one another. That is, the sheaf $\mathcal{E}_\Gamma(X)$ and the scheme X_{E_Γ} (together with its $\operatorname{Aut}\Gamma$-action) contain equivalent information.

Finally, we remark that E_Γ is valued in the correct category of modules so that the functor $\mathcal{E}_\Gamma$ constructed by Definition 1.3.4 takes values in the correct category of sheaves:

Lemma 1.4.9 ([72, Theorem 12]). *The functor E_Γ is valued in modules with a continuous action of the Lubin–Tate ring (i.e., in pro-systems of modules over the finite stages of the Lubin–Tate scheme). Additionally, these sheaves are equivariant against the action of the stabilizer group of Theorem 1.2.26, hence they descend to the Lubin–Tate stack* $\operatorname{Def}(\Gamma)\subseteq\mathcal{M}_{\mathrm{fg}}$. □

Picard-graded homotopy

Picard-graded homotopy groups are a recurrent theme in homotopy theory. For example, the $RO(G)$ grading in equivariant stable homotopy theory refers to the equivariant Picard grading, and the twists in twisted cohomology (e.g., twisted K-theory) refer to the Picard grading for parametrized spectra. It also appears elsewhere in mathematics: in algebraic geometry, one studies sections of a line bundle on a projective variety rather than mere functions (i.e., sections of the trivial bundle) in order to recover further interesting data. This, too, is an example of a Picard grading (and is where the phrase "Picard grading" comes from, as the the group of isomorphism classes of line bundles on a variety is called its Picard group). The behavior of the appropriate analogue of Picard gradings in chromatic homotopy theory is very telling, and in this subsection we will recount some of what is known.

Definition 1.4.10. The Picard category of a symmetric monoidal (∞-)category C is the maximal subgroupoid of the full subcategory spanned by the $\otimes$-invertible objects. Its connected components determine a group $\operatorname{Pic}\mathsf{C}$ called the Picard group of C.

Example 1.4.11 ([31, pg. 90] or [70, Theorem 2.2]). The Picard group of the global stable category Spectra is isomorphic to $\mathbb{Z}$ and generated by $\mathbb{S}^1$.

The Picard group of the Γ-local stable category is considerably more complicated. Its study was initiated by Hopkins, Mahowald, and Sadofsky [31] at height $d = 1$, but there are now a number of results at height $d = 2$ as well; see Remark 1.4.19. For our present purposes, the most important of the Hopkins–Mahowald–Sadofsky results is the following theorem:

Theorem 1.4.12 ([31, Theorem 1.3]). *A spectrum X is Γ-locally invertible if and only if $(K_\Gamma)_*X$ is 1-dimensional as a graded vector space.* ☐

Remark 1.4.13. Taking "Pic(C)" for a moment to mean the category in Definition 1.4.10, this theorem can also be interpreted as asserting that the following square is a pullback in monoidal categories:

$$\begin{tikzcd}
\mathsf{Spectra}_\Gamma \arrow[r] & \mathsf{VectorSpaces}_{K_*} \\
\mathsf{Pic}(\mathsf{Spectra}_\Gamma) \arrow[r] \arrow[u] & \mathsf{Lines}_{K_*}. \arrow[u]
\end{tikzcd}$$

We also note that Corollary 1.4.6 gives a factorization

$$\begin{tikzcd}
\mathsf{Spectra}_\Gamma \arrow[r, "\mathscr{E}_\Gamma"] & \mathsf{QCoh}(\mathrm{Def}(\Gamma)) \arrow[r, "i_0^*"] & \mathsf{VectorSpaces}_{K_*} \\
\mathsf{Pic}(\mathsf{Spectra}_\Gamma) \arrow[r] \arrow[u] & \mathsf{LineBundles}(\mathrm{Def}(\Gamma)) \arrow[r, "i_0^*"] \arrow[u] & \mathsf{Lines}_{K_*}. \arrow[u]
\end{tikzcd}$$

The left-hand square in this diagram is also a pullback square, but the categories in the middle column are considerably more rich than those in the right column; in particular, they contain more than one isomorphism class. In the case of LineBundles(Def(Γ)), this corresponds to tracking the 1-dimensional Aut Γ-representation given by $E_\Gamma(X)$ rather than the mere 1-dimensional vector space given by $(K_\Gamma)_*(X)$. The following result expresses just how much more information this encodes:

Lemma 1.4.14 ([31, Proposition 7.5]). *The map*

$$\mathsf{Pic}(\mathsf{Spectra}_\Gamma) \to \mathsf{LineBundles}(\mathrm{Def}(\Gamma))$$

is injective on objects for $2p - 2 \geq d^2$ and $p \neq 2$. ☐

Throughout this document, these theorems will be our main tool which we will use to furnish Γ-locally invertible spectra. Before proceeding to more complicated situations, we begin with a simple example in the $\widehat{\mathbb{G}}_m$-local category. Consider the following horizontal system of cofiber sequences in the global stable category:

$$\begin{tikzcd}
\cdots \arrow[r, equal] & \mathbb{S}^{-1} \arrow[r, equal] \arrow[d, "p^j"] & \mathbb{S}^{-1} \arrow[r, equal] \arrow[d, "p^{j+1}"] & \cdots \arrow[r, equal] & \mathbb{S}^{-1} \arrow[d] \\
\cdots \arrow[r, "p"] & \mathbb{S}^{-1} \arrow[r, "p"] \arrow[d] & \mathbb{S}^{-1} \arrow[r, "p"] \arrow[d] & \cdots \arrow[r] & p^{-1}\mathbb{S}^{-1} \arrow[d] \\
\cdots \arrow[r, dotted] & M^0(p^j) \arrow[r, dotted] & M^0(p^{j+1}) \arrow[r, dotted] & \cdots \arrow[r, dotted] & M^0(p^\infty).
\end{tikzcd}$$

The row-wise homotopy colimit, pictured on the far right, is also a cofiber sequence. The topmost object is the colimit of a sequence of identity morphisms, so is simply $\mathbb{S}^{-1}$. The middle object is the colimit along iterates of the map p, so is, by definition, the spectrum $p^{-1}\mathbb{S}^{-1}$ with the p-self-map inverted. Lastly, the spectrum on the bottom does not have a familiar name, so we call it $M^0(p^\infty)$.

When working p-locally (and so also when working $\widehat{\mathbb{G}}_m$-locally), we can see that the middle spectrum is contractible: the map p on $p^{-1}\mathbb{S}^{-1}$ is exactly multiplication by p in K-homology, but since the coefficient ring K_* is of characteristic p, this is the zero map. On the other hand, p is required to be invertible, which can only mean $K_* p^{-1}\mathbb{S}^{-1} = 0$. In turn, this means that the going-around map $M^0(p^\infty) \to \mathbb{S}^0$ is a p-local equivalence, and so $M^0(p^\infty)$ is an invertible spectrum, albeit not a very interesting one.[10]

A different way of at least detecting that $M^0(p^\infty)$ is $\widehat{\mathbb{G}}_m$-locally invertible is to apply K-homology to the bottom row: each object in the sequence becomes a 2-dimensional graded vector space over K_*, and each map from one to the next is $- \cdot p = - \cdot 0$ on the (-1)-graded piece and the identity on the 0-graded piece. Hence, the colimit is a K_*-line, and Theorem 1.4.12 thus assures us we have an invertible spectrum. This is portrayed in Figure 1.1. This suggests a way we can modify this construction: if we insert other maps which are K-homology isomorphisms, then we will not harm this proof that the colimit is an invertible spectrum. We will make use of the following results to furnish ourselves with such maps:

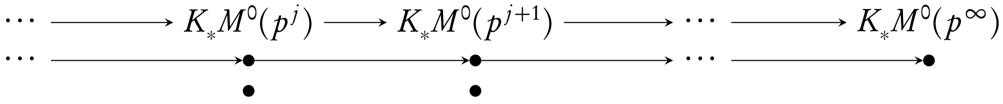

Figure 1.1: "Cell diagram" of $K_*M^0(p^\infty)$.

Definition 1.4.15 ([63, Definition 1.5.3]). A finite spectrum X is said to be of type d when it is Γ-locally acyclic for all Γ of height strictly less than d and Γ-locally nontrivial for a Γ of height exactly d. (In fact, it suffices to check that acyclicity condition for any single Γ of height $(d-1)$ [61, Theorem 2.11].)

---

[10] The dimension shift from $\mathbb{S}^{-1}$ to $\mathbb{S}^0$ is a homotopical reflection of the statement that the p-primary part of the circle group S^1 is the p-Prüfer group $\mathbb{Z}/p^\infty = \operatorname{colim}_j \mathbb{Z}/p^j$.

Theorem 1.4.16 (Devinatz–Hopkins–Smith [32, Theorem 9]). *A p-local finite spectrum X is of type d if and only if for $N \gg 0$ there is a map $v : \Sigma^N X \to X$ which is an isomorphism in K_Γ-homology for Γ of height d and which induces the zero map in K_Γ-homology for Γ not of height d.*

Lemma 1.4.17 (Adams [1, Lemma 12.5]). *The spectrum $M^0(p^{j+1})$ is type 1 and it admits a map*

$$v_1^{p^j} : M^{2p^j(p-1)}(p^{j+1}) \to M^0(p^{j+1})$$

which induces multiplication by $v_1^{p^j}$ in $K(1)$-homology.[11] *Moreover, the following square commutes:*

$$\begin{tikzcd}
M^{2p^j(p-1)}(p^j) \arrow[r, "\left(v_1^{p^{j-1}}\right)^p"] \arrow[d] & M^0(p^j) \arrow[d] \\
M^{2p^j(p-1)}(p^{j+1}) \arrow[r, "v_1^{p^j}"] & M^0(p^{j+1}).
\end{tikzcd}$$

Selecting a p-adic integer $a_\infty = \sum_{j=0}^\infty c_j p^j$ with $0 \le c_j < p$, one can now construct the system

$$\cdots \to M^{-|v_1|a_{j-1}}(p^j) \to M^{-|v_1|a_{j-1}}(p^{j+1}) \xrightarrow{v_1^{p^j c_j}} M^{-|v_1|a_j}(p^{j+1}) \to M^{-|v_1|a_j}(p^{j+2}) \to \cdots .$$

We define $\mathbb{S}^{-|v_1|a_\infty}$ to be its colimit. Theorem 1.4.12 is then sufficient to check that $\mathbb{S}^{-|v_1|a_\infty}$ is $\widehat{\mathbb{G}}_m$-locally invertible, but more is true:

Lemma 1.4.18 ([31, Proposition 2.1]). *The above construction defines an injective continuous homomorphism of groups*

$$\mathbb{Z}_p \to \operatorname{Pic}(\mathsf{Spectra}_{\widehat{\mathbb{G}}_m}).$$

When $p \ge 3$ (i.e., $p \ne 2$), the cosets of its image are represented by $\mathbb{S}^1, \ldots, \mathbb{S}^{|v_1|}$. Abstractly, there is an isomorphism

$$\operatorname{Pic}(\mathsf{Spectra}_{\widehat{\mathbb{G}}_m}) \cong \mathbb{Z}_p \rtimes \mathbb{Z}/|v_1|. \quad \square$$

Remark 1.4.19. This is the most thorough result of this kind that we know presently. We also know a calculation of $\operatorname{Pic}(\mathsf{Spectra}_{K(d)})$ for $d = 1$ at $p = 2$ [31, Theorem 3.3], for $d = 2$ at $p \ge 5$ [11, Theorem 8.1], and for $d = 2$ at $p = 3$ [22, Theorem 1.2]. We have partial information for $d = 2$ and $n = 2$ [70, pg. 50], and we know essentially nothing for $n \ge 3$ apart from the Hopkins–Gross analysis of the Brown–Comenetz dualizing spectrum [28, Theorem 6] and the analogue of Lemma 1.4.18 using "generalized Moore spectra" [32, Proposition 5.14], [31, Proposition 9.2-3].

That $\operatorname{Pic}(\mathsf{Spectra}_{\widehat{\mathbb{G}}_m}) \cong \mathbb{Z}_p$ carries a profinite topology is not an accident; this, too, is found to be an effect internal to algebraic topology.

---

[11] Here $K(1)$ is the close cousin of $K_{\widehat{\mathbb{G}}_m}$ described in Remark 1.3.23.

Lemma 1.4.20 ([35, Proposition 14.3.d]). *Let $F(I)$ denote the collection of Γ-local invertible spectra which become (noncanonically) isomorphic to $L_\Gamma \mathbb{S}^0$ after smashing with the generalized Moore spectrum $M_0(v^I)$. The $F(I)$ form a basis of closed neighborhoods at the identity which upon linear translation endow* $\operatorname{Pic}(\mathsf{Spectra}_\Gamma)$ *with the structure of a profinite group.* □

This computation of the Picard group pairs well with another classical calculation:

Theorem 1.4.21 ([1, Theorem 1.5]). *There are isomorphisms*

$$\pi_s L_{K(1)}\mathbb{S}^0 \cong \begin{cases} \mathbb{Z}_p & \text{when } s = 0, \\ \mathbb{Z}_p/(pk) & \text{when } s = k|v_1| - 1, \\ 0 & \text{otherwise.} \quad \square \end{cases}$$

The right-hand side of this formula can be interpreted through the p-adic valuations of the Bernoulli numbers — or, equivalently, through the special negative values of the Riemann ζ-function:

$$\mathbb{N} \xrightarrow{s \mapsto \zeta(1-s} \mathbb{Q} \xrightarrow{\text{denom}} \mathbb{Q}/\mathbb{Z}_{(p)} \cong \mathbb{Z}/p^\infty.$$

Number theorists have constructed p-adic analytic versions of the Riemann ζ-function by interpolating its special values at negative odd integers and have found such constructions to continue to hold interesting number theoretic data [38]. For our purposes, it is sufficient to note that the p-adic valuation of $\zeta_p^\wedge(1-s)$ for a p-adic integer $s = k|v_1| - 1$ agrees with that of $1/(pk)$ and is nonnegative otherwise, so that the formula in Theorem 1.4.21 needs no modification. However, because the variables s and k in the above formula are linked, taking k to be a general p-adic integer necessitates that we also take s to be general p-adic integer as well.

Corollary 1.4.22 (Hopkins, see also Strickland [70]). *Interpolating the homotopy groups $\pi_* L_{\widehat{\mathbb{G}}_m} \mathbb{S}^0$ using the spectra $\mathbb{S}^{-|v_1|a_\infty}$ constructed in Lemma 1.4.18 agrees with the number theoretic p-adic interpolation of ζ.*

Proof. Generally, the cofiber sequence

$$\mathbb{S}^n \xrightarrow{p^j} \mathbb{S}^n \to M_n(p^j) \to \mathbb{S}^{n+1} \xrightarrow{p^j} \mathbb{S}^{n+1}$$

induces a short exact sequence

$$0 \leftarrow (\pi_n X)[p^j] \leftarrow [M_n(p^j), X] \leftarrow (\pi_{n+1} X)/p^j \leftarrow 0,$$

and the diagram

$$\begin{tikzcd}
\mathbb{S}^n \arrow[r, "p^j"] \arrow[d, "1"] & \mathbb{S}^n \arrow[r] \arrow[d, "p"] & M_n(p^j) \arrow[r] \arrow[d] & \mathbb{S}^{n+1} \arrow[r, "p^j"] \arrow[d, "1"] & \mathbb{S}^{n+1} \arrow[d, "p"] \\
\mathbb{S}^n \arrow[r, "p^{j+1}"] & \mathbb{S}^n \arrow[r] & M_n(p^{j+1}) \arrow[r] & \mathbb{S}^{n+1} \arrow[r, "p^{j+1}"] & \mathbb{S}^{n+1}
\end{tikzcd}$$

induces the following map of short exact sequences:

$$\begin{tikzcd}
0 & (\pi_n X)[p^j] \arrow[l] & {[M_n(p^j), X]} \arrow[l] & (\pi_{n+1} X)/p^j \arrow[l] & 0 \arrow[l] \\
0 & (\pi_n X)[p^{j+1}] \arrow[l] \arrow[u, "p"] & {[M_n(p^{j+1}), X]} \arrow[l] \arrow[u] & (\pi_{n+1} X)/(p^{j+1}) \arrow[l] \arrow[u, "\text{quotient}"] & 0. \arrow[l]
\end{tikzcd}$$

This map of short exact sequences interacts with the Adams v_1–self–map of Lemma 1.4.17 according to the rectangular prism in Figure 1.2. The result follows immediately from the construction of $\mathbb{S}^{-|v_1|a_\infty}$. □

Remark 1.4.23. We caution the reader that the behavior of the Picard–graded homotopy of the Γ–local sphere for $\mathrm{ht}(\Gamma) > 1$ is considerably more strangely (i.e., poorly) behaved than that of the $\widehat{\mathbb{G}}_m$–local sphere. Hovey and Strickland prove a partial "continuity" result [35, Proposition 14.6] but also provide details on the remaining bad behavior [35, Theorem 15.1]. The punchline of the bad news is as follows: take Γ to be of height at least 2, and define F to be the set

$$F := \{\lambda \in \mathrm{Pic}(\mathsf{Spectra}_\Gamma) : |\pi_\lambda L_\Gamma M^0(p)| < \infty\}.$$

Then there is a nonempty open U for which $U \cap F$ is Haar–negligible. Nonetheless, all but finitely many of the standard spheres belong to F — a curious situation.

Remark 1.4.24. Having set up some of the groundwork of chromatic homotopy theory, we pause to make a remark on the philosophy of the rest of this document. The other homotopical context in which Picard–graded homotopy groups have taken central relevance is equivariant homotopy theory, which concerns itself with spaces and spectra with a suitable notion of a "G–action", G a compact Lie group. The notion of "G–action" turns out to be somewhat complex, and the correct notion enjoys a notion of cellular approximation, where the cells are formed as follows: for a G–representation V we form the representation sphere S^V by compactifying V with a single point at ∞. Cellular approximation then states that any map of G–spaces can be G–equivariantly weakly replaced by a map of "G–CW–complex", which are suitably built from the spheres S^V as V ranges.

The analogous construction in chromatic homotopy theory has not appeared before. Although Picard–graded phenomena in the Γ–local category have been studied, "Picard–cellular" constructions have escaped attention. The primary goal of the remainder of the present work is to construct and study a certain Picard–cellular filtration of Γ–localized Eilenberg–Mac Lane spaces.

$$\begin{tikzcd}
& 0 & & (\pi_n X)[p^j] \arrow[ll] \arrow[dl] & & {[M_n(p^j),X]} \arrow[ll] \arrow[dl] & & (\pi_{n+1}X)/p^j \arrow[ll] \arrow[dl] \arrow[dlllll, bend right=10, "\alpha^p_{j-1/j-1}"'] & 0 \arrow[l] \\
0 & & (\pi_{n-|v_1|p^j}X)[p^j] \arrow[ll] & & {[M_{n-|v_1|p^j}(p^j),X]} \arrow[ll] & & (\pi_{n-|v_1|p^j+1}X)/p^j \arrow[ll] & & 0 \arrow[ll] \\
& 0 & & (\pi_n X)[p^{j+1}] \arrow[ll] \arrow[dl] \arrow[uu] & & {[M_n(p^{j+1}),X]} \arrow[ll] \arrow[dl] \arrow[uu] & & (\pi_{n+1}X)/p^{j+1} \arrow[ll] \arrow[dl] \arrow[uu] \arrow[dlllll, bend right=10, "\alpha_{j/j}"] & 0 \arrow[l] \\
0 & & (\pi_{n-|v_1|p^j}X)[p^{j+1}] \arrow[ll] \arrow[uu] & & {[M_{n-|v_1|p^j}(p^{j+1}),X]} \arrow[ll] \arrow[uu] & & (\pi_{n-|v_1|p^j+1}X)/p^{j+1} \arrow[ll] \arrow[uu] & & 0. \arrow[ll]
\end{tikzcd}$$

Figure 1.2: Interaction of Adams's v_1–self–maps with Moore spectra of different indices.

Chapter 2

Ravenel–Wilson and E–theory

Algebraic topologists restrict attention to their usual context of homotopy theory is that this is the setting in which it is possible to prove representability results. The most basic of these is Brown Representability:

Theorem (Brown). *If* $\widetilde{E}_* : h\mathsf{Spaces} \to \mathsf{GradedAbelianGroups}$ *is a functor which preserves infinite wedges and cofiber sequences, then there is a spectrum E satisfying the formula*

$$\widetilde{E}_* X \cong \pi_*(E \wedge X). \quad \square$$

With additional hypotheses, Brown's theorem can be enriched with some functoriality properties and can be made to give a similar formula in cohomology:

$$\widetilde{E}^* X \cong \pi_{-*} F(X, E).$$

The primary utility of such representability results are Yoneda–type arguments: extra algebraic structure appearing on the image of a homology functor induces structure maps on the representing object. In this way, one can investigate the algebraic structure by re-applying tools from algebraic topology to the representing objects.

Separately, homotopy theory also appears to be the correct context in which one can lift results from pure algebra to objects intrinsic to topology. A significant such phenomenon was discussed at length in Section 1.3, including our preferred homology theory E_Γ as constructed in Example 1.3.20 via Theorem 1.3.16. Through these two avenues, certain objects from algebraic topology and from algebraic geometry find a mutual context in homotopy theory, where one can then study their interactions.

It is in this context that Ravenel and Wilson [64] considered the value of Morava K–theories on Eilenberg–Mac Lane spaces.

- In Section 2.1, we recall the definition of a Hopf ring and give an example of the unstable cooperations on ordinary mod–2 homology. With the language established, we also state the result of Ravenel and Wilson and indicate how its proof is connected to the presented example.

- Given Corollary 1.4.6 and Lemma 1.4.9, it is then natural to ask for a description of the Morava E-theories of Eilenberg–Mac Lane spaces, a richer invariant. In Section 2.2, we lift Ravenel and Wilson's result from a computation in K_Γ-homology to a computation in continuous E_Γ-homology, together with all of the structure maps described in Section 2.1. In short, the E_Γ-homology of the Eilenberg–Mac Lane space $K(\mathbb{Z}, q+1)$ appears as the q^{th} exterior power of $E_\Gamma K(\mathbb{Z}, 2)$, considered as a Hopf algebra.
- In Section 2.3, we give a geometric perspective on the calculation presented in Section 2.2. Calling on existing results in Dieudonné theory, we determine that $K(\mathbb{Z}, q+1)_{E_\Gamma}$ is the q^{th} exterior power *as group schemes* of $K(\mathbb{Z}, 2)_{E_\Gamma}$, which was one of the central objects of interest in Section 1.3.

2.1 Recollections on Hopf rings

Before beginning, we recall the definition of a Hopf ring.

Definition 2.1.1. A Hopf ring $A_{*,*}$ over a graded ring R_* is itself a graded ring object in the category $\mathsf{Coalgebras}_{R_*}$, sometimes called an R_*-coalgebraic graded ring object. It has the following structure maps:

$$\begin{aligned}
+ \colon & A_{s,t} \times A_{s,t} \to A_{s,t} && (A_{s,t} \text{ is an abelian group}) \\
\cdot \colon & R_{s'} \otimes_{R_*} A_{s,t} \to A_{s+s',t} && (A_{*,t} \text{ is a } R_*\text{-module}) \\
\Delta \colon & A_{s,t} \to \bigoplus_{s'+s''=s} A_{s',t} \otimes_{R_*} A_{s'',t} && (A_{*,t} \text{ is a } R_*\text{-coalgebra}) \\
* \colon & A_{s,t} \otimes_{R_*} A_{s',t} \to A_{s+s',t} && (\text{addition for the ring in } R_*\text{-coalgebras}) \\
\eta_* \colon & R_* \to A_{*,0} && (\text{null element for ring addition}) \\
\chi \colon & A_{s,t} \to A_{s,t} && (\text{negation for the ring in } R_*\text{-coalgebras}) \\
\circ \colon & A_{s,t} \otimes_{R_*} A_{s',t'} \to A_{s+s',t+t'} && (\text{multiplication map for the ring in } R_*\text{-coalgebras}) \\
\eta_\circ \colon & R_* \to A_{*,0} && (\text{null element for ring multiplication}).
\end{aligned}$$

These are required to satisfy various commutative diagrams. The least obvious is displayed in Figure 2.1, encoding the distributivity of $\circ$–"multiplication" over $*$–"addition".

Hopf rings arise in algebraic topology in the theory of unstable cooperations, i.e., in the calculation of $E_* \underline{F}_*$ where E and F are spectra and $\underline{F}_t$ is the t^{th} space in the Ω-spectrum representing F. This space is determined by the property

$$F^t(X) \cong \mathsf{Spaces}(X, \underline{F}_t) \cong \mathsf{Spectra}(\Sigma^{-t+\infty} X, F).$$

If E and F are *ring* spectra which for all t and t' satisfy the Künneth formula

$$E_* \underline{F}_t \otimes_{E_*} E_* \underline{F}_{t'} \xrightarrow{\cong} E_*(\underline{F}_t \times \underline{F}_{t'}),$$

$$\begin{tikzcd}
A_{s,t} \otimes_{R_*} (A_{s',t'} \otimes_{R_*} A_{s'',t'}) \arrow[r, "1\otimes *"] \arrow[d, "\Delta\otimes(1\otimes 1)"] & A_{s,t} \otimes_{R_*} A_{s'+s'',t'} \arrow[dddd, "\circ"] \\
\left(\bigoplus_{s_1+s_2=s} A_{s_1,t} \otimes_{R_*} A_{s_2,t}\right) \otimes_{R_*} (A_{s',t'} \otimes_{R_*} A_{s'',t'}) \arrow[d, "\simeq"] & \\
\bigoplus_{s_1+s_2=s} \left(A_{s_1,t} \otimes_{R_*} A_{s_2,t} \otimes_{R_*} A_{s',t'} \otimes_{R_*} A_{s'',t'}\right) \arrow[d, "1\otimes\tau\otimes 1"] & \\
\bigoplus_{s_1+s_2=s} \left(A_{s_1,t} \otimes_{R_*} A_{s',t'} \otimes_{R_*} A_{s_2,t} \otimes_{R_*} A_{s'',t'}\right) \arrow[d, "\circ\otimes\circ"] & \\
\bigoplus_{s_1+s_2=s} \left(A_{s_1+s',t+t'} \otimes_{R_*} A_{s_2+s'',t+t'}\right) \arrow[r, "*"] & A_{s+s'+s'',t+t'},
\end{tikzcd}$$

Figure 2.1: The distributivity axiom for $*$ over $\circ$ in a Hopf algebra.

then

$$A_{s,t} = \bigoplus_t E_s \underline{F}_t$$

forms a ring object in the category of E_*–coalgebras. Taking E–homology of spaces with Künneth isomorphisms gives the first three maps in the definition, and the maps representing the ring structure on $\underline{F}_*$ give the remaining two.

Example 2.1.2. Because $H\mathbb{Z}/2$ has field coefficients, it always has Künneth isomorphisms, and in particular we can study the Hopf ring

$$A_{s,t} = (H\mathbb{Z}/2)_s \underline{H\mathbb{Z}/2}_t = H_s(K(\mathbb{Z}/2,t);\mathbb{F}_2)$$

for $E = H\mathbb{Z}/2$ and $F = H\mathbb{Z}/2$. We compute this object in three steps.

First, recall that there is a canonical isomorphism $H^*(\mathbb{R}\mathrm{P}^\infty;\mathbb{F}_2) \cong \mathbb{F}_2[x]$ for $|x| = 1$. In turn, the homology groups $H_*(\mathbb{R}\mathrm{P}^\infty;\mathbb{F}_2)$ can be written as

$$H_*(\mathbb{R}\mathrm{P}^\infty;\mathbb{F}_2) \cong \mathbb{F}_2\{a_0, a_1, a_2, \ldots, a_n, \ldots\}$$

for generators a_n of degree n with Kronecker pairing $\langle x^m, a_n\rangle = \delta^m{}_n$. For degree reasons, the diagonal on cohomology takes the form $\Delta x = x \otimes 1 + 1 \otimes x$ (cf. Example 1.3.6), from which it follows that the $*$–algebra structure on homology takes the form of a divided power algebra on a single class:

$$H_*(\mathbb{R}\mathrm{P}^\infty;\mathbb{F}_2) \cong \Gamma[a_\varnothing] \cong \bigotimes_{j=0}^{\infty} \mathbb{F}_2[a_{(j)}]/(a_{(j)})^2.$$

Mysteriously, we have labeled the generating class "$a_\varnothing$" and defined $a_\varnothing^{[2^j]} =: a_{(j)}$. This notation will align with complications later in the example.

Second, using the fact that $\mathbb{R}\mathrm{P}^\infty \simeq K(\mathbb{Z}/2,1) \simeq B\mathbb{Z}/2$ has a bar filtration, we study the associated bar spectral sequence, which has the signature

$$E^2_{*,*} = \mathrm{Tor}^{H_*(\mathbb{Z}/2;\mathbb{F}_2)}_{*,*}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow H_*(K(\mathbb{Z}/2,1);\mathbb{F}_2).$$

The homology of the discrete group $\mathbb{Z}/2$ can be expressed as an algebra as

$$H_*(\mathbb{Z}/2;\mathbb{F}_2) \cong \mathbb{F}_2[\underline{0},\underline{1}] \Big/ \left(\begin{array}{c} \underline{0} = 1 \\ \underline{1}^{*2} = \underline{0} \end{array} \right) \cong \mathbb{F}_2[\underline{1} - 1]/(\underline{1} - 1)^{*2} =: \mathbb{F}_2[a_\varnothing]/a_\varnothing^{*2},$$

where $\underline{n}$ denotes the degree–zero class representing the point n in $\mathbb{Z}/2$ and we have additionally defined $a_\varnothing := \underline{1} - 1$. The Tor groups of this truncated polynomial algebra form a divided power algebra [79]:

$$\mathrm{Tor}^{H_*(\mathbb{Z}/2;\mathbb{F}_2)}_{*,*} \cong \Gamma[a_\varnothing].$$

It follows that the bar spectral sequence collapses at $E^2 = E^\infty$, and there are no differentials.

Third, we use the second step as the base of an inductive argument, powered by Ravenel and Wilson's key lemma [64, Theorem 2.2], to analyze the other bar spectral sequences:

$$\mathrm{Tor}^{H_*(K(\mathbb{Z}/2,t);\mathbb{F}_2)}_{*,*}(\mathbb{F}_2;\mathbb{F}_2) \Rightarrow H_*(K(\mathbb{Z}/2,t+1);\mathbb{F}_2).$$

Inductively assume that the t^{th} level $A_{*,t}$ of the Hopf ring is an exterior $*$–algebra on classes which are t–fold $\circ$–products of the classes $a_{(j)}$. It follows that the Tor groups of the bar spectral sequence computing $A_{t+1,*}$ form a divided power algebra generated by the same t–fold $\circ$–products. An analogue of another Ravenel–Wilson lemma [64, Lemma 9.5] gives a congruence[1]

$$(a_{(j_1)} \circ \cdots \circ a_{(j_t)})^{[2^{j_{t+1}}]} \equiv a_{(j_1)} \circ \cdots \circ a_{(j_t)} \circ a_{(j_{t+1})} \quad \text{(mod decomposables)}.$$

It follows from the key lemma [64, Theorem 2.2], which lets us transport differentials from earlier bar spectral sequences to the current one by applying the $\circ$–product, that the differentials vanish:

$$\begin{aligned} d((a_{(j_1)} \circ \cdots \circ a_{(j_t)})^{[2^{j_{t+1}}]}) &\equiv d(a_{(j_1)} \circ \cdots \circ a_{(j_t)} \circ a_{(j_{t+1})}) \quad \text{(mod decomposables)} \\ &= a_{(j_1)} \circ d(a_{(j_2)} \circ \cdots \circ a_{(j_{t+1})}) = 0. \end{aligned}$$

Hence, the spectral sequence collapses and the induction holds. It follows that

$$A_{*,*} \xleftarrow{\cong} \bigoplus_{t=0}^{\infty} (H_*(\mathbb{R}\mathrm{P}^\infty;\mathbb{F}_2))^{\wedge t},$$

where $(-)^{\wedge t}$ denotes the t^{th} exterior power in the category of Hopf algebras. The leftward direction of this isomorphism is realized by the $\circ$–product.

---

[1] It's conceivable that this congruence can be repaired to an equality, since the 2–series for $\widehat{\mathbb{G}}_a$ is so abbreviated. I have not worked this out.

Definition 2.1.3. We spell out some of the Hopf algebra constructions named above. For a cocommutative R–coalgebra C, we define its free commutative and cocommutative Hopf R–algebra [77] to have underlying algebra

$$\frac{\operatorname{SymmetricAlgebra}(C \otimes_R (\chi C))}{\left(c \otimes \chi c = 1 \right)}$$

with diagonal

$$\Delta(c_1 \otimes \cdots \otimes c_k \otimes \chi c_1' \otimes \cdots \otimes \chi c_{k'}') = \Delta c_1 \otimes \cdots \otimes \Delta c_k \otimes \chi(\Delta c_1') \otimes \cdots \otimes \chi(\Delta c_{k'}')$$

and antipode

$$\chi(c_1 \otimes \cdots \otimes c_k \otimes \chi c_1' \otimes \cdots \otimes \chi c_{k'}') = \chi c_1 \otimes \cdots \otimes \chi c_k \otimes c_1' \otimes \cdots \otimes c_{k'}'.$$

Then, given a Hopf R–algebra A, we define the free Hopf ring [36, Definition 4.2, Proposition 2.16] to be

$$\bigoplus_{k=0}^{\infty} A^{\wedge_R k} \Bigg/ \left(x \wedge y = \sum_i (x_i' * y') \wedge (x_i'' * y'') \,\middle|\, \begin{array}{c} y = y' * y'', \\ \Delta x = \sum_i x_i' \otimes x_i'' \end{array} \right)$$

with $\circ$–product given by the natural maps $A^{\wedge_R n} \otimes_R A^{\wedge_R m} \to A^{\wedge_R (n+m)}$.

Remark 2.1.4. The odd-primary analogue of this result appears in Wilson's book [82, Theorem 8.5]. In that situation, the bar spectral sequences do not degenerate but rather have a single family of differentials, and the result imposes a single relation on the free Hopf ring.

This example is enlightening because it captures most of the features of the other known calculations along these lines. These Hopf algebras $A_{*,t}$ are accessed by iterating a bar spectral sequence and transporting differentials from one to the next. In the end, the Hopf ring as a whole is freely generated by its first level. The real example we will use is Ravenel and Wilson's calculation of the Hopf ring for the choice $E = K(d)$ and $F = H\mathbb{Z}/p^j$ [64]. This is more complicated than the one presented above in two ways:

1. The base-case bar spectral sequence does have a family of nonvanishing differentials, because $K(d)_* \underline{H\mathbb{Z}/p^j}_1$ is known to be finitely generated as a module by Remark 1.3.25.
2. It is important to understand the interrelations between the subexamples in this family as j varies, i.e., to understand the effect of the maps $H\mathbb{Z}/p^j \to H\mathbb{Z}/p^{j+1}$.

With this in mind, they prove the following result:

Theorem 2.1.5 (Ravenel–Wilson [64]). *Take the ambient prime p to satisfy $p \geq 3$. There is a Hopf ring isomorphism*

$$\bigoplus_{t=0}^{\infty} K(d)_* K(\mathbb{Z}/p^j, t) \cong \bigoplus_{t=0}^{\infty} (K(d)_* K(\mathbb{Z}/p^j, 1))^{\wedge t}.$$

Moreover, as j tends to ∞, for fixed $t \geq 1$ the system $\{K(\mathbb{Z}/p^j, t)_{K(d)}\}_j$ has the structure of a connected p-divisible group of dimension $\binom{d-1}{t-1}$. Algebraically, this is equivalent data to a formal group of the same dimension by Remark 1.3.24). Topologically, that formal group is given by $K(\mathbb{Z}_p, t+1)_{K(d)}$.

Corollary 2.1.6. *Because $K(d)_*B\mathbb{Z}/p^j$ is generated by d elements as a Hopf algebra, it follows that $A_{*,t}$ vanishes for $t > d$. It follows that stably $K(d)_*H\mathbb{Z}/p^j = 0$.* □

Remark 2.1.7. At $t = 0$, the system $\{\mathbb{Z}/p^j_{K(d)}\}_j$ has the structure of the constant p-divisible group:

$$\{\mathbb{Z}/p^j_{K(d)}\}_j \cong \mathbb{Q}_p/\mathbb{Z}_p.$$

However,

$$K(d)^*K(\mathbb{Z}_p, 1)_{K(d)} \cong K(d)^*S^1 \cong K(d)^*[\varepsilon]/\varepsilon^2$$

does not have the corresponding structure of the ring of functions on a p-divisible formal group.

2.2 The calculation in E-theory

Our goal in this section is to compute the E-theory Hopf ring for Eilenberg–Mac Lane spaces. Corollary 1.4.6 describes the module structure of $E_\Gamma\underline{H\mathbb{Z}/p^j}_*$, and we are left to track the rest of the Hopf ring structure on our own. We will follow the proof of Corollary 1.4.6 and begin by considering the short exact sequence of coefficients:

$$0 \to \bigoplus_{\substack{t_1,\ldots,t_n \geq 0 \\ t_+ = r}} K_* \to E_*/\mathfrak{m}^{r+1} \to E_*/\mathfrak{m}^r \to 0.$$

Because we know that $E_*\underline{H\mathbb{Z}/p^j}_t$ is a pro-free (and thus flat [35, Theorem A.9]) E_*-module deforming the original $K_*\underline{H\mathbb{Z}/p^j}_t$ by Corollary 1.4.6, we tensor $E_*\underline{H\mathbb{Z}/p^j}_t$ against the above short exact sequence to get a new short exact sequence:

$$0 \to \bigoplus_{\substack{t_1,\ldots,t_n \geq 0 \\ t_+ = r}} K_*\underline{H\mathbb{Z}/p^j}_t \to E_*\underline{H\mathbb{Z}/p^j}_t \otimes_{E_*} E_*/\mathfrak{m}^{r+1} \to E_*\underline{H\mathbb{Z}/p^j}_t \otimes_{E_*} E_*/\mathfrak{m}^r \to 0.$$

We can also build the free alternating Hopf ring $(E_*\underline{H\mathbb{Z}/p^j}_1)^{\wedge *}$; tensoring the above short exact sequence of coefficients with any graded piece of this ring gives the exact sequence (which is not, a priori, left exact)

$$\bigoplus_{\substack{t_1,\ldots,t_n \geq 0 \\ t_+ = r}} (K_*\underline{H\mathbb{Z}/p^j}_1)^{\wedge t} \to (E_*\underline{H\mathbb{Z}/p^j}_1 \otimes_{E_*} E_*/\mathfrak{m}^{r+1})^{\wedge t} \to (E_*\underline{H\mathbb{Z}/p^j}_1 \otimes_{E_*} E_*/\mathfrak{m}^r)^{\wedge t} \to 0.$$

Then, the cup product map $(\underline{H\mathbb{Z}/p^j}_1)^{\wedge t} \to \underline{H\mathbb{Z}/p^j}_t$ induces a map on homology

$$(E_*\underline{H\mathbb{Z}/p^j}_1)^{\otimes t} \to (E_*\underline{H\mathbb{Z}/p^j}_1)^{\wedge t} \xrightarrow{\circ} E_*\underline{H\mathbb{Z}/p^j}_t.$$

Bifunctoriality of the tensor product induces a map between these rows by $\circ$-product:

$$\begin{tikzcd}
& \bigoplus_{\substack{t_1,\ldots,t_n\geq 0\\ t_+=r}}(K_*\underline{H\mathbb{Z}/p^j}_1)^{\wedge t} \arrow[r] \arrow[d] & (E_*\underline{H\mathbb{Z}/p^j}_1/\mathfrak{m}^{r+1})^{\wedge t} \arrow[r] \arrow[d] & (E_*\underline{H\mathbb{Z}/p^j}_1/\mathfrak{m}^{r})^{\wedge t} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bigoplus_{\substack{t_1,\ldots,t_n\geq 0\\ t_+=r}} K_*\underline{H\mathbb{Z}/p^j}_t \arrow[r] & E_*\underline{H\mathbb{Z}/p^j}_t/\mathfrak{m}^{r+1} \arrow[r] & E_*\underline{H\mathbb{Z}/p^j}_t/\mathfrak{m}^{r} \arrow[r] & 0.
\end{tikzcd}$$

Theorem 2.2.1. *There is an isomorphism of Hopf rings*

$$\bigoplus_{t=0}^{\infty}(E_*\underline{H\mathbb{Z}/p^j}_1)^{\wedge t} \cong \bigoplus_{t=0}^{\infty} E_*\underline{H\mathbb{Z}/p^j}_t.$$

Proof. We perform an induction on r. In the base case of $r=1$, Theorem 2.1.5 states that

$$(E_*\underline{H\mathbb{Z}/p^j}_1/\mathfrak{m}^1)^{\wedge t} = (K_*\underline{H\mathbb{Z}/p^j}_1)^{\wedge t} \xrightarrow{\cong} K_*\underline{H\mathbb{Z}/p^j}_t = E_*\underline{H\mathbb{Z}/p^j}_t/\mathfrak{m}^1$$

is an isomorphism, where the outer two equalities follow from Corollary 1.4.6. This also tells us that the left-hand vertical map in the above diagram is always an isomorphism. In particular, this map is injective, as is the first nontrivial horizontal map on the second row, so their composite is injective. It follows that the first horizontal map on the first row

$$\bigoplus_{\substack{t_1,\ldots,t_n\geq 0\\ t_+=r}} (K_*\underline{H\mathbb{Z}/p^j}_1)^{\wedge t} \to (E_*\underline{H\mathbb{Z}/p^j}_1/\mathfrak{m}^{r+1})^{\wedge t}$$

is also injective, and thus that the top sequence is short exact.

Then, assume that $\circ$-multiplication induces an isomorphism modulo $\mathfrak{m}^r$ for some fixed r, i.e., that the right-hand vertical map in the above diagram is an isomorphism of modules. As the left-hand and right-hand vertical maps are isomorphisms, the center map must be as well. As t varies, the center maps additionally assemble into a map of graded Hopf rings, and so furthermore induce an isomorphism of such. Induction provides isomorphisms for all r, and the Milnor sequence of Lemma 1.4.5 finishes the argument:

$$\begin{aligned}
E_*\underline{H\mathbb{Z}/p^j}_t &= \lim_r E_*\underline{H\mathbb{Z}/p^j}_t/\mathfrak{m}^r \\
&= \lim_r (E_*\underline{H\mathbb{Z}/p^j}_1/\mathfrak{m}^r)^{\wedge t} \\
&= (E_*\underline{H\mathbb{Z}/p^j}_1)^{\wedge t}.
\end{aligned}$$

□

2.3 Connections to Dieudonné theory

Ravenel and Wilson originally performed their calculation in the language of Hopf algebras, as K-homology is naturally valued in these, but it is most easily understood through the lens of Dieudonné theory. In particular, analyzing the features of the exterior power of a Hopf algebra is a somewhat intricate task. However, Hopf algebras over a field form an abelian category with a set of projective generators, implying that they are categorically equivalent to modules over some ring. Dieudonné theory describes this ring and the functors inducing the equivalence.

Theorem 2.3.1 ([17, Sections III.8-9]). *Let k be a perfect field of characteristic p, and define the Cartier ring as follows:*

$$\mathrm{Cart}(k) := \mathbb{W}(k)\langle F, V\rangle \Big/\!\!\Big/ \left(\begin{array}{c} Fw = w^{\varphi} F, \\ V w^{\varphi} = w V, \\ FV = p = VF \end{array} \right).$$

(Here φ is a lift of the Frobenius to $\mathbb{W}(k)$, as in Theorem 1.2.26.) Then there is an equivalence of categories from formal groups over k of finite height and arbitrary dimension to modules M over Cart_k which satisfy the following properties:

1. *M is finite and free as a $\mathbb{W}(k)$-module.*
2. *The associated-graded with respect to V is "uniform": $V^j : M/VM \xrightarrow{\simeq} V^j M/V^{j+1}M$.*
3. *M is complete with respect to V: $M \cong \lim_j M/V^j M$.*

In fact, if the final condition is dropped, the equivalence enlarges to one between p-divisible groups (as in Remark 1.3.24) over k and such Cart_k-modules.

Corollary 2.3.2 ([17, Sections III.8-9]). *The invariants defined in Definition 1.2.1 and Definition 1.2.20 can be read off from a Dieudonné module M as follows:*

- *There is a natural isomorphism of M/VM with the tangent space of the formal group associated to M by the above equivalence. In particular, $\dim_k(M/VM)$ gives the dimension of the formal variety.*
- *When the associated formal group is 1-dimensional, the rank $\dim_{\mathbb{W}(k)} M$ of the Dieudonné module encodes the height of the associated formal group.*

Remark 2.3.3 ([17, Chapter III]). For the uninitiated, we include a very brief sketch for how Dieudonné modules are built for formal groups over a perfect field of positive characteristic. To a 1-dimensional formal group $\widehat{\mathbb{G}}$ one associates the collection of curves

$$C\widehat{\mathbb{G}} := \mathsf{FormalSchemes}(\widehat{\mathbb{A}}^1, \widehat{\mathbb{G}}).$$

This functor supports Verschiebung operators V_n and Frobenius operators F_n defined by the following two formulas:

$$V_n\left(\sum_{j=1}^{\infty} a_j x^j\right) := \sum_{j=1}^{\infty} a_j x^{jn},$$
$$F_n\left(\sum_{j=1}^{\infty} a_j x^j\right) := \sum_{i=0}^{n-1}{}^{\widehat{\mathbb{G}}}\left(\left(\sum_{j=1}^{\infty} a_j \zeta_n^j x^{j/n}\right)\right),$$

where ζ_n is a primitive nth root of unity and $\sum^{\widehat{\mathbb{G}}}$ denotes iterated summation in the formal group. (Despite the roots of unity and fractional powers of x, F_n can be shown to actually take values in curves on $\widehat{\mathbb{G}}$.) The subset $D\widehat{\mathbb{G}}$ of curves $\gamma(x)$ satisfying $F_n\gamma = 0$ whenever n is not a power of p are called the "p-typical" curves, and $D\widehat{\mathbb{G}}$ naturally forms a module for the Cartier ring with V acting by V_p and F by F_p. (Indeed, these V and F operators are precisely the ones used by Ravenel and Wilson at the level of Hopf algebras in the course of proving Theorem 2.1.5 [64, Section 7].)

Remark 2.3.4. Because the words involved are so similar, we should pause to note that the p-typicality of curves discussed in Remark 2.3.3 is not wholly identical to the use of "p-typical" in Lemma 1.2.16. Namely, the logarithm in Lemma 1.2.16 is thought of as a map

$$\widehat{\mathbb{G}} \overset{F}{\cong} \widehat{\mathbb{A}}^1 \xrightarrow{\log_F} \widehat{\mathbb{A}}^1 \cong \widehat{\mathbb{G}}_a,$$

i.e., as a curve on $\widehat{\mathbb{G}}_a$ rather than on $\widehat{\mathbb{G}}$. The vanishing Frobenius condition on $\widehat{\mathbb{G}}_a$ is what enforces the p-power-degree concentration of p-typical logarithms.

Remark 2.3.5. If M is a Dieudonné module (without assuming completeness), then one can construct a dual module $\mathbb{D}M$ whose underlying $\mathbb{W}(k)$-module is $M^\vee = \mathsf{Modules}_{\mathbb{W}(k)}(M, \mathbb{W}(k))$ and whose structure maps $V_{\mathbb{D}M}$ and $F_{\mathbb{D}M}$ are the $\mathbb{W}(k)$-linear duals of the original structure maps F_M and D_M. This duality reflects a duality on the level of p-divisible groups: the k^{th} stage $G[p^j]$ of a p-divisible group G is a finite flat group scheme, corresponding to a commutative and cocommutative Hopf algebra $\mathscr{O}_{G[p^j]}$. The k-linear dual $(\mathscr{O}_{G[p^j]})^\vee$ is also a commutative and cocommutative Hopf algebra, called the Cartier dual (or Serre dual) of $\mathscr{O}_{G[p^j]}$, and collectively these form the Cartier dual p-divisible group $(\mathbb{D}G)[p^j] = \mathbb{D}(G[p^j])$. In geometric language, this duality appears as

$$\mathbb{D}(G[p^j]) = \underline{\mathrm{Hom}}(G[p^j], \mathbb{G}_m),$$

where $\underline{\mathrm{Hom}}$ denotes the group scheme of homomorphisms (or "internal Hom object"). These dualities of group schemes and their Dieudonné modules are compatible in that

$$D(\mathbb{D}G) = \mathbb{D}(DG).$$

Remark 2.3.6 ([40, Section V]). There is also a contravariant notion of Dieudonné theory, defined by

$$D^\vee\widehat{\mathbb{G}} := PH^1_{\mathrm{dR}}(\widehat{\mathbb{G}}/\mathbb{W}(k)),$$

where P denotes the primitive elements and "$\widehat{\mathbb{G}}/\mathbb{W}(k)$" denotes any lift of $\widehat{\mathbb{G}}$ to $\mathbb{W}(k)$.[2] As suggested by the notation, one can prove that D and $D^\vee$ are suitably dual to one another:

$$D^\vee G \cong D(\mathbb{D}G).$$

---

[2] The crystalline nature of algebraic de Rham cohomology means that $D^\vee$ is insensitive to which lift is chosen (up to canonical isomorphism).

Example 2.3.7. The Dieudonné module associated to the height d Honda formal group $\widehat{\mathbb{G}}$ of Theorem 1.2.23 can be calculated as follows: by Corollary 2.3.2, the height of $\widehat{\mathbb{G}}$ forces $D\widehat{\mathbb{G}}$ to be a free $\mathbb{W}(k)$-module on the basis $\{\gamma, V\gamma, \ldots, V^{d-1}\gamma\}$, where γ is the p-typical coordinate with p-series x^{p^d}. It follows from the p-series calculation that $px = V^d\gamma$, and hence the action of F is

$$\begin{aligned} p\gamma &= V^d\gamma \\ VF\gamma &= VV^{d-1}\gamma \\ F\gamma &= V^{d-1}\gamma. \end{aligned}$$

Finally, we have $D\widehat{\mathbb{G}} = \mathrm{Cart}(k)/(V^{d-1} = F)$. Since $D\widehat{\mathbb{G}}$ is module–isomorphic to a quotient ring of $\mathrm{Cart}(k)$, we can easily calculate its $\mathrm{Cart}(k)$-linear endomorphisms by examining where the unit element is sent (cf. Theorem 1.2.26):

$$\mathrm{End}(D\widehat{\mathbb{G}}) \cong \mathrm{Cart}(k)/(V^{d-1} = F) \cong \mathbb{W}(k)\langle V\rangle \Big/ \left(\begin{array}{c} Vw^{\varphi} = wV, \\ V^d = p \end{array} \right).$$

In any event, the equivalence of categories in Theorem 2.3.1 suggests that the theory of "multilinear algebra" we seek for finite rank Hopf algebras (or, equivalently, for finite group schemes) can be compared well with the actual multilinear algebra of vector spaces. Theorems of various authors make this precise:

Theorem 2.3.8 ([23, Section 7], [25], [30, Construction 3.2.11], [3, Section 2], [36, Proposition 2.6]). *Let A be the Hopf algebra associated to a 1-dimensional p-divisible formal group over a perfect field k, and let M be its associated Dieudonné module. Then, the Dieudonné module associated to the t^{th} exterior power of A has as its underlying $\mathbb{W}(k)$-module an exterior power of the underlying $\mathbb{W}(k)$-module of M. The operators F and V extend to $M^{\wedge j}$ by the formulas*

$$\begin{aligned} V(v_1 \wedge \cdots \wedge v_t) &:= V(v_1) \wedge \cdots \wedge V(v_t), \\ F(V(v_1) \wedge v_2 \wedge \cdots \wedge v_t) &:= v_1 \wedge F(v_2 \wedge \cdots \wedge v_t). \quad \square \end{aligned}$$

Remark 2.3.9. We note that this existence theorem only applies to the Dieudonné modules of formal dimension 1, and it is prohibited in higher dimensions by a slope argument. See Hedayatzadeh [25, Section 0] for a few more details.

This strong comparison between Hopf algebras, finite group schemes, and Dieudonné modules is special to the case of a ground field. Over a more general ring, this comparison is more complicated (and, as we've seen in Definition 1.1.18, already imperfect for the comparison between Hopf algebras and group schemes). Dieudonné modules were generalized by Zink's theory of Dieudonné displays [84, 85] which model p-divisible groups over very general rings. We find ourselves in the situation of the finite stages in the Lubin–Tate deformation space (cf. Theorem 1.2.24), where Zink's theory applies, and Hedayatzadeh works through the multilinear algebra and algebraic geometry necessary to prove the following theorem:

Theorem 2.3.10. *[24, Proposition 3.12] Let R be a complete local Noetherian ring with residue field k of positive characteristic p and G a p-divisible group over R with connected special fiber of dimension* 1 *and height n. Then the t^{th} exterior powers of G and of the subgroups $G[p^j]$ exist, and they are mutually compatible*

$$(G^{\wedge t})[p^j] = (G[p^j])^{\wedge t}$$

and further compatible with Dieudonné theory

$$D(G[p^j]^{\wedge t}) = (D\widehat{\mathbb{G}}[p^j])^{\wedge t}.$$

Moreover, $G^{\wedge t}$ is smooth of dimension $\binom{n-1}{t-1}$ at the special fiber.[3] □

Corollary 2.3.11. *There is a factorization*

$$\begin{tikzcd}
(\mathbb{C}\mathrm{P}^{\infty}_{E})^{\times t} \arrow[rr, "\circ"] \arrow[dr] & & (\underline{H\mathbb{Z}/p^{\infty}}_{t})_E \\
& (\mathbb{C}\mathrm{P}^{\infty}_{E})^{\wedge t}. \arrow[ur, dotted, "\cong"] &
\end{tikzcd}$$

Proof. The existence of the factorization follows from Theorem 2.3.10, and that it is an isomorphism follows from Lemma 1.1.16. □

We also note that our methods also extend the results of Buchstaber and Lazarev to the E-theoretic setting.

Corollary 2.3.12. *The Dieudonné module associated to $(\underline{H\mathbb{Z}/p^{\infty}}_{d})_K$ is* 1*-dimensional over $\mathbb{W}(k)$, generated by the element*

$$\gamma_* = \gamma \wedge V\gamma \wedge \cdots \wedge V^{d-1}\gamma$$

with action $V\gamma_* = (-1)^{d-1}p\gamma_*$.

Proof. This is a direct calculation from Example 2.3.7 and Theorem 2.3.8, together with the observation that

$$V\gamma_* = V(\gamma \wedge V\gamma \wedge \cdots \wedge V^{d-2}\gamma \wedge V^{d-1}\gamma) = V\gamma \wedge V^2\gamma \wedge \cdots \wedge V^{d-1}\gamma \wedge p\gamma$$

requires $(d-1)$ transpositions to return to the form of γ_*. □

Theorem 2.3.13 (Theorem 2.1.5). *The ∘-product gives a perfect pairing*

$$(\underline{H\mathbb{Z}/p^{\infty}}_{t})_K \times (\underline{H\mathbb{Z}/p^{\infty}}_{d-t})_K \to (\underline{H\mathbb{Z}/p^{\infty}}_{d})_K. \quad \square$$

---

[3] We remark that Hedayatzadeh's results are ultra-recent, contemporary with the present work.

Corollary 2.3.14 ([14, Theorem 12.7.4]). *When d is odd, Corollary 2.3.12 shows*

$$\mathbb{D}(\underline{H\mathbb{Z}/p^{\infty}}_{t})_K \cong (\underline{H\mathbb{Z}/p^{\infty}}_{d-t})_K,$$

where "$\mathbb{D}$" is used in the sense of Remark 2.3.5. In general, $(\underline{H\mathbb{Z}/p^{\infty}}_{t})_K$ is the "twisted dual" of $(\underline{H\mathbb{Z}/p^{\infty}}_{d-t})_K$. ☐

Corollary 2.3.15 (Theorem 2.2.1). *As it suffices to check nondegeneracy on the special fiber, the same is true for $(\underline{H\mathbb{Z}/p^{\infty}}_{t})_E$.* ☐

Remark 2.3.16. Hopkins and Lurie [30, Sections 3.3-4] have accomplished a direct construction of the alternating powers in our specific situation. They apply this to give a coordinate–free calculation of the Morava E–theory of Eilenberg–Mac Lane spaces [30, Section 3.5], though as we will need to understand its action of the stabilizer group in Chapter 3 it's to our advantage to have selected a coordinate. Their approach differs from our own in two other important ways:

- Ravenel and Wilson perform their K–theoretic calculation away from 2, which obstructs our induction. Hopkins and Lurie go to some length to correct for this; the construction of alternating objects has extra snares at that prime. The Hopf algebra version of this assertion has appeared previously in work of Johnson–Wilson [39, Appendix].
- Their basic induction takes place at the $j = \infty$ stage, where the bar spectral sequence differentials all vanish. To counterbalance this slickness, it takes substantial work to recover from that calculation the finite subschemes visible at stages with finite j and check that they agree with the data coming from algebraic topology.

Remark 2.3.17. Dieudonné modules were also generalized by the the Grothendieck–Messing theory of Dieudonné crystals [51] in the situation of a nilpotent thickening of a field. This framework allows one to analyze the behavior of the Lubin–Tate formal group over the divided power envelope of (or, more delicately, a rigid analytification of) Lubin–Tate space; see work of Hopkins–Gross [27] and of Devinatz–Hopkins [18] for more details relevant to the E–theorist.

Remark 2.3.18. The cobar (or Eilenberg–Moore) spectral sequence for this situation has been investigated by Tilman Bauer [9] and others. For F the homotopy pullback of a span of spaces $E_1 \to B \leftarrow E_2$, one might hope for a spectral sequence of the form

$$E^2_{*,*} \cong \mathrm{Cotor}^{K_*B}_{*,*}(K_*E_1, K_*E_2) \Rightarrow K_*F.$$

However, it is clear from the calculations of this chapter that such a spectral sequence cannot have good convergence properties in general. The path-loops fibration

$$\begin{tikzcd}
\Omega\underline{\mathbb{Z}/p^j}_{n+1} \arrow[r] \arrow[d, equal] & P\underline{H\mathbb{Z}/p^j}_{n+1} \arrow[r] \arrow[d, equal] & \underline{H\mathbb{Z}/p^j}_{n+1} \arrow[d, equal] \\
\underline{H\mathbb{Z}/p^j}_{n} \arrow[r] & * \arrow[r] & \underline{H\mathbb{Z}/p^j}_{n+1}
\end{tikzcd}$$

would beget a cobar spectral sequence

$$\mathrm{Cotor}_{*,*}^{K_*\underline{H\mathbb{Z}/p^j}_{n+1}}(K_*, K_*) \Rightarrow K_*\underline{H\mathbb{Z}/p^j}_n.$$

However, $\underline{H\mathbb{Z}/p^j}_{n+1}$ is K-acyclic, while $\underline{H\mathbb{Z}/p^j}_n$ is not. Bauer's main result is that this is the only thing that can go wrong: if π_*B has finite torsion exponent in each degree and $\pi_{n+1}B = 0$, then a reasonable such spectral sequence does exist [9, Theorem 1.1].

Chapter 3

The cotangent space construction

In this chapter is we seek out a source of spectra X with $\mathscr{E}(X)$ a line bundle, so that we can leverage Theorem 1.4.12 to study the Picard groups of the Γ–local categories. One source of line bundles, motivated by the previous chapter, is the cotangent space of a smooth point on a curve over $\operatorname{Def}(\Gamma)$. In light of the examples in Section 1.3, there are spectra like $\mathbb{C}\mathrm{P}^\infty$ for which $\mathbb{C}\mathrm{P}^\infty_E$ is a pointed formal line. It follows that $T_0^* \mathbb{C}\mathrm{P}^\infty_E$ is a line bundle over $\operatorname{Def}(\Gamma)$. These ideas assemble into the following diagram:

$$\begin{tikzcd}
\{\text{appropriate spectra}\} \arrow[r, dashed, "T_\eta^*"] \arrow[d, "(-)_E"] & \mathsf{Pic}(\mathsf{Spectra}_\Gamma) \arrow[d, "\mathscr{E}_\Gamma"] \\
\left\{ \begin{array}{c} \text{formal affine varieties} \\ \text{of dimension 1} \end{array} \right\} \arrow[r, "T_0^*"] & \mathsf{Lines}(\operatorname{Def}(\Gamma)).
\end{tikzcd}$$

The primary goal of this chapter is to decide on an appropriate category for the top-left node and to construct the dashed map T_η making the square commute, i.e., a spectrum-level version of the cotangent space construction.

It will turn out that this construction is part of a more general pattern, which builds spectral analogues of the cotangent space for spectra which "look like" formal varieties of any dimension. The same methods also apply to study j–jets rather than merely 1–jets (cf. Remark 1.1.23), and the resulting decomposition into infinitesimals we call the annular tower. The annular tower for projective space recovers the cellular decomposition, but yields new information for more exotic choices of input — namely, drawing from Chapter 2, the spaces $\underline{H\mathbb{Z}/p^\infty}_q$. In Section 3.3 we find that the spectral cotangent space of $L_\Gamma \Sigma^\infty_+ \underline{H\mathbb{Z}/p^\infty}_d$ is a (choice-free) model of the determinantal sphere, and the annular tower shows that the Γ–localization of this Eilenberg-Mac Lane space admits an interesting decomposition into "generalized cells," as viewed through the Γ–local Picard group.

- In Section 3.1, we give an account of a "homotopical" theory of coalgebras, abbreviated to the parts relevant for this document. We focus especially on coalgebra objects in the category of Γ–local spectra. This will provide us the basic constructions needed to assemble the functor "T_η^*" above, and we will use their associated computational tools through the remainder of the chapter.

- In Section 3.2, we specialize to the case that the coalgebra is presented as $C = \Sigma_+^\infty X$, where X is a space such that X_{K_Γ} is a formal variety. This situation has exceptionally good properties, and we can use the computational tools in Section 3.1 to check that T_η^* is well-behaved enough to have the desired interchange law on these spectra.
- In Section 3.3, we apply T_η^* to two classical examples, the real and complex projective spaces, and one nonclassical example, $K(\mathbb{Z}, n+1)$. This last example results in a choice–free model of the element of $\mathrm{Pic}(\mathsf{Spectra}_\Gamma)$ called the "determinantal sphere".
- In Section 3.4, we reinterpret the definition of T_η^* in terms of Koszul duality. This allows us to show that $G(C) = \Sigma^{-1} T_\eta^* C$ carries an A_∞–multiplication for each of the example coalgebras C above, and C itself is presented by the bar construction on $G(C)$. This exhibits the multiplication map on $G(C)$ as a nonclassical analogue of the Hopf invariant one elements.
- In Section 3.5, we relate our methods to those of Westerland [81]. We show that our methods are powerful enough to internally construct a spectrum $K^{\det}$ which is weakly equivalent to his spectrum "R_d".

3.1 Cotensors of coassociative spectra

In this section we define a cotensor product of comodules over a fixed coalgebra in a homotopical context. Before arriving at the right definition, it is instructive to review the classical theory and make a naive calculation.

To begin, consider a k–algebra A, a right–module M with action map α_M, and a left–module N with action map α_N. Taking as given the tensor product of modules over k, the tensor product of modules over A is defined by the following exact sequence:

$$0 \leftarrow M \otimes_k N \to M \otimes_A N \xleftarrow{\alpha_M \otimes 1 - 1 \otimes \alpha_N} M \otimes_k A \otimes_k N$$

or equivalently by the following coequalizer

$$\begin{tikzcd} M \otimes_A N & M \otimes_k N \arrow[l] & M \otimes_k A \otimes_k N. \arrow[l, shift left, "\alpha_M \otimes 1"'] \arrow[l, shift right, "1 \otimes \alpha_N"] \end{tikzcd}$$

Dually, one can consider a k–coalgebra C, a right–comodule M with coaction map ψ_M, and a left–comodule N with coaction map ψ_N. The dual definition for the cotensor product is then given by the exact sequence

$$0 \to M \square_C N \to M \otimes_k N \xrightarrow{\psi_M \otimes 1 - 1 \otimes \psi_N} M \otimes_k C \otimes_k N$$

or the equalizer

$$\begin{tikzcd} M\square_C N \arrow[r] & M \otimes_k N \arrow[r, shift left, "\psi_M \otimes 1"] \arrow[r, shift right, "1 \otimes \psi_N"'] & M \otimes_k C \otimes_k N. \end{tikzcd}$$

Example 3.1.1. As in Example 1.1.14, consider once more the k–coalgebra $C = k\{\beta_0, \beta_1, \beta_2, \ldots\}$ with comultiplication given by $\Delta\beta_n = \sum_{i=0}^{n} \beta_i \otimes \beta_{n-i}$, dual to the power series k–algebra in a single variable selected by β_1. We form a right–C–comodule M by projecting away from β_0, which then has coaction map $\psi \colon M \to M \otimes_k C$ specified by

$$\psi\beta_n = \sum_{i=1}^{n} \beta_i \otimes \beta_{i-n}.$$

(Since C is cocommutative, M can also be interpreted as a left–C–comodule.) The map in the kernel sequence is specified by the formula

$$\beta_n \otimes \beta_m \mapsto \sum_{i=1}^{n} \beta_i \otimes \beta_{n-i} \otimes \beta_m - \sum_{j=1}^{m} \beta_n \otimes \beta_{m-j} \otimes \beta_j.$$

This formula gives rise to a graded map using the grading $|\beta_n| = 2n$, and so when studying the kernel it suffices to consider one graded component at a time. Routine calculation shows that the kernel in degree n is spanned by $\sum_{j=1}^{n-1} \beta_j \otimes \beta_{n-j}$.

Let us proceed naively in pursuit of our spectral cotangent construction T^*_η and suppose that we have spectra C, M, and N equipped with maps satisfying the coalgebra, right–comodule, and left–comodule diagrams respectively in the homotopy category. Given this data, we can define the naive cotensor product by lifting the above definition:

$$M \overset{\text{naive}}{\square}{}_C N \xrightarrow{\text{fiber}} M \wedge N \xrightarrow{\psi_M \wedge 1 - 1 \wedge \psi_N} M \wedge C \wedge N.$$

However, this definition is poorly behaved. Even when K is a Morava K–theory and M and N are chosen as nicely as can be, we may still arrive at the following situation:

$$K_*\left(M \overset{\text{naive}}{\square}{}_C N\right) \not\cong (K_*M)\square_{K_*C}(K_*N). \tag{3.1.2}$$

Example 3.1.3. Let's continue the previous example by supposing that C, M, and N are designed so that K_*C is the power series coalgebra (over the graded field K_*) and that K_*M and K_*N realize the coideal of series vanishing to zeroth order. Writing T for the naive cotensor product and applying K_* to its definition, we find an exact sequence

$$0 \to K_0T \to K_0M \otimes K_0N \xrightarrow{\psi_M \otimes 1 - 1 \otimes \psi_N} K_0M \otimes K_0C \otimes K_0N \to K_{-1}T \to 0.$$

It is thus clear that K_0T recovers the expected value of $K_0M \square_{K_0C} K_0N$, but $K_{-1}T$ is visibly nontrivial, since $\beta_1 \otimes \beta_0 \otimes \beta_1$ is not in the image of the named coaction map and hence survives to the cokernel. In order to correct Equation (3.1.2), we must modify this definition so as to remove this odd-degree class.

In the remainder of this section we adopt the language of ∞–categories (specifically, we take quasicategories as a model), as this negates the headache of keeping track of fibrancy conditions at the cost of some additional language. The reader hesitant to embrace ∞–categories should be able to rework the arguments in terms of a model category of spectra for which Σ^∞_+ is a strict oplax symmetric monoidal functor (e.g., orthogonal spectra [65, Section III.1]). A sufficiently enterprising such reader should be aware that the lion's share of the extant work on cotensor products in model categories of comodule spectra is due to Hess and Shipley [26].

We begin by motivating our definition of a coassociative coalgebra in spectra. The classical algebraic structure of a monoid appears because, given an object X in a category C, the subcategory of endomorphisms $\mathrm{End}_{\mathsf{C}}(X)$ naturally forms a monoid under composition of arrows. Moreover, given an abstract monoid M, an M–action on X (or the structure of an M–representation on X) is specified by a monoid map $M \to \mathrm{End}_{\mathsf{C}}(X)$. Given a monoidal category C, one can associate a multicategory $\mathsf{C}^{\mathrm{multi}}$ (i.e., a category-esque structure whose arrows have many sources and one target) by setting

$$\mathsf{C}^{\mathrm{multi}}(A_1,\ldots,A_n;B) := \mathsf{C}(A_1 \otimes \cdots \otimes A_n, B).$$

For an object X of C, the correct endomorphism multicategory $\mathrm{End}^{\mathrm{multi}}(X)$ has the single object X and the morphism sets specified by $\mathsf{C}(X^{\otimes n}, X)$. An operad O is a minimal abstraction of this structure, and a morphism $O \to \mathrm{End}^{\mathrm{multi}}(X)$ endows X with the structure of an O–algebra (or of an O–representation). More generally, the multicategory $\mathsf{C}^{\mathrm{multi}}$ can be thought of as an endomorphism operad which is "delocalized" away from any particular object, or as a multicolored endomorphism operad. Then, the definition of ∞–operads is precisely what is required to lift the classical notion of the multicolored endomorphism operad to the setting of quasicategories (cf. work of Lurie [47, Section 2.1.1]).

Definition 3.1.4 ([47, Definition 2.1.1.10, Remark 2.1.1.14]). A monoidal structure on an ∞–category C is a morphism $\mathsf{C} \to \mathsf{PointedFiniteSets}$ of ∞–operads which is a co-Cartesian fibration and whose fiber $\mathsf{C}^{\otimes}_{\langle n\rangle}$ over the $(n+1)$–point set in Fin_* is isomorphic to $\mathsf{C}^{\times n}$.

Definition 3.1.5 ([47, Remark 4.1.1.4, Definition 4.1.1.6, Notation 4.1.1.9]). A monoidal ∞–category is said to be planar if its structure morphism factors through the ∞–operad $\mathsf{Ass}^{\otimes}$. For C a planar monoidal ∞–category, an associative algebra object of C is a section A of ∞–operads in

$$\begin{tikzcd}
\mathsf{C}^{\otimes} \arrow[d] \arrow[dr] & \\
\mathsf{Ass}^{\otimes} \arrow[u, bend left, dotted, "A"] \arrow[r] & \mathsf{PointedFiniteSets}.
\end{tikzcd}$$

Definition 3.1.6 ([47, Remarks 4.3.1.8-10, Definition 4.3.1.12, Example 4.3.1.15]). Let R and S be assocative algebra objects of a planar monoidal ∞–category C. An (R,S)–bimodule in C is a section M of ∞–operads in the following diagram

$$\begin{tikzcd}
\mathsf{C}^{\otimes} \arrow[d] \arrow[rrrd] & & & & & \\
\mathsf{Ass}^{\otimes} \arrow[u, dotted, bend left, "R"] \arrow[rrd] & \mathsf{C}^{\otimes} \arrow[rr] \arrow[dd] & & \mathsf{C}^{\otimes} \times_{\mathsf{Ass}^{\otimes}} \mathsf{BM}^{\otimes} \arrow[r] \arrow[dd] & \mathsf{C} \arrow[dd] \arrow[rdd] & \\
& & \mathsf{LM}^{\otimes} \arrow[rd] & & & \\
& \mathsf{Ass}^{\otimes} \arrow[uu, dotted, bend left, "S"] \arrow[r] & \mathsf{RM}^{\otimes} \arrow[r] & \mathsf{BM}^{\otimes} \arrow[uu, dotted, bend left, "M"] \arrow[r] & \mathsf{Ass}^{\otimes} \arrow[r] & \mathsf{PointedFiniteSets}
\end{tikzcd}$$

which by restriction induces the sections R and S.

Remark 3.1.7 ([47, Example 4.3.1.16]). Pulling back an associative algebra R along the map

$$\mathsf{BM}^{\otimes} \to \mathsf{Ass}^{\otimes}$$

realizes R as an (R,R)–bimodule in the expected way.

Remark 3.1.8 ([47, Definition 4.3.1.12, Constructions 4.4.2.7 and 4.4.2.10, Example 4.4.2.11]). Denote the ∞–category of (R,S)–bimodules described above by $\mathsf{Bimodules}_{(R,S)}(\mathsf{C})$. In the case that the monoidal structure on C is compatible with geometric realization, there is a tensor product functor

$$\otimes_S\colon \mathsf{Bimodules}_{(R,S)}(\mathsf{C}) \times \mathsf{Bimodules}_{(S,T)}(\mathsf{C}) \to \mathsf{Bimodules}_{(R,T)}(\mathsf{C}).$$

Moreover, for $M \in \mathsf{Bimodules}_{(R,S)}$ and $N \in \mathsf{Bimodules}_{(S,T)}$, the section $M \otimes_S N \in \mathsf{Bimodules}_{(R,T)}$ restricts along the module coordinate to give the cone point of the simplicial object

$$B(M;S;N)\colon \Delta^{\mathrm{op}} \to \mathsf{C}.$$

Important Warning 3.1.9. Our base categories C of interest include $\mathsf{Spaces}^{\mathrm{op}}$, $\mathsf{Spectra}^{\mathrm{op}}$, and $\mathsf{Spectra}_{\Gamma}^{\mathrm{op}}$, where geometric realization does *not* commute with the monoidal structure. This means that, while we can always define the object $M \otimes_S N \in \mathsf{C}$ by the geometric realization of the bar construction, imbuing it with the structure of an (R,T)–bimodule is nontrivial — and, indeed, not always possible.

Lemma 3.1.10. *A space X determines an associative algebra object in the monoidal ∞–category* $\mathsf{Spectra}_{\Gamma}^{\mathrm{op}}$.

Proof. Sets (and hence spaces) already determine strictly coassociative algebra objects on the 1–categorical level. By localizing away from the weak equivalences and passing to the associated ∞–category, we recover an associative algebra object in $\mathsf{Spaces}^{\mathrm{op}}$. Because the stabilization functor $\Sigma^{\infty}_{+} : \mathsf{Spaces} \to \mathsf{Spectra}$ respects the Cartesian monoidal structure on Spaces and the smash monoidal structure on $\mathsf{Spectra}$ [47, Propositions 4.8.2.9 and 4.8.2.18] and because the monoidal structure on Γ–local spectra is defined so that the localization functor L_{Γ} is monoidal, it follows that $L_{\Gamma}\Sigma^{\infty}_{+}X$ is a coaugmented coassociative $L_{\Gamma}\mathbb{S}^0$–coalgebra spectrum. □

The category $\mathsf{Spectra}_\Gamma$ comes with a totalization functor:

$$\mathrm{Tot} : (\mathsf{Spectra}_\Gamma)^\Delta \to \mathsf{Spectra}_\Gamma$$

defined by taking the homotopy limit of the Δ–indexed diagram.

Definition 3.1.11. Let R be an algebra spectrum, let C be an R–coalgebra spectrum, and let M and N be left– and right–C–comodule spectra respectively. Then the C–cotensor product of M against N is defined by the following totalization:

$$M \square_C N := \mathrm{Tot}\,\Omega(M; C; N).$$

In light of Important Warning 3.1.9, we record the following lemma, which is an "at a point" version of Remark 3.1.8.

Lemma 3.1.12. *Let M be a (B, C)–cobimodule spectrum and N a (C, D)–cobimodule spectrum. If the natural map*

$$\mathrm{Tot}\,(X \wedge \Omega(M; C; N) \wedge Y) \to X \wedge (\mathrm{Tot}\,\Omega(M; C; N)) \wedge Y$$

is a weak equivalence, naturally in choice of X and Y, then $\Omega(M; C; N)$ receives the structure of a cobimodule. □

We now recite some standard results about (co)tensor products as defined through (co)bar constructions.

Lemma 3.1.13. *When M is a right–C–comodule, there is a natural equivalence $M \square_C C \simeq M$. (Dually, if N is a left–C–comodule, there is a natural equivalence $C \square_C N \simeq N$.)* □

Lemma 3.1.14. *The diagonal map $C \xrightarrow{\Delta} C \wedge C$ of a coalgebraic spectrum C factors as*

$$C \xrightarrow{\simeq} C \square_C C \to \mathrm{Tot}^0\,\Omega(C; C; C) \simeq C \wedge C. \quad \square$$

Corollary 3.1.15. *If $\pi : C \to M$ presents M as a C–coideal (cf. Example 1.1.28), then the diagonal map $C \xrightarrow{\Delta} C \square_C C$ induces a map $M \xrightarrow{\tilde{\Delta}} M \square_C M$.*

Proof. There is a diagram

$$\begin{tikzcd}
C \arrow[r, "\pi"] \arrow[d, "{\Delta,\simeq}"] & M \arrow[d, "{\psi,\simeq}"] \arrow[dr, "\tilde{\Delta}"] & \\
C \square_C C \arrow[r, "\pi \square_C C"] & M \square_C C \arrow[r, "M \square_C \pi"] & M \square_C M.
\end{tikzcd}$$

Using Lemma 3.1.13, the square square commutes, giving $\tilde{\Delta}$ its name. □

Lemma 3.1.16. *Suppose that $M' \to M \to M''$ is a pair of composable maps of C-comodule spectra for an R-coalgebra C which is also a fiber sequence of underlying R-modules. Then there is an induced fiber sequence*

$$M' \square_C N \to M \square_C N \to M'' \square_C N.$$

Proof. This is immediate from the definitions: the cotensor product $M \square_C N$ is first defined by the limit of a cosimplicial diagram whose levels are given by smashing M against some other fixed spectra dependent upon C and N. Limits commute with finite limits, as does smashing against fixed spectra. □

Remark 3.1.17 (Lemma A.7). There is a forgetful–cofree adjunction:

$$U\colon \mathsf{RightComodules}_C \leftrightarrows \mathsf{Modules}_R\colon -\wedge_R C.$$

It follows that fiber sequence of right C-comodules are detected by fiber sequences of the underlying R-modules. Thus, Theorem 3.2.1 is not just further evidence that this is the "correct" derived cotensor product, but it is a fraction of a proof that the functor $-\square_D C$ induced by a map $C \to D$ of coalgebras participates in a push-pull adjunction. We omit proofs of these assertions, since they do not affect the rest of the paper, and we invite the reader interested in foundations to pursue an Elmendorf–Kriz–Mandell–May [20] style exploration of the theory of coalgebraic spectra.

We now turn to the problem of computing the K-homology groups $K_*(M \square_C N)$. To begin, note that the category Δ comes with a natural filtration by the subcategories $i_n : \Delta^n \to \Delta$ with nondegenerate cosimplices only up to dimension n. The restriction map i_n^* of diagram categories admits an adjoint $(i_n)_*$ by right Kan extension, and the associated push–pull object has a standard name:

Definition 3.1.18. For a cosimplicial object Y the composite $(i_n)_* i_n^* Y =: \operatorname{cosk}^n Y$ is called the n^{th} coskeleton of Y.

This begets a natural cofiltration of totalized objects:

Lemma 3.1.19. *Again for a cosimplicial object Y, the natural diagram*

$$\operatorname{Tot} Y \to \left(\cdots \to \operatorname{Tot}\operatorname{cosk}^{n+1} Y \to \operatorname{Tot}\operatorname{cosk}^n Y \to \cdots\right)$$

expresses $\operatorname{Tot} Y$ *as a sequential limit.*

Remark 3.1.20. Because L_Γ is a left-adjoint, when the objects in the diagram are Γ-local spectra it does not matter if these limits are taken in the ∞-category of spectra or in the ∞-category of Γ-local spectra.

Using this sequential presentation of totalization as a cofiltration, we can build a spectral sequence computing the homotopy groups $\pi_* \operatorname{Tot} Y$. Its E^1-page is given by the homotopy groups of the fibers F_n in the sequences

$$F_n \to \operatorname{Tot}\operatorname{cosk}^{n+1} Y \to \operatorname{Tot}\operatorname{cosk}^n Y,$$

which in the case $Y = \Sigma^\infty_+ \Omega(*;X;*)$ take the form

$$F_n = \Omega^n \Sigma^\infty X^{\wedge n}. \tag{3.1.21}$$

In the next theorem, we construct a naive homology spectral sequence from this tower:

Theorem 3.1.22. *Let C be a coalgebra spectrum, M a right C–comodule spectrum, and N a left C–comodule spectrum. When K is a spectrum with enough Künneth isomorphisms so that*

$$K_*(M \wedge C^{\wedge n} \wedge N) \cong K_*M \otimes (K_*C)^{\otimes n} \otimes K_*N,$$

then there are spectral sequences

$$\begin{aligned} E^1_{*,*} \cong K_*M \otimes_{K_*} (K_*C)^{\otimes(*\leq n)} \otimes_{K_*} K_*N, &\;\Big\} \Rightarrow K_* \operatorname{Tot} \operatorname{cosk}^n \Omega(M;C;N), && (\mathit{partial}) \\ \left.\begin{aligned} E^1_{*,*} &\cong K_*M \otimes_{K_*} (K_*C)^{\otimes *} \otimes_{K_*} K_*N, \\ E^2_{*,*} &\cong \operatorname{Cotor}^{K_*C}_{*,*}(K_*M, K_*N) \end{aligned}\right\} &\Rightarrow \lim_n K_* \operatorname{Tot} \operatorname{cosk}^n \Omega(M;C;N), && (\mathit{full}) \end{aligned}$$

with the right–hand limit taken in the category of K_*–modules. The partial spectral sequences are strongly convergent, and the full spectral sequence is conditionally strongly convergent [12, Theorem 7.1]. ☐

However, we are most interested in the K–homology of the limit rather than the limit of the K–homologies. Interchanging these functors requires a considerably more delicate argument, and in particular relies on exceptionally good properties of K. To begin, we reproduce the following theorem, which will be proven in the appendix:

Theorem 3.1.23 (See Theorem A.1). *Let $K = K(d)$, $1 \leq d < \infty$, be a standard Morava K–theory with formal group Γ and let $\{X_\alpha\}_\alpha$ be a sequential inverse system of Γ–local spectra such that $\{K_*X_\alpha\}_\alpha$ is a Mittag-Leffler system of K_*–modules. There is then a convergent spectral sequence of signature*

$$R^* \lim_\alpha \{K_*X_\alpha\}_\alpha \Rightarrow K_* \lim_\alpha \{X_\alpha\}_\alpha,$$

where the right–derived inverse limit on the left is taken in the category of $E(d)_*E(d)$–comodules. ☐

Remark 3.1.24. This theorem finally explains our interest in coalgebras: it is homology that has a natural such spectral sequence rather than cohomology. In light of Remark 1.3.28, an obvious way to proceed to analyze the problem presented at the beginning of this chapter is to replace the scheme $\widehat{\mathbb{A}}^1$ with the cohomology ring E^*X of some space, then to replace the cohomology ring with the function spectrum $F(\Sigma^\infty_+ X, E)$ whose homotopy is E^*X. Starting from this more familiar vantage point, one can attempt to recast the constructions given so far in terms of cohomology — but at some point one will be forced to compare $F(\Sigma^\infty_+ X, E)$ and $D\Sigma^\infty_+ X \wedge E$. Working instead with homology and with coalgebras avoids the Spanier–Whitehead dual of an infinite complex, and instead the complexity of the situation is successfully trapped by the spectral sequence of Theorem 3.1.23.

We now use this spectral sequence to analyze the sequence of partial spectral sequences of Theorem 3.1.22.

Theorem 3.1.25. *When M, N, and C have even-concentrated K-homology, there is a further spectral sequence*

$$R^* \lim_t \{K_* \operatorname{Tot} \operatorname{cosk}^t \Omega(M;C;N)\}_t \Rightarrow K_*(M \square_C N),$$

where the derived inverse limit is taken in a category of comodules appropriate to K (cf. Theorem A.1).

Proof. In order to make the comparison between $\lim_t K_* \operatorname{Tot} \operatorname{cosk}^t$ and $K_* \operatorname{Tot}$, we analyze the homology of the coskeletal tower itself, i.e., the sequence of partial spectral sequences of Theorem 3.1.22. This is mostly an exercise in homological algebra, so will require some tedious bookkeeping but is otherwise fairly straightforward. Throughout, we will consider the degree $s+t$ part of the t-cochains C^t_{s+t}, coboundaries B^t_{s+t}, and cocycles Z^t_{s+t} of the normalized cobar complex $\Omega(K_*M;K_*C;K_*N)$. With an induction beginning at $t = 0$ and $t = 1$, we claim the K-homology of the tth level of the tower (i.e., the target of the t^{th} partial spectral sequence) is

$$K_s \operatorname{Tot} \operatorname{cosk}^t \Omega(M;C;N) = D^1_{s,t} = \begin{cases} D^1_{s,t-1} \oplus \dfrac{C^t_{\frac{s+t}{2}}}{B^t_{\frac{s+t}{2}}} & \text{when } s+t \text{ is even, and} \\ D^1_{s,t-2} \oplus H^{t-1}_{\frac{s+t-1}{2}} & \text{when } s+t \text{ is odd,} \end{cases} \tag{3.1.26}$$

where $D^1_{*,*}$ denotes the rear of the exact couple of the full spectral sequence. Induction then proceeds by considering one triangle in that exact couple:

$$\begin{tikzcd}
K_s \operatorname{Tot} \operatorname{cosk}^{t-1} \Omega(M;C;N) \arrow[rd, "{[-1]}"'] & K_s \operatorname{Tot} \operatorname{cosk}^t \Omega(M;C;N) \arrow[l] & & D^1_{s,t-1} \arrow[rd, "{[-1]}"'] & D^1_{s,t} \arrow[l] \\
& K_s \Omega^t(M \wedge C^{\wedge t} \wedge N) \arrow[u] & = & & E^1_{s,t}. \arrow[u]
\end{tikzcd}$$

The bottom vertex in the triangle is the K_*-module of cochains, and we take s to be a degree in which $K_{s+t}(M \wedge C \wedge \cdots \wedge C \wedge N)$ is nonvanishing, i.e., $s+t$ is even. Then, the triangle unrolls into an exact sequence which, using the inductive hypothesis, takes the form

$$\begin{tikzcd}
0 \arrow[r] & D^1_{s+1,t} \arrow[r] \arrow[d, equal] & D^1_{s+1,t-1} \arrow[r] \arrow[d, equal] & E^1_{s,t} \arrow[r] \arrow[d, equal] & D^1_{s,t} \arrow[r] \arrow[d, equal] & D^1_{s,t-1} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & H^{t-1}_{\frac{s+t}{2}} \oplus X \arrow[r, "i \oplus 1"] & \dfrac{C^{t-1}_{\frac{s+t}{2}}}{B^{t-1}_{\frac{s+t}{2}}} \oplus X \arrow[r, "\partial \oplus 0"] & C^t_{\frac{s+t}{2}} \arrow[r, "\pi \oplus 0"] & \dfrac{C^t_{\frac{s+t}{2}}}{B^t_{\frac{s+t}{2}}} \oplus Y \arrow[r, "0 \oplus 1"] & Y \arrow[r] & 0
\end{tikzcd}$$

for some modules X and Y to be determined. We then splice three of these long sequences together to form Figure 3.1, which is labeled in terms of the cobar cohomology groups in Figure 3.2. The zig-zag containing $D^1_{s,t+1}$ through $D^1_{s,t-2}$ is where the action is: $D^1_{s,t-1}$ and $D^1_{s,t-2}$ are assumed

$$\begin{tikzcd}
 & & & & 0 \arrow[d] & & \\
 & & & & D^1_{s,t+1} \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & D^1_{s+1,t} \arrow[r] & D^1_{s+1,t-1} \arrow[r] & E^1_{s,t} \arrow[r] & D^1_{s,t} \arrow[r] \arrow[dl, bend left] & D^1_{s,t-1} \arrow[r] \arrow[d] & 0 \\
0 & D^1_{s-1,t} \arrow[l] & D^1_{s-1,t+1} \arrow[l] & E^1_{s-1,t+1} \arrow[l] & & D^1_{s,t-2} \arrow[dl, bend left] & \\
 & 0 & D^1_{s-1,t-2} \arrow[l] & D^1_{s-1,t-1} \arrow[l] & E^1_{s-1,t-1} \arrow[l] & &
\end{tikzcd}$$

Figure 3.1: Three interacting exact sequences.

$$\begin{tikzcd}
 & & & & 0 \arrow[d] & & \\
 & & & & H^t_{\frac{s+t}{2}} \oplus D^1_{s,t-1} \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & D^1_{s+1,t} \arrow[r] & D^1_{s+1,t-1} \arrow[r] & C^t_{\frac{s+t}{2}} \arrow[r] & \frac{C^t_{\frac{s+t}{2}}}{B^t_{\frac{s+t}{2}}} \oplus D^1_{s,t-1} \arrow[r] \arrow[dl, bend left] & D^1_{s,t-1} \arrow[r] \arrow[d] & 0. \\
0 & D^1_{s-1,t} \arrow[l] & D^1_{s-1,t+1} \arrow[l] & C^{t+1}_{\frac{s+t}{2}} \arrow[l] & & D^1_{s,t-2} \arrow[dl, bend left] & \\
 & 0 & D^1_{s-1,t-2} \arrow[l] & D^1_{s-1,t-1} \arrow[l] & E^1_{s-1,t-1} \arrow[l] & &
\end{tikzcd}$$

Figure 3.2: The interacting sequences with cobar complex names.

to be as claimed, then the observation involving Y shows that $D^1_{s,t-1}$ is as well, which in turn determines $D^1_{s,t+1}$ as the kernel of the map $D^1_{s,t} \to E^1_{s-1,t+1}$, i.e., as the cocycle subgroup of the cochain group, taken modulo the coboundaries. This proves the inductive claim.

We are now in a position to use the spectral sequence of Theorem 3.1.23, as applied to the

K-homology of the local system

$$\{\operatorname{Tot}\operatorname{cosk}^t \Omega(M;C;N)\}_t.$$

The inductive claim shows that the tower $\{D^1_{s,t}\}$ as t grows is given by projections onto sums of the groups specified by Equation (3.1.26), hence is Mittag-Leffler. It then follows that there exists a derived inverse limit spectral sequence, as desired. □

3.2 The main calculation

Recall that we are searching for a functor T^*_η which operates on pointed coalgebra spectra $\mathbb{S} \xrightarrow{\eta} C$ and which participates in the interchange law

$$T^*_{K_*\eta} \operatorname{Sch} K_* C = K_* T^*_\eta C.$$

It will be easiest to prove this law if we can show that K_* commutes with taking (derived) cotensor products.

Theorem 3.2.1. *Let C be a pointed $L_K\mathbb{S}$-coalgebra spectrum with* $\operatorname{Sch} K_* C$ *a formal variety. Setting M to be the C-bicomodule $M =$* $\operatorname{cofib}\eta$, *the objects $M^{\Box_C j}$ all exist and there is an isomorphism*

$$K_*(M^{\Box_C j}) = (K_* M)^{\Box_{K_* C} j}.$$

Proof. Our first goal is to compute $\operatorname{Cotor}^{K_*C}_{*,*}(K_*M, K_*(M^{\Box_C j}))$ as input to our coalgebraic Künneth spectral sequence, which we do inductively. This turns out to be much simpler on the level of algebras, since there we have access to Koszul complexes [80, Corollary 4.5.5]. Using duality of graded finite-type algebras and coalgebras over a field, we would equivalently like to compute $\operatorname{Tor}^A_{*,*}(I, I)$, for $A = K_*[\![x_1, \ldots, x_n]\!]$ and $I = \langle x_1, \ldots, x_n \rangle$. Let $K^A(x_i)$ denote the Koszul complex $A \xrightarrow{\cdot x_i} A$, and set $K^A(x_1, \ldots, x_n) = \bigotimes_{i=1}^n K^A(x_i)$. The shifted subcomplex $K^A(x_1, \ldots, x_n)^{\geq 1}[-1]$ forms an A-free resolution of I, which we use to compute our Tor groups. First, because the x_i form a regular sequence, the higher $\operatorname{Tor}^A_{\geq 1,*}(K^A(x_1, \ldots, x_n); I^j)$ vanish, and so this must also be true of $\operatorname{Tor}^A_{\geq 1,*}(K^A(x_1, \ldots, x_n)^{\geq 1}[-1]; I^j)$. This leaves only $\operatorname{Tor}^A_{0,*}$, which is the ordinary tensor product, yielding $\operatorname{Tor}^A_{*,*}(K^A(x_1, \ldots, x_n)^{\geq 1}[-1]; I^j) = I^{j+1}$ just as desired.

Together with Theorem 3.1.22 and Theorem 3.1.25, this shows first that the inverse tower of K_*-groups in question are a sequence of projections of direct sums of cobar cohomology groups, and second that all but one of these cohomology groups vanishes. This means that each tower $\{D^1_{s,t}\}_t$ is pro-constant, and hence that the derived inverse limits vanish in the obstructing spectral sequence. This gives the desired identification $K_*(M \Box_C M^{\Box_C j}) \cong K_* M \Box_{K_* C} K_* M^{\Box_C j}$. It also shows that the hypotheses of Lemma 3.1.12 are satisfied, as tensoring a pro-constant tower with a constant tower preserves pro-constancy, and hence the natural map of spectral sequences from Theorem 3.1.23 is an isomorphism. □

Definition 3.2.2. Using Corollary 3.1.15, define $T^*_\eta C$ to be

$$T^*_\eta C = \mathrm{fib}(M \xrightarrow{\Delta} M \square_C M).$$

Corollary 3.2.3. *Under the same hypotheses as Theorem 3.2.1, there is an interchange formula* $K_* T^*_\eta C = T^*_0 \,\mathrm{Sch}\, K_* C$*, and hence* $T^*_\eta C$ *is a Picard element of the* Γ*–local category.* □

Our primary interest in this construction will come from examples. First, however, we are prepared to prove one interesting general feature now: the existence of the "annular tower".

Definition 3.2.4. In the following diagram, each angle is a fiber sequence and each horizontal map is $\Delta \square_C 1 \square_C \cdots \square_C 1$:

$$\begin{tikzcd}
C_0^\infty \arrow[d, equal] & C_1^\infty \arrow[d, equal] & C_2^\infty \arrow[d, equal] & C_3^\infty \arrow[d, equal] & C_4^\infty \arrow[d, equal] & \cdots \\
M^{\square_C 0} \arrow[r] & M^{\square_C 1} \arrow[r] & M^{\square_C 2} \arrow[r] & M^{\square_C 3} \arrow[r] & M^{\square_C 4} \arrow[r] & \cdots \\
C_0^0 = \mathbb{S} \arrow[u] & C_1^1 = T^*_\eta C \arrow[u] & C_2^2 \arrow[u] & C_3^3 \arrow[u] & C_4^4 \arrow[u] & \cdots .
\end{tikzcd}$$

We call this sequence the annular tower, with C_j^j the jth annulus and C_n^∞ the n^{th} punctured disk. In the first two stages, $D_0 = \mathbb{S}$ selects the point from our pointed coalgebra spectrum C, and $D_1 = T^*_\eta C$ is our definition of the cotangent space. For indices $0 \leq i \leq j < k \leq \infty$, these spectra extend to a family of thickened annuli by means of the cofiber sequences

$$C_i^j \to C_i^k \to C_{j+1}^k.$$

Remark 3.2.5. Because the natural coalgebra structure on spaces is cocommutative as well as coassociative, the homotopy type of the objects C_j^j does not depend upon the position of Δ in the above formula. Since we have not worked hard enough to develop cocommutative objects and because we don't use this fact below, we will not prove it.

The algebraic analogue of the spectrum C_j^j is the module

$$k\{x_1^{j_1} \cdots x_n^{j_n} \mid j_1 + \cdots + j_n = j\} = I^{\otimes_k j} / I^{\otimes_k (j+1)},$$

where I is the augmentation ideal of the power series ring $k[\![x_1, \ldots, x_n]\!]$. In the 1–dimensional case, there is the additional identification

$$I^{\otimes_k j} / I^{\otimes_k (j+1)} \cong (I/I^2)^{\otimes j}.$$

Before proceeding to the examples, we prove a topological analogue of this fact:

Theorem 3.2.6. *When* $\operatorname{Sch} K_* C$ *is* 1*–dimensional, there is a* Γ*–local equivalence* $C^j_j \simeq (T^*_\eta C)^{\wedge j}$.

Proof. Consider the following diagram of short exact sequences of k–modules, where the dashed line is filled in by the universal property of the quotient:

$$\begin{tikzcd}
0 \arrow[r] & \langle x\rangle \otimes_k \langle x^{n+1}\rangle \oplus \langle x^2\rangle \otimes_k \langle x^n\rangle \arrow[r] \arrow[d] & \langle x\rangle \otimes_k \langle x^n\rangle \arrow[r] \arrow[d] & \frac{\langle x\rangle}{\langle x^2\rangle} \otimes_k \frac{\langle x^n\rangle}{\langle x^{n+1}\rangle} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \langle x^{n+2}\rangle \arrow[r] & \langle x^{n+1}\rangle = \langle x\rangle \otimes_R \langle x^n\rangle \arrow[r] & \frac{\langle x^{n+1}\rangle}{\langle x^{n+2}\rangle} \arrow[r] & 0.
\end{tikzcd}$$

In perfect analogy, we construct a diagram of fiber sequences of spectra, modeling the situation for coalgebras and comodules:

$$\begin{tikzcd}
(M \wedge M^{\square_C (j+1)}) \times (M^{\square_C 2} \wedge M^{\square_C j}) & M \wedge M^{\square_C j} \arrow[l] & T^*_\eta C \wedge C^j_j \arrow[l] \\
M^{\square_C (j+2)} \arrow[u] & M^{\square_C (j+1)} \arrow[l] \arrow[u] & C^{j+1}_{j+1}. \arrow[l] \arrow[u, dashed]
\end{tikzcd}$$

The solid vertical maps are induced by the natural map

$$M \square_C N = \operatorname{Tot} \Omega(M; C; N) \to \operatorname{Tot}^0 \Omega(M; C; N) = M \wedge N,$$

and the dashed map exists by the extension property for fiber sequences. Applying K_* to the diagram and chasing the left-hand square shows the dashed map to be a Γ–local equivalence. $\square$

3.3 Examples

In this section, we apply the machinery developed in Section 3.1 to some examples, first familiar and then unfamiliar.

Complex projective space

We select $R = \mathbb{S}$ as our algebra, $C = \Sigma^\infty_+ \mathbb{C}\mathrm{P}^\infty$ our $\mathbb{S}$–coalgebra, and $\eta : \mathbb{S} \to \Sigma^\infty_+ \mathbb{C}\mathrm{P}^\infty$ its natural pointing by the disjoint basepoint. In this case, we apply the version of Definition 3.2.2 for $K = H\mathbb{Z}$. In order the analyze the coskeletal tower, we appeal to the auxiliary spectra $K = H\mathbb{Q}$ and $K = H\mathbb{F}_p$, where we find

$$H\mathbb{Q}_* T_+ \Sigma^\infty_+ \mathbb{C}\mathrm{P}^\infty \cong \Sigma^2 \mathbb{Q},$$
$$(H\mathbb{F}_p)_* T_+ \Sigma^\infty_+ \mathbb{C}\mathrm{P}^\infty \cong \Sigma^2 \mathbb{F}_p.$$

Since $T_+\Sigma^\infty_+\mathbb{C}\mathrm{P}^\infty$ is a connective spectrum, it follows from early work of Sullivan [75, Proposition 3.20] on adèlic reconstruction that the integral homology is

$$H\mathbb{Z}_* T_+\Sigma^\infty_+\mathbb{C}\mathrm{P}^\infty \cong \Sigma^2\mathbb{Z},$$

and hence[1] that its homotopy type is

$$T_+\mathbb{C}\mathrm{P}^\infty \simeq \mathbb{S}^2.$$

To verify that the annular tower records the cellular decomposition of $\mathbb{C}\mathrm{P}^\infty$, we transfer to the dual of the annular tower by considering the following diagram, whose columns form fiber sequences:

$$\begin{tikzcd}
C_0^\infty \arrow[r] & C_1^\infty \arrow[r] & C_2^\infty \arrow[r] & C_3^\infty \arrow[r] & \cdots \\
C_0^\infty \arrow[r] \arrow[u] & C_0^\infty \arrow[r] \arrow[u] & C_0^\infty \arrow[r] \arrow[u] & C_0^\infty \arrow[r] \arrow[u] & \cdots \arrow[u] \\
\mathrm{pt} \arrow[r] \arrow[u] & C_0^0 \arrow[r] \arrow[u] & C_0^1 \arrow[r] \arrow[u] & C_0^2 \arrow[r] \arrow[u] & \cdots. \arrow[u]
\end{tikzcd}$$

By connectivity, it again follows that $C^n_m \simeq \mathbb{C}\mathrm{P}^n_m$.

Real projective space

We can also apply our tools in the case $C = \mathbb{R}\mathrm{P}^\infty_+$ and $K = K(\infty) = H\mathbb{F}_2$, but we must first note that the proof of Theorem 3.1.25 does not work as given: $(H\mathbb{F}_2)_*\mathbb{R}\mathrm{P}^\infty$ is not even–concentrated. However, using the fact that the groups

$$\mathrm{Cotor}^{(H\mathbb{F}_2)_*\mathbb{R}\mathrm{P}^\infty}_{*,*>0}\left(\widetilde{(H\mathbb{F}_2)}_*\mathbb{R}\mathrm{P}^\infty; \widetilde{(H\mathbb{F}_2)}_*\mathbb{R}\mathrm{P}^\infty\right)$$

all vanish, we can conclude that the truncated spectral sequences of Theorem 3.1.22 have no differentials beyond after the first page. It thus follows that there are still exact sequences of the form

$$0 \to D^1_{s+1,t} \to D^1_{s+t,t-1} \to E^1_{s,t} \to D^1_{s,t} \to D^1_{s,t-1} \to 0,$$

and hence the rest of the proof of Theorem 3.1.25 goes through. By similar reasoning to the complex projective case, performing the tangent space construction in the 2–adic stable category (in fact, because our objects are connective, it suffices to consider the $\widehat{\mathbb{G}}_a$–local category over $\mathbb{F}_2$) yields

$$T_+L_{H\mathbb{F}_2}\Sigma^\infty_+\mathbb{R}\mathrm{P}^\infty \simeq L_{H\mathbb{F}_2}\mathbb{S}^1,$$

and the annular tower recovers the cellular decomposition of $\mathbb{R}\mathrm{P}^\infty$.

It's worth remarking that the ambient category chosen to perform the tangent space construction is very important. It can neither be too localized nor too delocalized:

---

[1] The reader may compare with Example 1.4.11.

- Passing to the $\widehat{\mathbb{G}}_m$–local category (or, generally, the Γ–local category for Γ defined over k of characteristic 2) factors through 2–completion. By work of Ravenel [61, Theorem 9.1], there is an equivalence
$$L_{\widehat{\mathbb{G}}_m} \mathbb{R}\mathrm{P}^{\infty}_{8k+1} \simeq L_{K(1)} \mathbb{S}^{-1}$$
for any k. Taking $k = 0$, one sees $L_{\widehat{\mathbb{G}}_m} \mathbb{R}\mathrm{P}^{\infty} \simeq L_{K(1)} \mathbb{S}^{-1}$, so its $K(1)$–cohomology is not a power series ring, and it is furthermore too small to have the correct Cotor groups. Letting k range, it's also plain that the bar filtration looks wildly different from the behavior of any sort of expected annular filtration.
- On the other hand, $\mathbb{R}\mathrm{P}^{\infty}$ is p–locally acyclic for $p \neq 2$. It follows that the integral homology of $T_+ \Sigma^{\infty}_+ \mathbb{R}\mathrm{P}^{\infty}$, as performed in the global stable category, will not have the freeness property required by Example 1.4.11.

"Determinantal projective space": $\underline{H\mathbb{Z}/p^{\infty}}_d$

Our final and most interesting example comes from Ravenel and Wilson's calculation, presented in Theorem 2.1.5. Selecting a formal group Γ of finite height d with associated Morava K–theory K, they show that the formal scheme $(\underline{H\mathbb{Z}/p^{\infty}}_t)_K$ is a formal Lie group of dimension $\binom{d-1}{t-1}$. The value of $t = 1$ corresponds to $\Sigma^{\infty}_+ \mathbb{C}\mathrm{P}^{\infty}$, an interesting spectrum, and the symmetry of Pascal's triangle suggests that the value $t = d$ may also be interesting. Lemma 1.4.14 provides a tool we can use to identify what Picard element Definition 3.2.2 constructs. Collecting Theorem 1.2.26, Theorem 2.2.1, and Theorem 2.3.8 we record the following calculation:

Lemma 3.3.1. *Let $\beta_1 \in E_* \underline{H\mathbb{Z}/p^{\infty}}_1$ be an element dual to a coordinate on $\mathbb{C}\mathrm{P}^{\infty}_E[p^{\infty}]$. The tangent space of $(\underline{H\mathbb{Z}/p^{\infty}}_d)_E$ is spanned by the dual to the element*

$$\beta_* = S^0 \beta_1 \otimes S^1 \beta_1 \otimes \cdots \otimes S^{d-1} \beta_1$$

pushed forward along the $\circ$-product map. □

Inspired by this formula, we also record the following standard representation of the stabilizer group $\mathbb{S}_d$:

Definition 3.3.2. By Theorem 1.2.26, the group $\mathbb{S}_d$ arises as the group of units of a d–dimensional $\mathbb{W}(k)$–algebra $\operatorname{End} F_d$. Left–multiplication gives a map

$$\mathbb{S}_d \to GL(\operatorname{End} F_d)$$

and postcomposing with the determinant map gives definition of the determinant representation:

$$\mathbb{S}_d \to GL(\operatorname{End} \mathbb{F}_d) \xrightarrow{\det} \mathbb{W}(k)^{\times}.$$

The choice of basis $\{1, S, \ldots, S^{d-1}\}$ gives a presentation

$$\begin{tikzcd}
\mathbb{S}_d \arrow[r] & GL(\mathrm{End}\,\mathbb{F}_d) \arrow[r, "\det"] \arrow[d, equal] & \mathbb{W}(k)^\times \arrow[d, equal] \\
& GL(\mathbb{W}(k)\{1, S, \ldots, S^{d-1}\}) \arrow[r] \arrow[d, equal] & GL(\mathbb{W}(k)\{1 \wedge S \wedge \cdots \wedge S^{d-1}\}) \arrow[d, equal] \\
& GL_d\mathbb{W}(k) \arrow[r] & GL_1\mathbb{W}(k).
\end{tikzcd}$$

Theorem 3.3.3. *The stabilizer group acts as the determinant on* $\mathscr{E}_\Gamma(T_+^*\Sigma_+^\infty \underline{H\mathbb{Z}/p^\infty}_d)$.

Proof. The cup product map

$$E_*\underline{H\mathbb{Z}/p}_1^{\times d} \xrightarrow{\circ} E_*\underline{H\mathbb{Z}/p}_d$$

is surjective and respects the $\mathbb{S}_d$-action, as it is induced by a map of spaces. There is a Künneth formula

$$(E_\Gamma)_*\underline{H\mathbb{Z}/p}_1^{\times d} \cong ((E_\Gamma)_*\underline{H\mathbb{Z}/p}_1)^{\otimes d},$$

and the stabilizer group intertwines with Künneth splittings

$$E_*(A \times B) \cong E_*A \otimes_{E_*} E_*B$$

as

$$g \cdot (a \otimes b) = (g \cdot a) \otimes (g \cdot b).$$

In turn, an element $g \in \mathbb{S}_d$ acts on β_* by the formula

$$\begin{aligned}
g \cdot (\circ\beta_*) &= \circ(g \cdot \beta_*) \\
&= \circ(g \cdot (S^0\beta_1 \otimes \cdots \otimes S^{d-1}\beta_1)) \\
&= \circ(g \cdot (S^0 \otimes \cdots \otimes S^{d-1})\beta_1) \\
&= \circ(\det g(S^0 \otimes \cdots \otimes S^{d-1})\beta_1) \\
&= (\det g)(\circ\beta_*).
\end{aligned}$$

□

Corollary 3.3.4. *For* $2p - 2 \geq d^2$ *and* $p \neq 2$, *the spectrum* $T_+\Sigma_+^\infty \underline{H\mathbb{Z}/p^\infty}_d$ *models the determinantal sphere* $\mathbb{S}[\det]$.

Proof. Couple the above with Lemma 1.4.14. □

Remark 3.3.5. The object $\mathbb{S}[\det]$ is fairly familiar to chromatic homotopy theorists. Its first appearance was in work of Hopkins and Gross on describing the Γ–local homotopy type of the Brown–Comenetz dualizing spectrum [28, Theorem 6]. It has subsequently played a prominent role in the study of chromatic homotopy theory at the height 2. For instance, coupled with a variant of Lemma 1.4.18 it has been shown to span the rest of the torsion–free part of the $\widehat{C}_{\mathrm{ss}}$–local Picard group [11, Theorem 8.1]. It also has been used to study duality phenomena relating to topological modular forms; see for example work of Behrens [10, Proposition 2.4.1] and of Stojanoska [69, Corollary 13.3].

Remark 3.3.6. As there is an algebraic factorization

$$\begin{tikzcd}
T_0(\underline{H\mathbb{Z}/p^\infty}_d)_E \arrow[r] \arrow[d, equal] & (\underline{H\mathbb{Z}/p}_d)_E \arrow[r] \arrow[d, equal] & (\underline{H\mathbb{Z}/p^\infty}_d)_E \arrow[d, equal] \\
T_0(\mathbb{CP}^\infty_E[p^\infty]^{\wedge d}) \arrow[r] & \mathbb{CP}^\infty_E[p]^{\wedge d} \arrow[r] & \mathbb{CP}^\infty_E[p^\infty]^{\wedge d},
\end{tikzcd}$$

the above claims can be calculated in $E_*\underline{H\mathbb{Z}/p}_d$, where $\circ\beta_*$ agrees with $\beta_1 \circ \beta_p \circ \cdots \circ \beta_{p^{d-1}}$ modulo the maximal ideal of the Lubin–Tate ring.

Remark 3.3.7. The rest of the annular tower gives a remarkable filtration of the Γ–local homotopy type of $\underline{H\mathbb{Z}/p^\infty}_d$, as though it were a cell complex built out of Picard–graded cells from Section 1.4 with a simple inductive structure. The global homotopy type $\underline{H\mathbb{Z}/p^\infty}_d$ of course also comes with a cellular decomposition by global cells — after all, it is presented simplicially by an iterated bar construction — but this information is dramatically more complex. Morally, passing to the Γ–local category has simultaneously enlarged our notion of "cell" and simplified the homotopy type of a complicated space, at once resulting in a simple pattern not globally visible.

In particular, it follows that there is *no* global finite complex with a map to $\underline{H\mathbb{Z}/p^\infty}_d$ which in E–cohomology projects to precisely the 1–jets. The reader should compare with the situation with $\mathbb{CP}^\infty$ and ordinary homology, where $\mathbb{CP}^1 \simeq \mathbb{S}^2$ performs this selection of the 1–jets, but selecting the second annulus is obstructed by $\eta \in \pi_1\mathbb{S}$, available only after coning off $\mathbb{CP}^1$ or inverting 2. By enlarging the variety of spheres available to us, we have given ourselves more tools by which we can carefully select certain individual homology classes in Γ–local spectra.

3.4 Koszul duality and a new Γ–local Hopf map

In the construction of the annular tower in Definition 3.2.4, we made repeated use of the fiber sequence

$$\mathbb{S} \xrightarrow{\eta} C \to M,$$

where C was a coalgebra spectrum pointed by η, by using the identification $M \simeq M \square_C C$ to stitch together these fiber sequences at successive nodes.

In this section, we will further exploit this by considering the quotient map $C \to M$ to be a kind of resolution of $\mathbb{S}$ in the category of C–comodules.

Corollary 3.4.1. *There is a natural equivalence* $\mathbb{S} \square_C M \simeq T_\eta C$.

Proof. This is a direct consequence of Definition 3.2.2 and Lemma 3.1.16. □

Theorem 3.4.2. *When* $\mathbb{S} \to C$ *is a pointed coalgebra spectrum so that* $\operatorname{Sch} K_* C$ *is a* 1*–dimensional formal variety, the spectrum* $\mathbb{S} \square_C \mathbb{S}$ *has a filtration of the form*

$$\Sigma^{-1} T_\eta C \to \mathbb{S} \square_C \mathbb{S} \to \mathbb{S}.$$

Additionally, if the pointing has a splitting $C \to \mathbb{S}$*, this filtration is split.*

Proof. This follows by cotensoring the C-comodule $\mathbb{S}$ by the resolution sequence specified above to get the new fiber sequence

$$\cdots \to \Sigma^{-1}\mathbb{S}\square_C M \to \mathbb{S}\square_C \mathbb{S} \to \mathbb{S}\square_C C \to \mathbb{S}\square_C M \to \cdots .$$

By Lemma 3.1.13, we identify $\mathbb{S}\square_C C$ with $\mathbb{S}$, and by Corollary 3.4.1, we identify $\mathbb{S}\square_C M$ with $T_\eta C$, giving the fiber sequence

$$\Sigma^{-1} T_\eta C \to \mathbb{S}\square_C \mathbb{S} \to \mathbb{S}. \qquad \square$$

Our interest in the object $\mathbb{S}\square_C\mathbb{S}$ is that it interacts well with Koszul duality. Namely, there is the following general result:

Theorem 3.4.3 ([16, Proposition 7.26]). *Let $\mathcal{O}$ be an operad in k-modules, where k is some algebra spectrum, and let A be a left-module for the operad $\mathcal{O}$ (i.e., an $\mathcal{O}$-algebra). It follows that the arboreal bar construction $B(k;\mathcal{O};k)$ provides a co-operad $\mathcal{O}^\vee$ for which $B(A;\mathcal{O};k)$ is a left-co-module, and dually. Additionally, the operad so-produced from the coassociative co-operad is the associative operad.* ☐

Corollary 3.4.4. *The spectrum $\mathbb{S}\square_C\mathbb{S}$ is an associative algebra spectrum.* ☐

Theorem 3.4.5 ([16, Proposition 6.4]). *When $\mathcal{O}$ and A are suitably of finite type (and the associative operad is such an operad), this operation is involutive: there is a natural equivalence of algebras*

$$\operatorname{Tot}\Omega(k;|B(k;A;k)|;k) \to A. \quad \square$$

Corollary 3.4.6. *For C a Γ-local coalgebraic formal variety spectrum, the natural map*

$$|B(L_\Gamma\mathbb{S};L_\Gamma\mathbb{S}\square_C L_\Gamma\mathbb{S};L_\Gamma\mathbb{S})| \to C$$

is a Γ-local equivalence.

Proof. Though Theorem 3.4.5 is sufficient to deduce the equivalence of co-operads, the equivalence of co-operadic left-modules (i.e., algebras for the co-operads) is not immediate since C does not have good finiteness properties. Nonetheless, the coalgebraic formal schemes $\operatorname{Sch} K_* C$ and $\operatorname{Sch} K_* |B(L_\Gamma\mathbb{S};L_\Gamma\mathbb{S}\square_C L_\Gamma\mathbb{S};L_\Gamma\mathbb{S})|$ are both formal lines, and the natural map is an isomorphism on tangent spaces. Lemma 1.1.16 thus applies. ☐

Remark 3.4.7. The bar filtration agrees with the annular filtration, since the natural maps

$$|\operatorname{sk}^n B(L_\Gamma\mathbb{S};L_\Gamma\mathbb{S}\square_C L_\Gamma\mathbb{S};L_\Gamma\mathbb{S})| \to C_n^\infty$$

are equivalences in K-homology. Additionally, the algebra multiplication on $\mathbb{S}\square_C\mathbb{S}$ encodes the attaching map in C_1^2, which is to be thought of as a kind of determinantal Hopf map.

3.5 Comparison of K^{det} with Westerland's R_d

The calculation of the previous section suggests the presence of an interesting Γ–local spectrum K^{det}, which we now describe. First consider the dual to the annular decomposition, as presented in the following diagram of cofiber sequences:

$$\begin{tikzcd}
C \arrow[r] & M \arrow[r] & M^{\square_C 2} \arrow[r] & M^{\square_C 3} \arrow[r] & \cdots \\
C \arrow[r] \arrow[u] & C \arrow[r] \arrow[u] & C \arrow[r] \arrow[u] & C \arrow[r] \arrow[u] & \cdots \\
* \arrow[r] \arrow[u] & \mathbb{S}^0 \arrow[r] \arrow[u] & XP^1 \arrow[r] \arrow[u, "\beta"] & XP^2 \arrow[r] \arrow[u] & \cdots .
\end{tikzcd}$$

The Γ–local spectra give an increasing filtration of $XP^\infty = C$ generalizing the filtration of infinite complex projective space $\mathbb{C}\mathrm{P}^\infty$ by the finite projective spaces $\mathbb{C}\mathrm{P}^n$. In the case of $C = \mathbb{C}\mathrm{P}^\infty$, the labelled arrow β is the generator of $\pi_2\mathbb{C}\mathrm{P}^\infty$, i.e., the Bott class.

Definition 3.5.1. We take the β appearing in the above diagram to be the "generalized Bott class" of C.

Definition 3.5.2. The K–theory for C is defined to be the localization away from β. In particular, for $C = L_\Gamma\Sigma^\infty_+\underline{H\mathbb{Z}/p^\infty}_d$, we set "determinantal K-theory" to be

$$K^{\mathrm{det}} := \left(L_\Gamma\Sigma^\infty_+\underline{H\mathbb{Z}/p^\infty}_d\right)[\beta^{-1}].$$

Remark 3.5.3. When $d = 1$, $\underline{H\mathbb{Z}/p^\infty}_1$ is p–adically (and hence $\widehat{\mathbb{G}}_m$–locally) indistinguishable from $\underline{H\mathbb{Z}}_2 \simeq \mathbb{C}\mathrm{P}^\infty$. It follows that β agrees with the usual Bott class for $\mathbb{C}\mathrm{P}^\infty$, and hence by Snaith's theorem [66] that K^{det} agrees with p–adic K–theory $KU^\wedge_p$.

Craig Westerland has recently considered a spectrum related to K^{det}:

Definition 3.5.4 ([81, Definition 3.11])**.** Take $p > 2$ and consider the action of $\mathbb{Z}/p^\times$ on $\underline{H\mathbb{Z}/p}_d$ by field multiplication. Averaging this action gives rise to an idempotent in K–homology which splits the suspension spectrum as follows:

$$L_\Gamma\Sigma^\infty_+\underline{H\mathbb{Z}/p}_d \simeq \bigvee_{k=0}^{p-1} Z^{\otimes k},$$

where Z has the property $Z^{\wedge p} \simeq Z$. The spectrum R_d is defined by inverting the element of Picard–graded homotopy determined by the composite

$$Z \to \bigvee_{k=0}^{p-1} Z^{\otimes k} \simeq L_\Gamma\Sigma^\infty_+\underline{H\mathbb{Z}/p}_d \to L_\Gamma\Sigma^\infty_+\underline{H\mathbb{Z}/p^\infty}_d \xrightarrow{\text{Bockstein}} L_\Gamma\Sigma^\infty_+\underline{H(\mathbb{Z}_p)}_{d+1}.$$

Upon picking a coordinate on $\mathbb{CP}^\infty_K$ and applying K–homology to this composite, one sees that it selects the dual of the induced coordinate on $(\underline{H(\mathbb{Z}_p)}_{d+1})_K$.

Theorem 3.5.5. *There is an equivalence* $K^{\det} \simeq R_d$.

Proof. This is an immediate consequence of the definitions and of Remark 3.3.6. □

Remark 3.5.6 ([81, Section 3.9]). Westerland exposes a variety of remarkable features of R_d, the grandest of which is the E_∞–equivalence

$$R_d \simeq E_d^{hS\mathbb{G}_d^\pm},$$

where $\mathbb{G}_d$ is the extension of $\mathbb{S}_d$ by $\mathrm{Gal}(k/\mathbb{F}_p)$ and $S\mathbb{G}_d^\pm$ lies in the fiber sequence

$$1 \to S\mathbb{G}_d^\pm \to \mathbb{G}_d \xrightarrow{\det^\pm} \mathbb{Z}_p^\times \to 1,$$

i.e., the "special" elements. It follows that there is a short resolution:

$$L_\Gamma \to E_d^{hS\mathbb{G}_d^\pm} \xrightarrow{\psi-1} E_d^{hS\mathbb{G}_d^\pm},$$

where ψ is a certain Adams–type operation inherited from the action of $\mathbb{Z}_p^\times$ on $\underline{H(\mathbb{Z}_p)}_{d+1}$ (or, equivalently, the lingering action of $\mathbb{Z}_p^\times$ on the fixed point spectrum).

Remark 3.5.7 ([81, Proposition 4.18]). Westerland also uncovers the cellular filtration dual to the annular filtration described in Definition 3.2.4, but by wholly different means. He demonstrates the existence of a map

$$(\Omega^\infty K^{\det})[1,\infty) \xrightarrow{J^{\det}} BGL_1 L_\Gamma \mathbb{S},$$

an analogue of the classical complex J–homomorphism

$$(\Omega^\infty KU)[1,\infty) \simeq BU \xrightarrow{J_{\mathbb{C}}} BGL_1 \mathbb{S}.$$

He then checks that the Thom spectrum of the restricted "canonical bundle" on $\underline{H(\mathbb{Z}_p)}_{d+1}$ has the following Γ–local homotopy type:

$$\mathrm{Thom}(J^{\det} \downarrow \underline{H(\mathbb{Z}_p)}_{d+1}) \simeq (\mathbb{S}[\det])^{-1} \wedge L_\Gamma \Sigma^\infty \underline{H(\mathbb{Z}_p)}_{d+1}.$$

This is in analogy to the following classical fact about projective spaces [7, Proposition 4.3]:

$$\mathrm{Thom}(m(\mathscr{L}-1) \downarrow \mathbb{CP}^{n-m}) \simeq \Sigma^{-2m} \mathbb{CP}^n_m,$$

which specializes to the directly analogous fact

$$\mathrm{Thom}(\mathscr{L}-1 \downarrow \mathbb{CP}^\infty) \simeq \Sigma^{-2} \Sigma^\infty \mathbb{CP}^\infty.$$

Appendix A

The Morava K–theory of inverse limits

[1]In this section we describe a result, attributed to Hal Sadofsky and expressed in a talk by Mike Hopkins [57, Section 14], concerning the homology of inverse limits certain local systems. Sadofsky's theorem is stated as follows:

Theorem A.1 (Sadofsky). *Let k be a field spectrum, i.e., let k be a ring spectrum with k_* a graded field. Furthermore let $\{X_\alpha\}_\alpha$ be a sequential inverse system of k–local spectra such that $\{k_* X_\alpha\}_\alpha$ is Mittag-Leffler as a system of k_*–modules. There is then a spectral sequence of signature*

$$R^* \lim_\alpha \{k_* X_\alpha\}_\alpha \Rightarrow k_* \lim_\alpha \{X_\alpha\}_\alpha,$$

where the derived inverse limit on the left is taken in an appropriate category of comodules. The spectral sequence is at least *conditionally convergent.*

Remark A.2. The locality is *the* essential assumption. For instance, set $k = H\mathbb{Q}$ and consider the system $\{\mathbb{S}^0/p^j\}_j$, with maps the natural projections. The constituent spaces in this system are all rationally acyclic, but the inverse limit is given by the p-adic sphere $(\mathbb{S}^0)^\wedge_p$. Its rational homology is $H\mathbb{Q}_*(\mathbb{S}^0)^\wedge_p \cong \mathbb{Q}_p$, and hence no such convergent spectral sequence can exist. On the other hand, first rationalizing this system produces the trivial system of zero spectra, and thus the rational homology of the system — which is empty — compares well to the rational homology of the inverse limit — which is also empty. Noting that $\mathbb{S}^0/p^j$ is also known as the Moore spectrum $M_0(p^j)$, similar systems can also be constructed for any Morava K–theory by employing the generalized Moore spectra of Hopkins and Smith [32, Proposition 5.14].

Proof of Sadofsky's theorem for $k = H\mathbb{Q}$. In the rational case, there is a simple proof of Sadofsky's theorem, even without the Mittag-Leffler hypothesis. Much of the complication disappears in this situation, as the homology of an $H\mathbb{Q}$-local system agrees with its homotopy and the algebra of cooperations $H\mathbb{Q}_* H\mathbb{Q} = \mathbb{Q}$ is trivial. Hence, the proposed spectral sequence takes the form

$$R^* \lim_\alpha \{\pi_* X_\alpha\}_\alpha \Rightarrow \pi_* \lim_\alpha \{X_\alpha\}_\alpha,$$

---

[1] None of the material in this section is original; all of it was known to (at least) Hal Sadofsky and Mike Hopkins, and is of "folk lore" status. Additionally, Tobias Barthels lent a hand with some of the proofs.

with the left-hand derived limit interpreted in the category of $\mathbb{Q}$-modules.

The spectrum $\lim_\alpha \{X_\alpha\}_\alpha$ can be written as the fiber in the sequence

$$\lim_\alpha \{X_\alpha\}_\alpha \xrightarrow{\text{fiber}} \prod_\alpha X_\alpha \xrightarrow{\Delta} \prod_\alpha X_\alpha,$$

$$\Delta(x_\alpha)_\alpha = (x_\alpha - x_{\alpha+1})_\alpha.$$

Applying π_* to this fiber sequence begets a long exact sequence of the form

$$\cdots \to \pi_{*+1} \prod_\alpha X_\alpha \xrightarrow{\Delta_{*+1}} \prod_\alpha X_\alpha \to \pi_* \lim_\alpha \{X_\alpha\}_\alpha \to \pi_* \prod_\alpha X_\alpha \to \pi_* \prod_\alpha X_\alpha \to \cdots,$$

whose associated short exact sequence at the middle term is definitionally the Milnor sequence

$$0 \to \lim_\alpha{}^1 \{\pi_{*+1} \prod_\alpha X_\alpha\}_\alpha \to \pi_* \lim_\alpha \{X_\alpha\}_\alpha \to \lim_\alpha \{\pi_* X_\alpha\}_\alpha \to 0.$$

One may reinterpret the groups $\pi_* \prod_\alpha X_\alpha$ as the E^1-page of the cofiltration spectral sequence associated to the diagram

$$\begin{tikzcd}
\lim_\alpha \{X_\alpha\}_\alpha \arrow[r] & \prod_\alpha X_\alpha \arrow[r] \arrow[d, "\Delta"] & \mathrm{pt} \arrow[d] \\
 & \prod_\alpha X_\alpha & \Sigma^{-1} \prod_\alpha X_\alpha,
\end{tikzcd}$$

where the vertical arrows are fibrations and the horizontal arrows are their associated fibers. This spectral sequence degenerates at E^2 with the derived limit groups concentrated in the 0- and 1-lines, and the Milnor sequence appears as the extension problem. □

This proof works just as well in the category of $\mathbb{Z}$-modules, where it recovers the usual $\lim^1$ phenomenon appearing in the homotopy of an inverse limit. In particular, we gain the following lemma:

Lemma A.3. *If $\{X_\alpha\}_\alpha$ is an inverse system of spectra such that $\{\pi_* X_\alpha\}_\alpha$ is a Mittag-Leffler system of abelian groups, then*

$$\pi_* \lim_\alpha \{X_\alpha\}_\alpha \cong \lim_\alpha \{\pi_* X_\alpha\}_\alpha. \quad \square$$

Remark A.4. Thought of as a homology theory, stable homotopy has the trivial coefficient spectrum $\mathbb{S}$, so that

$$\pi_* X = \pi_*(\mathbb{S} \wedge X) \cong \mathbb{S}_* X.$$

In general, a homology theory has some nontrivial coefficient spectrum, and what follows below can be thought of as an analysis of a very particular situation where interchanging inverse limits and those smash products can be controlled by derived inverse limits of comodules. In trade, however, we will impose conditions so that the $\lim^1$ appearing in the Milnor sequence above vanishes, allowing us to smooth out various complexities that obstruct the desired identification of the E_2 page.

Remark A.5. For any sequence of spectra (X_α), the inverse system $\{Y_\alpha := \prod_{\beta \le \alpha} X_\beta\}$ with maps given by projections is Mittag-Leffler. Then, using the fiber sequence $\lim_\alpha X_\alpha \to \prod_\alpha X_\alpha \to \prod_\alpha X_\alpha$, applying k-homology gives

$$\cdots \to k_* \lim_\alpha X_\alpha \to k_*\left(\prod_\alpha X_\alpha\right) \to k_*\left(\prod_\alpha X_\alpha\right) \to \cdots .$$

The middle and right-hand terms of this sequence are calculable by Sadofsky's theorem, even if the inverse system $\{X_\alpha\}$ is itself not Mittag-Leffler, giving some foothold on that case as well.

Remark A.6. A silly feature of this appendix is that we use it only to prove Theorem 3.2.1, which only applies Sadofsky's spectral sequence in a highly degenerate case, and then only to check the convergence of Theorem 3.1.22. The spectral sequences constructed in this section promise to encode a large amount of highly interesting information (cf. Hopkins's talk on the subject [57, Section 14]), but we do not make use of any of this in the body of this document.

Homological algebra for inverse systems of comodules

The above simplicial spectral sequence is the construction we seek to generalize, but we will first set about making sense of derived inverse limits of sequential systems of comodules for a Hopf algebroid so that we understand the source in question. The homological algebra of comodules is well documented elsewhere [62, Appendix 1], but the homological algebra of diagrams of comodules is more scarce. Though we are interested in the case of a Hopf algebroid (E_*, E_*E) stemming from Lemma 1.3.5, in this section we will refer to this pair opaquely as (A, Γ) with Γ flat over A. To begin, we recall some classical results.

Lemma A.7 ([62, Lemma A1.2.1-2]). *A comodule Y of the form $Y = \Gamma \otimes_A Y'$ for an A-module Y' is said to be an extended comodule. This construction gives an adjunction*

$$\mathsf{Modules}_A(M, Y') \cong \mathsf{Comodules}_{(A,\Gamma)}(M, Y).$$

Furthermore, if I is an injective A-module, then $\Gamma \otimes_A I$ is an injective Γ-comodule, and hence $\mathsf{Comodules}_{(A,\Gamma)}$ *has enough injectives.* □

Corollary A.8. *If M is a sequential inverse system of A-modules and $\Gamma \otimes_A M$ is the induced system of extended Γ-comodules, then there is an isomorphism*

$$\lim M \cong \lim(M \otimes_A \Gamma),$$

where the left- and right-hand limits are taken in $\mathsf{Modules}_A$ *and* $\mathsf{Comodules}_{(A,\Gamma)}$ *respectively.* □

Remark A.9. The adjunction in Lemma A.7 should be thought of as geometrically analogous to the adjunction

$$\mathsf{Sets}(X, Y) \cong G\text{-}\mathsf{Sets}(G \times X, Y).$$

Now consider the category $\mathsf{Comodules}^{\mathbb{N}}_{(A,\Gamma)}$ of sequential inverse systems of comodules, where $\mathbb{N}$ denotes the category associated to the natural numbers with their partial ordering.

Lemma A.10 (Sadofsky). *The category* $\mathsf{Comodules}^{\mathbb{N}}_{(A,\Gamma)}$ *has enough injectives. That is, for X a sequential inverse system of (A,Γ)-comodules, there is a levelwise injection to an injective system J.*

Proof. Begin by, for each $n \in \mathbb{N}$, choosing an injection $j'_n : X(n) \to J'_n$ with J'_n an injective comodule. We form a diagram J equipped with a levelwise injection $j : X \to J$ by setting $J(n) = \prod_{i=1}^{n} J'_i$, with the map $J(n+1) \to J(n)$ specified by

$$\prod_{i=1}^{n+1} J'_i \xrightarrow{\left(\prod_{i=1}^{n} 1_{J'_i}\right) \times 0_{J'_{n+1}}} \prod_{i=1}^{n} J'_i$$

and the structure map by

$$X(n) \xrightarrow{\prod_{i=1}^{n}(j'_i \circ x_i^n)} \prod_{i=1}^{n} J'_i,$$

where x_i^n is the morphism specified by the diagram of signature

$$x_i^n : X(n) \to X(i).$$

We now check that this diagram has the relevant lifting property:

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d] & J \\ Y \arrow[ur, dotted, "\exists k"'] & \end{tikzcd}$$

whenever the vertical arrow is a levelwise injection. We argue inductively, beginning with the case $n = 1$. In that case, the diagram reduces to

$$\begin{tikzcd} X(1) \arrow[r, "j'_1"] \arrow[d] & J(1) \\ Y, \arrow[ur, dotted, "\exists k'_1"'] & \end{tikzcd}$$

which is precisely the diagram describing the classical injectivity condition. Because $J(1)$ was selected to be an injective comodule under $X(1)$, such an extension exists. In the case of a general n, we have the following diagram:

$$\begin{tikzcd}
X(n-1) \arrow[r, "\prod_{i=1}^{n-1}(j'_i \circ x_i^{n-1})"] \arrow[dd] & \prod_{i=1}^{n-1} J'_i & \\
& X(n) \arrow[ul] \arrow[r, "\prod_{i=1}^{n}(j'_i \circ x_i^{n})"] \arrow[dd] & \prod_{i=1}^{n} J'_i \arrow[ul, "(\prod_{i=1}^{n-1} 1_{J'_i}) \times 0_{J'_n}"'] \\
Y(n-1) \arrow[uur, "k(n-1)"'] & & \\
& Y(n). \arrow[ul] \arrow[uur, dotted] &
\end{tikzcd}$$

The dashed map is specified by a pair of morphisms $Y(n) \to \prod_{i=1}^{n-1} J'_i$ and $Y(n) \to J'_n$. The former arrow is specified by restriction. We also have the following diagram of injective comodules

$$\begin{tikzcd}
X(n) \arrow[r, "j'_n"] \arrow[d] & J'_n \\
Y(n), \arrow[ur, dotted, "\exists"] &
\end{tikzcd}$$

which allows us to select the other arrow. □

Remark A.11. It is not true that a diagram whose objects consist of levelwise injective comodules is always an injective object in sequential inverse systems of comodules. The construction above is designed to skirt past questions of compatibility of levelwise lifts. Relatedly, the above proof generalizes to "well-founded" inverse systems, i.e., inverse systems indexed on diagram categores where each object is the source of finitely many morphisms, so that induction still applies.

Because we have enough injectives, the general machinery of homological applies to produce right-derived functors of the left-exact inverse limit functor

$$\lim : \mathsf{Comodules}_{(A,\Gamma)}^{\mathbb{N}} \to \mathsf{Comodules}_{(A,\Gamma)}.$$

Additionally, there is a cobar complex which performs this computation. To see this, we first remark on a consequence of the adjunction of Lemma A.7.

Lemma A.12 (Sadofsky)**.** *If M is a Mittag-Leffler sequential inverse system of A-modules, then the system $M \otimes_A \Gamma$ of extended Γ-comodules has no higher derived limits, i.e., it is flasque.*

Proof. In the usual way, M can be resolved by a double complex J such that $J(n,*)$ gives an injective resolution of $M(n)$ and $J(*,m)$ is itself a Mittag-Leffler sequential inverse system. Each system $J_m = J(*,m)$ has the desired property, which we can verify by constructing the following resolution:

$$\begin{tikzcd}
0 \arrow[r] & J_m(n) \arrow[r] \arrow[d] & 0 & \\
0 \arrow[r] & \prod_{n' \leq n} J_m(n') \arrow[r] & \prod_{n' < n} J_m(n') \arrow[r] & 0.
\end{tikzcd}$$

This is an exact resolution of J_m by injectives in $\mathsf{Modules}_A^{\mathbb{N}}$, hence tensoring up with the flat module Γ gives an exact resolution of $J_m \otimes A$ by injectives in $\mathsf{Comodules}_{(A,\Gamma)}^{\mathbb{N}}$. Applying lim to this resolution of comodules and using Corollary A.8, we have

$$\lim\left(\left(\prod_{n' \leq n} J_m(n')\right) \otimes_A \Gamma \to \left(\prod_{n' < n} J_m(n')\right) \otimes_A \Gamma \to 0\right) =$$

$$\left((\lim J_m) \oplus \prod_n J_m(n)\right) \otimes_A \Gamma \xrightarrow{\text{project onto second factor}} \left(\prod_n J_m(n)\right) \otimes_A \Gamma \to 0,$$

which is visibly flasque.

To compute the derived inverse limits of $M \otimes_A \Gamma$ itself, we use the resolution J_* of M:

$$R^* \lim(M \otimes_A \Gamma) = H^*(\lim(J_* \otimes_A \Gamma)) = H^*((\lim J_*) \otimes_A \Gamma).$$

Because M is Mittag-Leffer and J_* is a resolution of M in A-modules, $H^* \lim J_*$ is concentrated in degree zero. Finally, because Γ is a flat A-module, $- \otimes_A \Gamma$ preserves exact sequences, and so the complex of comodules $(\lim J_*) \otimes_A \Gamma$ also has cohomology concentrated in degree zero, i.e., $M \otimes_A \Gamma$ is flasque. ☐

Corollary A.13 (Sadofsky). *Write $\Omega(\Gamma; \overline{\Gamma}; -)$ for the usual one-sided cobar cochain complex with*

$$\Omega(\Gamma; \overline{\Gamma}; -)[n] = \Gamma \otimes_A \overline{\Gamma}^{\otimes_A n} \otimes_A -.$$

If X is a sequential inverse system of comodules satisfying the Mittag-Leffler condition, then

$$R^s \lim X = H^s \lim \Omega(\Gamma; \overline{\Gamma}; X),$$

where the right-hand object is interpreted as the total complex of the obvious double complex stemming from Ω and X. ☐

Sadofsky's theorem for finite height Morava K-theories

Now we will discuss a similar theorem communicated to the author by Mike Hopkins [57, Section 14] as a stepping stone toward Sadofsky's theorem for Morava K-theories.

Theorem A.14 (Hopkins). *Let $E(d)$ be a Johnson-Wilson spectrum and $\{X_\alpha\}_\alpha$ be a system of $E(d)$-local spectra such that $\{E(d)_* X_\alpha\}_\alpha$ is Mittag-Leffler. There is then a convergent spectral sequence of signature*

$$R^* \lim_\alpha \{E(d)_* X_\alpha\}_\alpha \Rightarrow E(d)_* \lim_\alpha \{X_\alpha\}_\alpha,$$

where the derived inverse limit on the left is taken in the category of $E(d)_* E(d)$-comodules.

The proof of this theorem relies on some shorter results, useful in their own right. Our first subgoal is to show that the $E(d)$-homology of $E(d)$-modules results in an extended comodule, which gives us access to the limit trick in Corollary A.8.

Lemma A.15 (Angeltveit)**.** *The Brown–Peterson spectrum BP and the Johnson–Wilson spectrum $E(d)$ are A_∞-ring spectra and the orienting map $BP \to E(d)$ is a map of A_∞-rings.*

Proof. First, we assemble results about A_∞-multiplications on chromatic spectra:

1. It follows from work of Lazarev [44, Theorem 5.18] that the Brown–Peteron spectrum BP has an A_∞-multiplication.
2. Then, work of Angeltveit [5, Corollary 3.7] applies to show that the truncated Brown–Peterson spectrum $BP\langle d\rangle$ supports an A_∞-map from BP.
3. Finally, work of May et al. [20, Proposition V.2.3], shows that spectrum $v_d^{-1}BP$ receives an A_∞-map from BP as well.

Putting these facts together, it then follows that $E(d) \simeq v_d^{-1}BP \wedge_{BP} BP\langle d\rangle$ is an A_∞-ring with A_∞-orientation by BP. □

Lemma A.16. *Let M be an $E(d)$-module spectrum; then there is a natural isomorphism*

$$E(d)_*M \cong E(d)_*E(d) \otimes_{E(d)_*} \pi_*M.$$

Proof. Using Lemma A.15, work of May et al. [20, Theorem IV.4.1] shows that for a right $E(d)$-module spectrum N and left $E(d)$-module spectrum M there is a strongly convergent spectral sequence

$$\operatorname{Tor}_{*,*}^{E(d)_*}(N, M) \Rightarrow \pi_*(N \wedge_{E(d)} M).$$

Taking $N = E(d) \wedge E(d)$ and noting that $\pi_*N = E(d)_*E(d)$ is a flat right $E(d)_*$-module, the specialized spectral sequence

$$\operatorname{Tor}_{*,*}^{E(d)_*}(E(d)_*E(d), E(d)_*M) \Rightarrow \pi_*((E(d) \wedge E(d)) \wedge_{E(d)} M)$$

is concentrated on the 0-line and collapses at E^2. Using the freeness of N, this collapse gives

$$E(d)_*E(d) \otimes_{E(d)_*} \pi_*M \cong \pi_*(E(d) \wedge E(d) \wedge_{E(d)} M) \cong \pi_*(E(d) \wedge M) = E(d)_*M. \qquad \square$$

Remark A.17. There is also an unstructured proof of Lemma A.16, using the fact that tensoring against a flat module preserves long exact sequences and hence takes Brown representable functors to Brown representable functors.

Corollary A.18. *If $\{M_\alpha\}_\alpha$ is a system of $E(d)$-module spectra which is Mittag-Leffer on homotopy, then there is an isomorphism*

$$E(d)_* \lim_\alpha \{M_\alpha\}_\alpha \cong \lim_\alpha \{E(d)_*M_\alpha\},$$

where the right-hand limit occurs in the category of $E(d)_*E(d)$-comodules.

Proof. Combine Lemma A.15 and Lemma A.16:

$$\begin{aligned}
E(d)_* \lim_\alpha \{X_\alpha\}_\alpha &\cong E(d)_*E(d) \otimes_{E(d)_*} \pi_* \lim_\alpha \{M_\alpha\}_\alpha && \text{(Lemma A.16)} \\
&\cong E(d)_*E(d) \otimes_{E(d)_*} \lim_\alpha \{\pi_* M_\alpha\}_\alpha && \text{(Mittag-Leffler assumption)} \\
&\cong \lim_\alpha \{E(d)_*E(d) \otimes_{E(d)_*} \pi_* M_\alpha\}_\alpha && \text{(Corollary A.8)} \\
&\cong \lim_\alpha \{E(d)_* M_\alpha\}_\alpha. && \text{(Lemma A.16)}
\end{aligned}$$

□

Our next goal is to find a topological object to which we can apply Corollary A.13. This will be a certain Adams–type spectral sequence, and because we have not really used that tool in this paper, we quickly remind the reader of its construction.

Definition A.19. For a ring spectrum E and spectrum X, the following diagram describes the E–Adams tower for X:

$$\begin{tikzcd}
X \arrow[d] & \overline{E} \wedge X \arrow[l] \arrow[d] & \overline{E}^{\wedge 2} \wedge X \arrow[l] \arrow[d] & \overline{E}^{\wedge 3} \wedge X \arrow[l] \arrow[d] & \cdots \arrow[l] \\
E \wedge X & E \wedge \overline{E} \wedge X & E \wedge \overline{E}^{\wedge 2} \wedge X & E \wedge \overline{E}^{\wedge 3} \wedge X & \cdots,
\end{tikzcd}$$

where $\overline{E} \to S \xrightarrow{\eta_E} E$ describes the fiber of the unit map.

Remark A.20. In good cases, this spectral sequence converges to homotopy of the E–nilpotent completion of X [13, Proposition 6.3]. In better cases, the homotopy of the E–nilpotent completion of X agrees with that of the Bousfield E–localization of X [13, Corollary 6.13]. In better cases still, the E^2–page of the spectral sequence can be identified ([62, Theorem 2.2.11], cf. also Remark 1.3.27) as

$$E^2_{*,*} \cong \operatorname{Cotor}^{E_*E}_{*,*}(E_*, E_*X).$$

We are now in a position to construct the spectral sequence in Hopkins's inverse limit theorem.

Definition A.21. Suppose that $\{X_\alpha\}_\alpha$ is an inverse system of $E(d)$–local spectra with the induced system $\{E(d)_*X_\alpha\}_\alpha$ Mittag-Leffler. We then have the interlocking fiber sequences

$$\begin{tikzcd}
\lim X_\alpha \arrow[d] & \lim \overline{E(d)} \wedge X_\alpha \arrow[l] \arrow[d] & \lim \overline{E(d)}^{\wedge 2} \wedge X_\alpha \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\lim E(d) \wedge X_\alpha & \lim E(d) \wedge \overline{E(d)} \wedge X_\alpha & \lim E(d) \wedge \overline{E(d)}^{\wedge 2} \wedge X_\alpha & \cdots
\end{tikzcd}$$

which upon applying $E(d)$–homology gives a spectral sequence with target $E(d)_* \lim X_\alpha$ and E^1–page

$$E^1_{*,t} = E(d)_* \lim\left(E(d) \wedge \overline{E(d)}^{\wedge t} \wedge X_\alpha\right).$$

We combine the preceding corollaries to compute

$$E(d)_* \lim\left(E(d) \wedge \overline{E(d)}^{\wedge t} \wedge X_\alpha\right) \cong E(d)_* E(d) \otimes_{E(d)_*} \left(E(d)_* \overline{E(d)}\right)^{\otimes_{E(d)_*} t} \otimes_{E(d)_*} \lim E(d)_* X_\alpha$$

and hence by Corollary A.13

$$E^2_{*,*} \cong R^* \lim E(d)_* X_\alpha.$$

Lemma A.22. *The $E(d)$–Adams resolution for the $E(d)$–local sphere $L_d\mathbb{S}^0$ is equivalent to a finite–dimensional cosimplicial resolution by $E(d)$–module spectra. (In particular, the $E(d)$–Adams spectral sequence has a horizontal vanishing line.)*

Proof. Work of Ravenel [63, Lemmas 8.3.7 and 8.3.1] gives an L_dBP–prenilpotent finite spectrum F whose ordinary homology is torsion–free. Since prenilpotent spectra form a thick subcategory, it follows from their classification [32, Theorem 9] that $\mathbb{S}^0$ is L_dBP–prenilpotent. Since L_dBP and $E(d)$ share a Bousfield class [63, Theorem 7.3.2b and Lemma 8.1.4], it follows that $L_d\mathbb{S}^0$ is thus $E(d)$–nilpotent. Also definitionally [63, Definition 7.1.6], this means that $L_d\mathbb{S}^0$ has a finite $E(d)$–Adams resolution. □

Corollary A.23. *Every $E(d)$–local spectrum X has a finite–dimensional cosimplicial resolution by $E(d)$–module spectra. The length of the resolution is independent of X and dependent only on the prime p and height n.*

Proof. Noting that $X \simeq L_dX \simeq X \wedge L_d\mathbb{S}^0$ (cf. the first part of Theorem 1.4.1), smash the finite resolution for $L_d\mathbb{S}^0$ guaranteed by Lemma A.22 with X. □

Proof of Theorem A.14. Having constructed the relevant spectral sequence in Definition A.21, we need only address convergence. Corollary A.23 shows that the homotopy inverse system in Definition A.21 is weakly equivalent to a finite inverse system. It follows that, upon applying $E(d)$–homology, the resulting spectral sequence is concentrated in a finite horizontal band and hence is strongly convergent to $E(d)_* \lim_\alpha X_\alpha$. □

Let M be a "finite" $E(d)$–module in the sense that it admits a presentation as $M \simeq \operatorname{Tot} M^\bullet$, where $M^\bullet$ is a levelwise–free finite–dimensional cosimplicial $E(d)$–module. The proof of Hopkins's theorem can be modified slightly to accommodate taking M–homology rather than $E(d)$–homology. As this is one extra finite limit, it does not introduce any real complications.

Proof of Sadofsky's theorem for $k = K(d)$. With this observation in hand, we need only show that $K(d)$ admits such a resolution. One definition of $K(d)$ is as the iterated cofiber:

$$\begin{aligned} E(0,d) &= E(d), \\ E(j+1,d) &= \operatorname{cofib}(\Sigma^{|v_j|} E(j,d) \xrightarrow{v_j} E(j,d)), \\ E(d,d) &= K(d). \end{aligned}$$

Since cofiber and fiber sequences agree in the categories of spectra and of $E(d)$–modules, tensoring these cofiber sequences together over $\mathbb{S}$ (i.e., constructing a Koszul complex) produces a cosimplicial object which resolves $K(d)$ and is levelwise free. □

Remark A.24. An odd wrinkle of this construction is that the inverse limit of $\{K(d)_*X_\alpha\}_\alpha$ is still taken in the category of $E(d)_*E(d)$-comodules, *not* of $K(d)_*K(d)$-comodules. However, the $E(d)_*E(d)$-comodule structure of $K(d)_*X_\alpha$ factors through the Hopf algebroid $(K(d)_*, \Gamma')$, where Γ' is given by

$$\begin{aligned}\Gamma' = K(d)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} K(d)_* = K(d)_*[t_1, t_2, \ldots]/(v_d t_j^{p^d} - v_d^{p^j} t_j \mid j > 0) \\ \subsetneq K(d)_*K(d) = \Gamma' \otimes \Lambda[\tau_1, \ldots, \tau_{d-1}].\end{aligned}$$

The reader should compare these stray τ_* cooperations with the Bockstein operations in Section 1.4.

Remark A.25. Sadofsky's theorem would follow immediately from Hopkins's theorem without further mention of $E(d)$-module spectra if Smith–Toda complexes were available to us. A Smith–Toda complex is a finite spectrum $V(d)$ with the property that

$$BP_*V(d) \cong BP_*/(p, v_1, \ldots, v_{d-1}),$$

and hence that

$$E(d)_*V(d) \cong E(d)_* \otimes_{BP_*} BP_*V(d) \cong E(d)/(p, v_1, \ldots, v_{d-1}) \cong K(d).$$

Since $V(d)$ is finite and since both inverse limits and localizations commute with smashing against finite spectra, we could replace our system $\{X_\alpha\}_\alpha$ with $\{X_\alpha \wedge V(d)\}_\alpha$ and apply Hopkins's theorem, pulling $V(d)$ to the outside where appropriate. However, such complexes do not exist in general; see Nave's thesis [55]. Nonetheless, it may still be possible to find a second proof of the above theorem by constructing a finite complex $F(d)$ such that $E(d) \wedge F(d)$ contains $K(d)$ as a wedge summand. The v_d-self-maps constructed in Hopkins–Smith [32, Section 3] are too evenly spaced to conclude the existence of such a complex, but they do not show that their construction is the "best possible" in this sense.

Sadofsky's theorem for ordinary homology with field coefficients

In the case $k = HK$ for a field K of positive characteristic, all of the above constructions can be re-done to produce a derived inverse limit spectral sequence for HK-homology. However, our convergence argument fails badly, as the p-complete sphere is no longer finitely resolvable by HK-module spectra — after all, the HK-Adams spectral sequence has both an infinite tower and a vanishing line of slope 1, rather than the horizontal vanishing line present in the $E(d)$-Adams spectral sequence. It follows that the resultant spectral sequence is merely conditionally convergent, and additional hypotheses on the system $\{X_\alpha\}_\alpha$ are required to do any better. In spite of the lack of topological finiteness, Sadofsky has proven the following theorem whose proof we will not recount here:

Theorem A.26 (Sadofsky)**.** *The HK-based inverse limit spectral sequence converges strongly in the case that $R^s \lim_\alpha\{H_*(X_\alpha)\}_\alpha = 0$ for $s \gg 0$.* □

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