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Multiplicative 2-cocycles at the prime 2

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ABSTRACT

Using a previous classification result on symmetric additive 2-cocycles, we collect a variety of facts about the Lubin–Tate cohomology of certain formal groups to produce a presentation of the 2-primary component of the scheme of symmetric multiplicative 2-cocycles. This scheme classifies certain kinds of highly symmetric multiextensions, generalizing those studied by Mumford or Breen. A low-order version of this computation has previously found application in homotopy theory through the σ-orientation of Ando, Hopkins, and Strickland, and the complete computation is reflective of certain structures found in the homotopy type of connective K-theory.

1. Introduction

In manifold geometry, there has been a history of marriages between certain structures on real vector bundles and quantities in analytic geometry, indexed by cohomology theories. For ordinary cohomology, the structure one requires is an orientation of the vector bundle in the classical sense; for real K-theory, one considers vector bundles with *Spin* structure, and for elliptic cohomology, one considers *String* bundles. On the level of homotopy theory, a vector bundle V over M is classified by a homotopy class $M \to BO$, and fitting the bundle with these extra structures corresponds to providing lifts $M \to BSO$, $M \to BSpin$, and $M \to BString$ respectively. The spaces BSO, $BSpin$, and $BString$ appear as spaces in the real connective K-theory Ω-spectrum kO, and by analogy one can tell a similar story for complex vector bundles by lifting a classifying map $M \to BU$ to spaces in the complex connective K-theory Ω-spectrum kU.

In order to produce characteristic classes for these bundles and to study ring spectra which are oriented against them, the homology $E_*\underline{kU}_{2k}$ and cohomology $E^*\underline{kU}_{2k}$ become objects of interest. Ando et al. [2] have described the group schemes $\operatorname{Spec} E_*\underline{kU}_{2k}$ in terms of certain schemes $C^k(G_E; \hat{\mathbb{G}}_m)$, a moduli of certain highly symmetric extensions of formal groups. For each k they produce a map

$$\operatorname{Spec} E_*\underline{kU}_{2k} \to C^k(G_E; \hat{\mathbb{G}}_m),$$

and they additionally demonstrate that this map is an isomorphism in the range $0 \le k \le 3$. For $k = 3$, the scheme $C^3(G_E; \hat{\mathbb{G}}_m)$ classifies cubical structures, which connects back to the geometry of elliptic curves through theorems in arithmetic geometry [3,11], and hence relays important information about elliptic cohomology.

Their proof relies on explicit calculation of the ring of functions on the affine scheme $C^k(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_m)$, which at the time could only be completed through this range of values of k up to 3. The main theorem of this paper is to describe $\mathcal{O}C^k(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_m) \otimes \mathbb{Z}_{(2)}$ for all $k \in \mathbb{N}$. To do so requires some elementary number theory and combinatorics, and in the end we arrive at the following.

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Theorem 1. *Set the following notation.*

1. *Let $\phi(n, k)$ denote the greatest common factor of the multinomial coefficients $\binom{n}{\lambda}$ where λ is a partition of n into k positive integers.*
2. *Writing a natural number c as $c = p^a d$ where $p \nmid d$, set $\nu_p c = a$, the number of times c is divisible by p.*
3. *Define $D_{n,k}$ to be the coefficient of the generating function $\prod_{i=0}^{\infty}(1 - tx^{2^i})^{-1} = \sum_{n,k} D_{n,k} x^n t^k$. This counts the number of ways n can be written as a sum of k many powers of 2.*
4. *Let $\sigma_p(n)$ be the p-adic digital sum of the integer n, so that for example $\sigma_2(2) = \sigma_2(10_2) = 1$ and $\sigma_2(3) = \sigma_2(11_2) = 2$. Then, let $\gamma_p(n, k)$ be the integer defined by $\gamma_p(n, k) = \max\{0, \min\{k - \sigma_p(n), \nu_p(n)\}\}$. The utility of this number is captured by Fig. 3; it counts up until the preconditions of Theorem 7.1 are satisfied.*
5. *Let $\Gamma[x]$ denote the divided power algebra on the generator x. This is defined by taking the module $\mathbb{Z}_{(2)}\{x^{[0]}, x^{[1]}, \ldots\}$ with algebra structure given by $x^{[i]} \cdot x^{[j]} = \binom{i+j}{i} x^{[i+j]}$. It comes with a power series $\exp_x(t) \in \Gamma[x][\![t]\!]$ given by $\sum_i x^{[i]} t^i$, satisfying $\exp_x(s + t) = \exp_x(s) \cdot \exp_x(t)$.*

Given this,

$$\begin{aligned} \mathcal{O}(\operatorname{Spec} \mathbb{Z}_{(2)} \times C^k(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_m)) &= \mathbb{Z}_{(2)}[z_n \mid \nu_2\phi(n, k) \le \nu_2 n] \\ &\quad \otimes \Gamma[b_{n,\gamma_2(n,k)} \mid \nu_2\phi(n, k) > \nu_2 n] \\ &\quad \otimes \mathbb{Z}_{(2)}[b_{n,i} \mid \gamma_2(n, k) < i < D_{n,k}]/\langle 2b_{n,i}, b_{n,i}^2 \rangle. \end{aligned}$$

The universal cocycle over this scheme is given by a product $\prod_i f_i \cdot \prod_j g_j \cdot \prod_\ell h_\ell$, corresponding to the three tensor factors. In this presentation, z_i corepresents the leading nonconstant coefficient of f_i, defined by the Artin–Hasse exponential in Theorem 7.1; see also [2, Corollary 3.22]. The element $b_{n,\gamma_2(n,k)}^{[1]}$ corepresents the leading nonconstant coefficient of the divided power exponential $\exp_{b_{n,\gamma_2(n,k)}}(\zeta_k^n)$ for a certain polynomial ζ_k^n given in Definition 4.7. Finally, $b_{n,i}$ corepresents the leading coefficient of the series $h_\ell = 1 + b_{n,i} \cdot h'_\ell$, where h'_ℓ can be taken to be a certain symmetric monomial.

1.1. Outline of the paper

The R-valued points u of $C^k(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_m)$ can be viewed as a power series satisfying certain criteria. In the previous paper [6], we produced a description of another scheme $C^k(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_a)$, whose R-valued points describe the leading nonconstant parts of such u. Thus, the main challenge of this paper is to investigate which of those polynomials can be viewed as "belonging to" some power series u, a question we approach by setting up an obstruction theory.

In Section 2 and Section 3, we recall various key definitions from algebraic geometry, including that of formal schemes, Lubin–Tate cohomology of formal Lie groups, multiextensions, and higher order cubical structures. Then, in Section 4, we compute the tangent spaces to formal Lie group cohomology $T_1 H^*(F; G)$ and the symmetric cocycle scheme $T_1 C^*(F; G)$ for the formal Lie groups $F = \hat{\mathbb{G}}_a$ and $G = \hat{\mathbb{G}}_m$; this first calculation is done in the style of Hopkins and the second is the content of our previous paper. The cohomological calculation is then used as input to a "tangent spectral sequence"

$$T_1 H^*(F; G) \Rightarrow H^*(F; G),$$

described in Section 5, where we produce a family of nonvanishing differentials on certain key classes. In Section 6, we recall some geometry related to Weil forms, which is certainly known to experts, but does not appear to be available in the literature. The important result for us is the existence of a certain asymmetric $(k - 1)$-variate cocycle e, which we call the half-Weil pairing, associated to any k-variate cocycle u. Together, e and u satisfy the two relations

$$\delta_1 e = u, \quad e = \prod_{i=1}^{p-1} u(ix_1, x_1, \ldots, x_{n-1}).$$

Section 7 is dedicated to proving the main theorem, which is a blend of everything that came before.

Proof (*Sketch*)**.** Suppose we start with such a 2-cocycle u over an $\mathbb{F}_2$-algebra. We then construct a point $[e_+]$ in the tangent space $T_1 H^k(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_m)$, which belongs to the sources of our family of differentials. Hence, $[e_+]$ is obstructed from lifting to e, which in turn is obstructed from satisfying $u = \delta_1 e$, unless certain conditions are met so that the differentials disappear. In the interesting case, this means that the leading coefficient must square to zero. With this in hand, we are then able to read off which classes in $T_1 C^k(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_m)$ lift unobstructed and which become obstructed, giving the description of $\mathcal{O}(\operatorname{Spec} \mathbb{F}_2 \times C^k(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_m))$. This immediately yields a description of $\mathcal{O}(\operatorname{Spec} \mathbb{Z}_{(2)} \times C^k(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_m))$ as a consequence of the previous calculation of $\operatorname{Spec} \mathbb{Z}_{(2)} \times C^k(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_a)$. □

These techniques also produce partial data at odd primes, and we discuss this and several other phenomena in Section 8.

2. Formal groups

Definition 2.1. Fix a commutative ring R with unit, and consider the category $\mathsf{AdicAlgebras}_{/R}$ of augmented R-algebras, complete and separated in the adic topology induced by powers of their augmentation ideal, with continuous, unity-preserving algebra homomorphisms. The category of "formal schemes over Spec R" is defined as the full subcategory of objects which are ind-representable in the following category of presheaves:

$$\mathsf{FormalSchemes}_{/\mathrm{Spf}\,R} \subseteq \mathrm{Hom}\,(\mathsf{AdicAlgebras}_{/R}, \mathsf{Sets}).$$

The Yoneda embedding $A \mapsto \mathsf{AdicAlgebras}^{cts}_{/R}(A, -)$ is denoted by Spf A.

Remark 2.2. Throughout, we will endeavor to be careful to distinguish actual formal schemes from presheaves that fail to satisfy ind-representability. Without ambiguity, we will refer to such objects simply as presheaves.

Lemma 2.3. *This categories satisfies the properties below.*

1. *This category is cocomplete.*
2. *This category has an internal function object:*

$$\underline{\mathrm{Maps}}\,(X, Y)(S) = \{(u, f) \mid u : \mathrm{Spf}\,S \to \mathrm{Spf}\,R, f : u^*X \to u^*Y\}.$$

3. *This satisfies the exponential relation*

$$\mathrm{Maps}\,(X \times_{\mathrm{Spf}\,R} Y, Z) \cong \mathrm{Maps}\,(X, \underline{\mathrm{Maps}}\,(Y, Z)).$$

Proof. A good reference for these facts — indeed, for this entire section — is Strickland [15, Section 2]. □

Definition 2.4. Formal affine k-space is defined to be $\hat{\mathbb{A}}^k = \mathrm{Spf}\,R[\![x_1, \ldots, x_k]\!]$. A formal variety V is a formal scheme noncanonically isomorphic to $\hat{\mathbb{A}}^n$ for some n. A coordinate on V is a selected such isomorphism $\hat{\mathbb{A}}^k \xrightarrow{\cong} V$. A formal Lie group is a commutative group object in the category of formal varieties; additionally, all formal Lie groups considered here will be 1-dimensional, i.e., isomorphic as formal schemes to $\hat{\mathbb{A}}^1$.

Definition 2.5. The Lubin–Tate cochains (see Lubin and Tate [9, Definition 2.2]) of a pair of formal Lie groups $(F; G)$ is defined as the ordinary affine scheme

$$A^k(F; G) = \underline{\mathrm{Maps}}\,(F^{\times k}, G).$$

There is a structure of cosimplicial object on A^* coming from the group operation in F; hence there are maps $\delta^k : A^k(F; G) \to A^{k+1}(F; G)$ forming a cochain complex upon evaluation on an algebra S. The kernel of δ^k is the group of Lubin–Tate k-cocycles, denoted by $Z^k(F; G)$, and the image is the group of Lubin–Tate k-coboundaries, denoted by $B^k(F; G)$. The object Z^k is again an ordinary affine scheme, but B^k may merely be a presheaf.

There is a further collection of presheaves $H^*(F; G)$ defined by the presheaf quotient

$$H^k(F; G)(S) = \frac{\ker \delta^k : A^k(F; G)(S) \to A^{k+1}(F; G)(S)}{\mathrm{im}\,\delta^{k-1} : A^{k-1}(F; G)(S) \to A^k(F; G)(S)} = \frac{Z^k(F; G)(S)}{B^k(F; G)(S)}.$$

Definition 2.6. The Lubin–Tate k-variate symmetric 2-cocycle group $C^k(F; G)$ is an affine subscheme of $\underline{\mathrm{Maps}}\,(F^{\times k}, G)$ consisting of points $f : F^{\times k} \to G$ with $f(\sigma x) = f(x)$ and

$$\begin{aligned} f(x_1, x_2, x_3, \ldots) &-_G \\ f(x_0 +_F x_1, x_2, x_3, \ldots) &+_G \\ f(x_0, x_1 +_F x_2, x_3, \ldots) &-_G \\ f(x_0, x_1, x_3, \ldots) &= 1_G. \end{aligned}$$

Definition 2.7. In the language of formal schemes, $\operatorname{Spf} R[\varepsilon]/\varepsilon^2$ plays the role of a point equipped with a tangent vector. The tangent bundle TX of a scheme X is then defined as

$$TX = \underline{\text{Maps}}\,(\operatorname{Spf} R[\varepsilon]/\varepsilon^2, X).$$

Note then that

$$\begin{aligned} TX(S) &= \text{Maps}\,(\operatorname{Spf} S, \underline{\text{Maps}}\,(\operatorname{Spf} R[\varepsilon]/\varepsilon^2, X)) \\ &= \text{Maps}\,(\operatorname{Spf} S \times_{\operatorname{Spf} R} \operatorname{Spf} R[\varepsilon]/\varepsilon^2, X) \\ &= \text{Maps}\,(\operatorname{Spf} S[\varepsilon]/\varepsilon^2, X). \end{aligned}$$

Given an R-valued point $x : \operatorname{Spf} R \to X$, the tangent space at x is defined to be the subscheme of TX restricting to x along the map $\operatorname{Spf} R \to \operatorname{Spf} R[\varepsilon]/\varepsilon^2$ induced by $\varepsilon \mapsto 0$. Equivalently, it is the pullback of the corner

$$T_xX := \lim\left(\operatorname{Spf} R \xrightarrow{x} X \leftarrow TX\right).$$

When X is a group scheme, we write T_1X for the tangent space of X at the identity point.

Definition 2.8. The most important formal Lie groups in this paper are $\hat{\mathbb{G}}_a$ and $\hat{\mathbb{G}}_m$, both isomorphic to $\hat{\mathbb{A}}^1$ as varieties. The functor $\hat{\mathbb{G}}_a$ is described on an I-adic R-algebra A by $\hat{\mathbb{G}}_a(R) = I$ with group law

$$x +_{\hat{\mathbb{G}}_a} y = x + y.$$

The functor $\hat{\mathbb{G}}_m$ is described on R by $\hat{\mathbb{G}}_m(R) = 1 + I$ with group law given by

$$(1 + x) +_{\hat{\mathbb{G}}_m} (1 + y) = (1 + x)(1 + y) = 1 + (x + y + xy).$$

The isomorphism $\hat{\mathbb{G}}_m \cong \hat{\mathbb{A}}^1$ is given by $1 + x \mapsto x$, and so the group law induced on the formal affine line is described by $x + y + xy$.

Lemma 2.9. $T_1G \cong \hat{\mathbb{G}}_a$.

Proof. Both T_1G and $\hat{\mathbb{G}}_a$ are isomorphic to $\hat{\mathbb{A}}^1$; the points $\varepsilon a \in T_1G(R)$ are in bijective correspondence with the points $a \in \hat{\mathbb{G}}_a(R)$. Moreover, this map respects the group laws, since every formal group law is of the form $x +_G y = x + y + o(2)$. $\square$

3. Lubin–Tate cohomology and multiextensions

For ordinary groups F and G, the cohomology groups $H^*(F; G)$ classify certain kinds of extensions of G by F. Because the Lubin–Tate cohomology of formal Lie groups is set up so similarly, it solves an identical extension problem, phrased in terms of torsors. This is discussed in generality by Demazure and Gabriel [4, III.6.1], though the new reader should note that the special case of formal Lie groups is simpler to work out by hand than to read in the reference. The groups H^2 are the original objects studied by Lubin and Tate [9, Section 2], which can also serve as exposition.

The C^k groups defined in the previous section also solve a sort of the extension problem, but because it is less well-known we take the time to seriously explore it here. As reference, the group C^3 makes an appearance in Breen's text [3, Sections 1 and 2], and the C^k groups are discussed in light tones in an introductory section of the previous paper [6, Section 2.3]. What we present is a straightforward generalization of what can be found at those sources.

First, we introduce torsors as a model for the "total space" of an extension of group schemes, along with constructions with them familiar to topologists.

Definition 3.1. Fix group S-schemes G and H and base S-schemes X and Y. A G-torsor $\mathscr{L}$ is an X-scheme with G-action such that the map $G \times_S \mathscr{L} \to \mathscr{L} \times_S \mathscr{L}$ described by $(g, \ell) \mapsto (g \cdot \ell, \ell)$ is an isomorphism of S-schemes. The torsor is additionally said to be trivializable when it is noncanonically G-equivariantly isomorphic to $G \times_S X$ as X-schemes, or equivalently when it admits a section $X \to \mathscr{L}$. If $\mathscr{L}$ is as above and $\mathscr{M}$ is an H-torsor over Y, then a map $\mathscr{L} \to \mathscr{M}$ of torsors is defined to be a triple of maps of schemes $(G \to H, \mathscr{L} \to \mathscr{M}, X \to Y)$ which commute with all the data present. Experienced readers will note that this is *not* the definition of map of torsors common to the rest of the literature, discussed in Remark 3.2 below.

Several common constructions for bundles translate to torsors.

- *Pullback.* Let $\mathscr{L}$ be a G-torsor over Y, and let $f : X \to Y$ be a map of schemes. Then we define the pullback $f^*\mathscr{L}$ to be the fiber product $\mathscr{L} \times_X Y$, which is easily seen to be a G-torsor over Y.
- *Pushforward.* Let $\mathscr{L}$ be as above, and let $\varphi : G \to H$ be a map of group schemes. Then we define the pushforward torsor $\varphi_*\mathscr{L}$ as the colimit of the diagram

$$\begin{array}{ccc} \mathscr{L} \times G \times H & \xrightarrow{\cdot \times \text{id}} & \mathscr{L} \times H \\ {\scriptstyle \text{id} \times \varphi \times \text{id}} \downarrow & & \| \\ \mathscr{L} \times H \times H & \xrightarrow[\text{id} \times \cdot]{} & \mathscr{L} \times H, \end{array}$$

encoding the Borel construction. Because formal schemes are not closed under arbitrary colimits, it is not clear if this formula is sane without assuming the trivializability of $\mathscr{L}$. This is one reason why we will restrict our attention to such torsors.

- *Tensor product.* Let $\mathscr{L}$ and $\mathscr{M}$ be two G-torsors over X. Then there exists a G-torsor $\mathscr{L} \otimes \mathscr{M}$ over X, with a natural isomorphism of fibers $(\mathscr{L} \otimes \mathscr{M})_x \cong \mathscr{L}_x \otimes_G \mathscr{M}_x$. Note that if we write $\Delta : X \to X \times X$ for the diagonal map and $\mu : G \times G \to G$ for the multiplication, then $\mathscr{L} \otimes \mathscr{M} = \Delta^* \mu_*(\mathscr{L} \times \mathscr{M})$, where the product $\times$ is the scheme-theoretic product with the diagonal G-action.
- *Dual.* Let 1 denote the trivial G-torsor $G \times X$ over the S-scheme X. Then any G-torsor $\mathscr{L}$ over X has a dual defined by $\mathscr{L}^{-1} = \underline{\mathrm{Maps}}_G(\mathscr{L}, 1)$. The dual comes with a natural isomorphism $\mathscr{L} \otimes \mathscr{L}^{-1} \to 1$ given by evaluation.

Remark 3.2. Given two trivializable G-torsors $\mathscr{L}$ and $\mathscr{M}$ over the S-schemes X and Y respectively, a $\star$-map $\mathscr{L} \to \mathscr{M}$ is a pair (f, t) of a map $f : X \to Y$ and a G-equivariant isomorphism of S-schemes $t : \mathscr{L} \to f^*\mathscr{M}$. This produces a different category of G-torsors over S-schemes than what is described above, and is by far the more common definition. Our notion of map of torsors is strictly weaker and designed to accommodate the pushforward, which we will use prominently in Section 6.

Now we turn to certain torsors over large bases, which are seen as parameterized families of extensions.

Definition 3.3. Fix a formal Lie group G to play the role of the structure group. We make a sequence of definitions leading up to that of a higher cubical structure.

- Select a family of formal Lie groups $F_1, \ldots, F_k$. A multiextension $\mathscr{L}$ is a G-torsor over $F_1 \times \cdots \times F_k$ so that for any point $f_{\hat{i}} = (f_1, \ldots, \hat{f}_i, \ldots, f_k) \in F_1 \times \cdots \times \hat{F}_i \times \cdots \times F_k$ the corresponding pullback $s^*\mathscr{L}$ along $s(f_{\hat{i}}) = (f_1, \ldots, f_i, \ldots, f_k)$ gives an extension of group schemes of F_i by G. After selecting trivializations, we see that these extensions are controlled by a family of 2-cocycles u in $Z^2(F; G)$, parameterized by the missing index i and the points $f_{\hat{i}}$:

  $$u_i(f_{\hat{i}}) : F_i \times F_i \to G.$$

  See Fig. 1 for an illustration for extensions contained in a multiextension when $k = 2$.

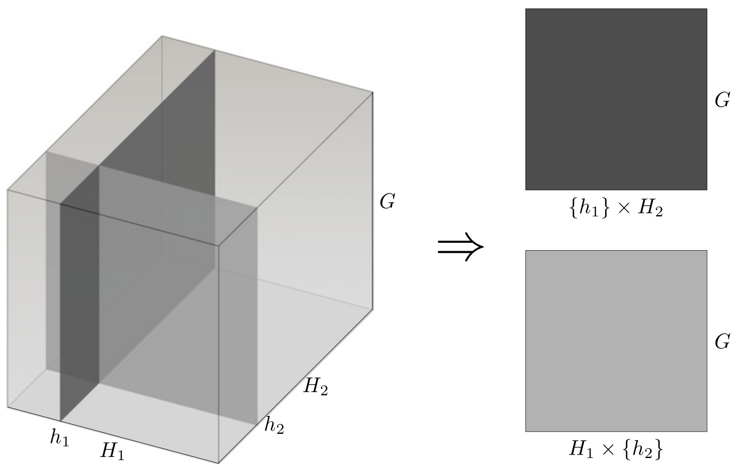


Fig. 1. Extensions contained in a biextension.

- In the case $F = F_1 = \cdots = F_k$, we can impose various symmetry conditions on such a multiextension. For the moment, let us just consider $k = 2$, where there is a flip map $\sigma : F \times F \to F \times F$. One could ask for a specified isomorphism $\alpha : \sigma^*\mathscr{L} \to \mathscr{L}$, but this may actually incorporate nontrivial isomorphisms of $\mathscr{L}$ in a way we do not want: specifically, the diagonal $\Delta : F \to F \times F$ has the property that $\Delta = \sigma\Delta$. Likewise, we should require that α pulls back to the identity $\mathrm{id} = \Delta^*\alpha : \Delta^*\sigma^*\mathscr{L} \to \Delta^*\mathscr{L}$.

  For a general k, we have a family of morphisms $\sigma : F^k \to F^k$ corresponding to permutations $\sigma \in \Sigma_k$, and to each permutation σ we can construct a map $\Delta_\sigma : F^{|\mathbf{k}/\langle\sigma\rangle|} \to F^k$ that populates the σ-orbits of F^k with diagonal values. We necessarily have $\Delta_\sigma = \sigma\Delta_\sigma$, and hence $\Delta_\sigma^*\mathscr{L}$ is canonically isomorphic to $(\sigma\Delta_\sigma)^*\mathscr{L}$. The most basic condition asserts that we fix a family of isomorphisms τ_σ of isomorphisms $\tau_\sigma : \sigma^*\mathscr{L} \to \mathscr{L}$ extending these given isomorphisms satisfying the coherence relations $\tau_{\sigma'\sigma} = (\sigma^*\tau_{\sigma'})\tau_\sigma$. A multiextension together with this symmetry data is called a symmetric multiextension.
- The most extreme symmetry we can request are trivializations such that the controlling cocycles satisfy

  $$u_i(f_1, \ldots, \hat{f}_i, \ldots, f_k)(f_i, f_{n+1}) = u_{\sigma i}(f_{\sigma 1}, \ldots, \hat{f}_{\sigma i}, \ldots, f_{\sigma k})(f_{\sigma i}, f_{\sigma(k+1)})$$

  for all choices of $\sigma \in \Sigma_{k+1}$. Under these conditions, we can simply write $u(f_1, \ldots, f_{k+1})$ without ambiguity, since all interpretations of this symbol produce the same point in G. A multiextension satisfying this condition is called a higher cubical structure. The name "cubical structure" stems from previous work on the case $k = 2$; see Mumford [11] and Breen [3].

Higher cubical structures can actually be produced en masse as sections of a certain functorially constructed torsor, which we now describe.

Definition 3.4. Denote the map $(f_1, \ldots, f_k) \mapsto \sum_{i \in I}^{F} f_i$ by μ_I, and select an extension $\mathscr{L}$ of F by G. We define $\Theta^k \mathscr{L}$ by

$$\Theta^k \mathscr{L} = \bigotimes_{I \subseteq \{1,\ldots,k\}} (\mu_I^* \mathscr{L})^{(-1)^{|I|}}, \qquad (\Theta^k \mathscr{L})_{\mathbf{x}} = \bigotimes_{I \subseteq \{1,\ldots,k\}} \mathscr{L}_{\sum_{i \in I} \mathbf{x}_i}^{(-1)^{|I|}}.$$

A Θ^k-structure on an extension $\mathscr{L}$ is a chosen trivialization of $\Theta^k \mathscr{L}$.

Remark 3.5. A Θ^{k+1}-structure on $\mathscr{L}$ corresponds to a higher cubical structure on $\Theta^k \mathscr{L}$. Both of these structures are classified by the Lubin–Tate cocycle groups $C^k(F; G)$. To indicate how this classification proceeds, we treat the special case of $k = 2$ for simplicity. Suppose that we have a Θ^3-structure on $\mathscr{L}$, i.e., a selected isomorphism $\Theta^3 \mathscr{L} \xrightarrow{\cong} 1$. Then, we produce a map on fibers as follows:

$$\begin{aligned} 1_{x,y,z} &\xrightarrow{\cong} \frac{\mathscr{L}_{x+y+z} \otimes \mathscr{L}_x \otimes \mathscr{L}_y \otimes \mathscr{L}_z}{\mathscr{L}_{x+y} \otimes \mathscr{L}_{x+z} \otimes \mathscr{L}_{y+z}}, \\ \frac{\mathscr{L}_{x+z} \otimes \mathscr{L}_{y+z}}{\mathscr{L}_x \otimes \mathscr{L}_y \otimes \mathscr{L}_z} &\xrightarrow{\cong} \frac{\mathscr{L}_{x+y+z}}{\mathscr{L}_{x+y}}, \\ \frac{\mathscr{L}_{x+z}}{\mathscr{L}_x \mathscr{L}_z} \otimes \frac{\mathscr{L}_{y+z}}{\mathscr{L}_y \mathscr{L}_z} &\xrightarrow{\cong} \frac{\mathscr{L}_{x+y+z}}{\mathscr{L}_{x+y} \otimes \mathscr{L}_z}, \\ (\Theta^2 \mathscr{L})_{x,z} \otimes (\Theta^2 \mathscr{L})_{y,z} &\xrightarrow{\cong} (\Theta^2 \mathscr{L})_{x+y,z}, \end{aligned}$$

which is a part of the biextension structure. Symmetries of the tensor product used in the definition of $\Theta^{k+1} \mathscr{L}$ ensure that the induced multiextension structure on $\Theta^k \mathscr{L}$ is a higher cubical structure. Such structures are classified by their controlling cocycle u, i.e., a point of $C^k(F; G)(S)$. This observation is recounted in great, careful detail in both Breen [3] and Mumford [11].

Remark 3.6. Finally, we make some remarks on how multiextensions interact with the torsor operations defined in Definition 3.1. All of these facts are proven by considering the naturality of the bijection between higher cubical structures and the group schemes C^k. Throughout, select a structure group S-scheme G, a base group S-scheme F, and $\mathscr{B}$ and $\mathscr{C}$ higher cubical structures over $F^{\times k}$.

- Select a map $f : X \to Y$ of group S-schemes. Then the pullback $f^* \mathscr{B}$ receives the structure of a symmetric multiextension so that the induced map $f^* \mathscr{B} \to \mathscr{B}$ is a map of multiextensions. If $u^{\mathscr{B}}$ is the controlling cocycle for $\mathscr{B}$, then we have $u^{f^* \mathscr{B}} = u^{\mathscr{B}} \circ f^{\times k}$.
- Select a map $\varphi : G \to H$ of group S-schemes. Then the pushforward $\varphi_* \mathscr{B}$ receives the structure of a higher cubical structure so that $\mathscr{B} \to \varphi_* \mathscr{B}$ is a map of higher cubical structures. If $u^{\mathscr{B}}$ is the controlling cocycle for $\mathscr{B}$, then we have $u^{\varphi_* \mathscr{B}} = \varphi \circ u^{\mathscr{B}}$.
- The dual torsor $\mathscr{B}^{-1}$ receives the structure of a higher cubical structure. If $u^{\mathscr{B}}$ is the controlling cocycle for $\mathscr{B}$, then we have $u^{\mathscr{B}^{-1}} = [-1]_G \circ u^{\mathscr{B}}$.
- The tensor product $\mathscr{B} \otimes \mathscr{C}$ also receives the structure of a higher cubical structure, with controlling cocycle described by $u^{\mathscr{B} \otimes \mathscr{C}} = u^{\mathscr{B}} +_G u^{\mathscr{C}}$.

4. Calculations tangent to the Lubin–Tate cohomology of $(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_m)$

Our ultimate goal is to understand the group scheme $C^k(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_m)$ and compute its coordinate ring. As in classical Lie theory, it is fruitful to first compute the tangent space at the identity as a means of understanding the local picture.

4.1. Calculation of $T_1 H^*(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_m)$

Let us begin by computing the tangent space to the cohomology groups.

Lemma 4.1. $T_1 H^*(F; G) = H^*(F; T_1 G)$.

Proof. We expand the definition of $H^*(F; G)$ to make the following calculation:

$$\begin{aligned} T_1 H^k(F; G)(S) &= \frac{\ker T_1 \delta^k : T_1 A^k(F; G)(S) \to T_1 A^{k+1}(F; G)(S)}{\operatorname{im} T_1 \delta^{k-1} : T_1 A^{k-1}(F; G)(S) \to T_1 A^k(F; G)(S)} \\ &= \frac{T_1 Z^k(F; G)(S)}{T_1 B^k(F; G)(S)}. \end{aligned}$$

Hence, we reduce to understanding $T_1 Z^k(F; G)$ and $T_1 B^k(F; G)$.

The point of $Z^k(F; G)(S)$ corresponding to the identity element is represented by the power series 0, sending F^k to the identity point of G. A point of $T_1 Z^k(F; G)(S)$ is then a power series u of the form $0 + \varepsilon u_+$ for some power series u_+. Since $\varepsilon^2 = 0$, we compute the G-inverse of εu_+ to be $-\varepsilon u_+$, and hence the 2-cocycle condition on u corresponds to the following condition on u_+:

$$\begin{aligned} &u_+(x_1, x_2, x_3, \ldots) - \\ u_+(&x_0 +_F x_1, x_2, x_3, \ldots) + \\ u_+(&x_0, x_1 +_F x_2, x_3, \ldots) - \\ &u_+(x_0, x_1, x_3, \ldots) = 0. \end{aligned}$$

These u_+ are exactly the elements of $Z^k(F; T_1 G)(S)$. We also have inclusion in the other direction; a point $u_+ \in Z^k(F; T_1 G)(S)$ corresponds to a point $0 + \varepsilon u_+ \in T_1 Z^k(F; G)(S)$. The argument for coboundaries is entirely similar. □

Corollary 4.2. *The tangent space $T_1 H^*(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_m)$ is $H^*(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_a)$.*

Theorem 4.3. *Letting a_i represent x^{2^i}, we calculate*

$$H^*(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_a)(\mathbb{F}_2) \cong \bigotimes_i \mathbb{F}_2[a_i].$$

Proof. It is equivalent to compute $\operatorname{Ext}_{\mathbb{F}_2[x]}(\mathbb{F}_2, \mathbb{F}_2)$ in the category of $\mathbb{F}_2[x]$-comodules. Then, as the P.D. algebra $\Gamma_{\mathbb{F}_2}[x]$ is the linear dual of $\mathbb{F}_2[x]$, it is again equivalent to compute $\operatorname{Ext}_{\Gamma_{\mathbb{F}_2}[x]}(\mathbb{F}_2, \mathbb{F}_2)$ in the category of $\Gamma_{\mathbb{F}_2}[x]$-modules. Using the splitting of the algebra $\Gamma_{\mathbb{F}_2}[x]$ in the tensor product $\Gamma_{\mathbb{F}_2}[x] \cong \bigotimes_{i=0}^{\infty} \Lambda[x^{[2^i]}]$, we see that it suffices to compute $\operatorname{Ext}_{\Lambda[y]}(\mathbb{F}_2, \mathbb{F}_2)$ for each factor individually then tensor together those results. Performing this last computation is straightforward with the Tate resolution [16]. The differential graded algebra described by Tate which computes $\operatorname{Ext}_{\Lambda[y]}(\mathbb{F}_2, \mathbb{F}_2)$ is given by $R_* = \Gamma[a] \cong \bigotimes_{i=0}^{\infty} \Lambda[a^{[2^i]}]$ with differential $da^{[j]} = a^{[j-1]}y$. Therefore,

$$\begin{aligned} \operatorname{Ext}_{\Gamma[x]}(\mathbb{F}_2, \mathbb{F}_2) &\cong \bigotimes_{i=0}^{\infty} \operatorname{Ext}_{\Lambda[x^{[2^i]}]}(\mathbb{F}_2, \mathbb{F}_2) \\ &\cong \bigotimes_{i=0}^{\infty} \operatorname{Hom}(\Gamma[a_i], \mathbb{F}_2) \\ &\cong \bigotimes_{i=0}^{\infty} \mathbb{F}_2[a_i^\vee]. \quad \square \end{aligned}$$

Remark 4.4. As defined, the objects $H^*(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_a)(\mathbb{F}_2)$ are only groups, but identifying their computation with that of an Ext , together with Tate's method for computing Ext , shows that they are in fact rings. The multiplication structure in fact exists on the level of A^*; we set $(f \cdot g)(x_1, \ldots, x_{p+q}) = f(x_1, \ldots, x_p) \cdot g(x_{p+1}, \ldots, x_{p+q})$, the product of power series. An exact mimic of the proofs for cup products of singular cocycles shows that $(f \cdot g) \in Z^{p+q}$ when $f \in Z^p$ and $g \in Z^q$, and moreover that this operation descends to cohomology.

4.2. Calculation of $T_1 C^*(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_m)$

The scheme $C^k(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_m)$ also has an interesting tangent space at its identity, which we describe now.

Lemma 4.5. *$T_1 C^k(F; G)$ is $C^k(F; T_1 G)$.*

Proof. This is identical to Lemma 4.1. □

Corollary 4.6. *$T_1 C^k(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_m) \cong C^k(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_a)$.*

Definition 4.7. Let ζ_k^n denote the integral polynomial

$$\zeta_k^n = \phi(n,k)^{-1} \sum_{X \subseteq \{x_1, \ldots, x_k\}} \left((-1)^{|X|} \cdot \left(\sum_{x \in X} x \right)^n \right) = \phi(n,k)^{-1} \sum_{\substack{\lambda \vdash n \\ \ell(\lambda) = k}} \binom{n}{\lambda} \mathbf{x}^\lambda,$$

where $\phi(n, k)$ is defined by

$$\phi(n,k) = \gcd_\lambda \binom{|\lambda|}{(\lambda_1, \ldots, \lambda_k)} = \gcd_\lambda \left(n! \prod_i (\lambda_i!)^{-1} \right).$$

Remark 4.8. The familiar reader will recognize ζ_2^n as Lazard's 2-cocycle [8].

Theorem 4.9 (*[6, Corollary 3.4.10 and Theorem 3.6.2]*). *Let $D_{n,k}$ count the number of power-of-2 partitions of n into k' parts, where k' is the smallest possible size equal to or greater than k. Then there is an isomorphism*

$$\mathcal{O}C^k(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_a) \times \mathbb{Z}_{(2)} \cong \mathbb{Z}_{(2)}[c_n \mid n \geq k] \otimes \frac{\mathbb{Z}_{(2)}\left[b_{n,i} \,\middle|\, \begin{array}{c} 1 \leq i < D_{n,k}, \\ n \geq k \end{array} \right]}{\langle 2b_{n,i} \rangle}.$$

The elements c_n corepresent ζ_k^n, and the elements $b_{n,i}$ corepresent all possible polynomials $\tau(\lambda)$, where λ is a multi-index consisting only of powers of 2.

Remark 4.10. In Fig. 3 below, we provide an excerpt from the previous paper [6, Appendix A.1] to give the reader a sense of what the space of additive 2-cocycles over $\mathbb{F}_2$ looks like.

Important remark 4.11. There is a gap in the proof of our theorem from the previous paper [6, Theorem 3.6.2] that we must remark on: the classification there correctly demonstrates this result for $\mathbb{F}_p$-algebras, but does not provide enough to conclude the result for $\mathbb{Z}_{(p)}$-algebras. To this end, it suffices to show that ζ_k^n is the only additive cocycle over $\mathbb{Z}/p^2$ with leading coefficient not divisible by p. The key is that when working in characteristic p we have $(a+b)^{p^j} = a^{p^j} + b^{p^j}$, whereas in $\mathbb{Z}/p^2$ we instead have $(a+b)^{p^j} = a^{p^j} + b^{p^j} + \sum_{i=1}^{p-1} \binom{p^j}{ip^{j-1}} a^{p^{j-1}i} b^{p^{j-1}(p-i)}$, where now $\binom{p^j}{ip^{j-1}}$ is nonzero mod p^2. In the language of the previous paper, this has the effect of enlarging our annihilator sets dramatically — namely, any carry-minimal partition contains in its annihilator set all other carry-minimal partitions, since we are now able to split and regather power-of-p entries.

5. The tangent spectral sequence

Now we use the information local to the origin computed in Section 4 to produce information about the entire scheme $H^*(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_m)$ through successive approximations. We organize this procedure into a spectral sequence and describe a family of nontrivial differentials for it.

Theorem 5.1. *There exists a filtration spectral sequence of signature*

$$T_1H^*(F; G)(R) = H^*(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_a) \Rightarrow H^*(F; G)(R).$$

Proof. Recall that picking coordinates gives $A^k(F; G)(R) = \{\sum_I a_I \mathbf{x}^I \mid \mathbf{x} = (x_1, \ldots, x_k)\}$ the set of k-variate power series, which can be identified with the space of scheme-theoretic maps $F^k \to G$ for a pair of 1-dimensional formal groups F and G. The cochain complex $A^*(F; G)$ admits a descending filtration by leading degree d, denoted by

$$A_n^k = A_n^k(F; G)(R) = \left\{ \sum_I a_I \mathbf{x}^I : a_I = 0 \text{ whenever } |I| < n \right\}.$$

Using standard machinery, this gives a convergent filtration spectral sequence, with E_1-page described by certain polynomials of homogeneous degree:

$$E_1^{k,n} \cong \left\{ f \in k[x_1, \ldots, x_k] : f = \sum_{|I|=n} a_I \mathbf{x}^I, \delta^k f = 0 \mod \langle x_1, \ldots, x_k \rangle^{n+1} \right\}.$$

Because $F(x, y) = x + y \pmod{\langle x, y\rangle^2}$ for all formal group laws F, we identify elements of these $E_1^{k,n}$-groups with those of $H^*(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_a)$ which are representable as k-variate polynomials of homogeneous degree n. □

So, the previous calculation of $H^*(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_a) = T_1H^*(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_m)$ serves as input to the tangent spectral sequences, with E_1-page illustrated in Fig. 2. With that description in hand, we turn to the differentials.

8	a_3	a_2^2	$a_1^2a_2$	$a_1^4, a_0^2a_1a_2$	$a_0^2a_1^3, a_0^4a_2$	$a_0^4a_1^2$
7			$a_0a_1a_2$	$a_0^3a_2, a_0a_1^2$	$a_0^3a_1^2$	$a_0^5a_1$
6		a_1a_2	$a_0^2a_2, a_1^3$	$a_0^2a_1^2$	$a_0^4a_1$	a_0^6
5		a_0a_2	$a_0a_1^2$	$a_0^3a_1$	a_0^5	
4	a_2	a_1^2	$a_0^2a_1$	a_0^4		
3		a_0a_1	a_0^3			
2	a_1	a_0^2				
1	a_0					
E_1	1	2	3	4	5	6

Fig. 2. The E_1-page of the tangent spectral sequence over $R = \mathbb{F}_2$.

Theorem 5.2. *Set $R = \mathbb{F}_2$, $F = \hat{\mathbb{G}}_a$, and $G = \hat{\mathbb{G}}_m$, and select $u_+ = ca_ia_j$ for $i \neq j$ and a coefficient c. Then there is a nonzero differential:*

$$d_{2^i+2^j}(ca_ia_j) = c^2(a_i^2a_{j+1} - a_{i+1}a_j^2).$$

Proof. The additive cohomology class $ca_ia_j = c[u]$ can be represented by the polynomial $u = cx^{2^i}y^{2^j}$. The first differentials in the tangent spectral sequence arise from applying the multiplicative coboundary map to polynomials, and so we can compute the smallest nonvanishing differential on $c[u]$ by computing

$$\frac{1 + cu(x, y)}{1 + cu(w + x, y)} \cdot \frac{1 + cu(w, x + y)}{1 + cu(w, x)},$$

provided that the result is not null-cohomologous. In our case, we have

$$\frac{\left(1 + cx^{2^i}y^{2^j}\right)\left(1 + cw^{2^i}(x + y)^{2^j}\right)}{\left(1 + c(w + x)^{2^i}y^{2^j}\right)\left(1 + cw^{2^i}x^{2^j}\right)} = 1 + c^2w^{2^i}x^{2^i}y^{2^{j+1}} - c^2w^{2^{i+1}}x^{2^j}y^{2^j} + o(2^{i+1} + 2^{j+1})).$$

Hence, $d_{2^i+2^j}(ca_ia_j) = c^2(a_i^2a_{j+1} - a_{i+1}a_j^2)$ as an equation on the $E_{2^i+2^j}$-page.

To show that this produces a nonvanishing differential, we must show that all these classes still exist on this page. The application of δ_2 to $1 + u_+$ has leading additive part of degree divisible by $|u_+|$; hence the first nonvanishing differential with source $a_i^2a_{j+1} - a_{i+1}a_j^2$ must be on the $E_{2^{i+1}+2^{j+1}}$-page. To check that it is also not the target of a differential, classes of degree below $2^i + 2^j$ are too far away from $a_i^2a_{j+1} - a_{i+1}a_j^2$ to hit it with a differential by the $E_{2^i+2^j}$-page. There is only one class in $E_1^{1,t}$ for $2^i + 2^j \leq t < 2^{i+1} + 2^{j+1}$: assuming $i > j$, it is a_{i+1}. We calculate the minimal differential on a_{i+1} similarly as

$$\begin{aligned}\frac{1 + cx^{2^{i+1}} + cy^{2^{i+1}}}{(1 + cx^{2^{i+1}})(1 + cy^{2^{i+1}})} &= (1 + cx^{2^{i+1}} + cy^{2^{i+1}}) \cdot (1 - cx^{2^{i+1}} + c^2x^{2^{i+2}} - o(3 \cdot 2^{i+1}))\\ &\quad \cdot (1 - cy^{2^{i+1}} + c^2y^{2^{i+2}} - o(3 \cdot 2^{i+1}))\\ &= 1 - c^2x^{2^{i+1}}y^{2^{i+1}} + o(3 \cdot 2^{i+1}).\end{aligned}$$

Hence $d_{2^{i+1}}(c[a_{i+1}]) = -c^2[a_{i+1}^2]$, and the degree of this class exceeds that of a_ia_j. □

Corollary 5.3. *To lift ca_ia_j to a multiplicative cocycle it is necessary that the coefficient c satisfy $c^2 = 0$.*

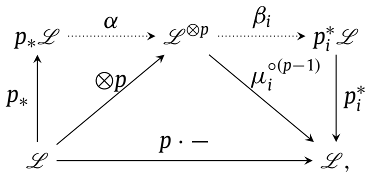
Remark 5.4. Note that these results obstruct particular asymmetric cohomology classes, and that the symmetric cocycles ζ_2^n do not support these differentials. For example, the cohomology class $[\zeta_2^{2^i+2^j}]$ is represented as $a_ia_j + a_ja_i = 0$, which has no obstruction.

6. Half-Weil forms

We will write the group structure of the fibers of our multiextensions using multiplicative notation, as the structure group $\hat{\mathbb{G}}_m$ will be the case of interest. In this section, we fix a prime p which can be taken to be 2 or an odd prime; the theory is identical either way. We will apply only the case $p = 2$ in the following section.

Definition 6.1. Fix a group scheme F and an integer k. For any $1 \leq i \leq k$, define p_i to be the map $F^k \to F^k$ defined by $1 \times \cdots \times 1 \times p \times 1 \times \cdots \times 1$, with the p occurring in the ith position.

Theorem 6.2. *Select a multiextension $\mathscr{L}$ over $F^{\times k}$ with structure group G. There is a diagram*



factoring the multiplication-by-p map on $\mathscr{L}$ so that the composition of the top row is an isomorphism of torsors.

Proof. Let $\mathscr{L}$ be trivialized with controlling cocycles $u_1, \ldots, u_k$ arising from coordinates on the base and structure groups; this proof then becomes completely computational. The map $\otimes p$ is described by the formula

$$\otimes p : (g \times (x_1, \ldots, x_k)) \mapsto (g \times (x_1, \ldots, x_k))^{\otimes p} = \left(g^p \times (x_1, \ldots, x_k)\right) \in \mathscr{L}^{\otimes p}.$$

Then, the map p_* acts as

$$p_* : (g \times (x_1, \ldots, x_k)) \mapsto (g^p \times (x_1, \ldots, x_k)),$$

which determines the map α to be

$$\alpha : (g \times (x_1, \ldots, x_k)) \mapsto (g \times (x_1, \ldots, x_k)).$$

Hence, α is an isomorphism.

We perform the same analysis on β_i. The map p_i^* acts by

$$p_i^* : (g \times (x_1, \ldots, x_k)) \mapsto g \times (x_1, \ldots, x_{i-1}, px_i, x_{i+1}, \ldots, x_k).$$

Then, writing $\mathbf{x}_{\hat{i}}$ for $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k)$, the map $\mu_i^{\circ(p-1)}$ acts by iterated multiextension addition in the ith factor, which gives the formula

$$\mu_i^{\circ(p-1)}(g \times (x_1, \ldots, x_k)) = \left(g \prod_{i=1}^{p-1} u_i(\mathbf{x}_{\hat{i}})(x_i, ix_i) \times (x_1, \ldots, px_i, \ldots, x_k) \right).$$

Hence, the map β_i is determined to be

$$\beta_i : (g \times (x_1, \ldots, x_k)) \mapsto \left(g \prod_{i=1}^{p-1} u_i(\mathbf{x}_{\hat{i}})(x_i, ix_i) \times (x_1, \ldots, x_k) \right).$$

Because the twist in the G-factor is invertible, as G is a group, β_i is also an isomorphism. $\square$

Definition 6.3. Given a 2-cocycle $u \in C^k(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_m)(R)$, we define the associated "half-Weil form"

$$e = \prod_{i=1}^{p-1} u(ix_1, x_1, x_2, \ldots, x_{k-1}).$$

Theorem 6.4. *The half-Weil form e associated to such a $u \in C^k(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_m)(R)$ for R an $\mathbb{F}_p$-algebra is a (not necessarily symmetric) $(k-1)$-variate multiplicative 2-cocycle satisfying*

$$\delta_1 e = u^p.$$

Proof. As $p = 0$ in R, the higher cubical structure $\mathscr{B}$ associated to u has trivial pullback $p_i^*\mathscr{B}$, since the cocycles associated to $p_i^*\mathscr{B}$ are of the two forms

$$u_j(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, \widehat{x_j}, \ldots, x_k)(x_j, x_j') = 1,$$
$$u_i(\mathbf{x}_{\hat{i}})(0, 0) = 1.$$

The isomorphism $p_i^*\mathscr{B} \cong p_*\mathscr{B}$ can be reinterpreted as a trivialization of the tensor $p_*\mathscr{B}/p_i^*\mathscr{B}$, i.e., a 1-cocycle whose image under δ_1 is the 2-cocycle associated to $p_*\mathscr{B}/p_i^*\mathscr{B}$ — but, since the 2-cocycle associated to $p_i^*\mathscr{B}$ is 1, we are really trivializing $p_*\mathscr{B}$, which has associated 2-cocycle $(u(x_1, \ldots, x_k))^p$.

We can produce an explicit formula for this 1-cocycle by chasing points around the diagram

$$\begin{array}{ccc} p_*\mathscr{B} \otimes p_*\mathscr{B} & \xrightarrow{\mu_j} & p_*\mathscr{B} \\ \beta\alpha \otimes \beta\alpha \downarrow & & \downarrow \beta\alpha \\ p_i^*\mathscr{B} \otimes p_i^*\mathscr{B} & \xrightarrow{\mu_j} & p_i^*\mathscr{B}. \end{array}$$

We make the following two computations, writing $\mathbf{x} = (x_1, \ldots, x_j, \ldots, x_k)$ and $\mathbf{x}' = (x_1, \ldots, x_j', \ldots, x_k)$:

$$\begin{aligned} &\beta\alpha \circ \mu_j \left((g \times \mathbf{x}) \otimes (g' \times \mathbf{x}') \right) = \beta\alpha \left((gg'u(\mathbf{x}_{\hat{j}})(x_j, x_j')^p) \times (x_1, \ldots, x_j + x_j', \ldots, x_k) \right) \\ &\quad = (gg'u(\mathbf{x}_{\hat{j}})(x_j, x_j')^p e(x_1, \ldots, x_j + x_j', \ldots, x_k)) \times (x_1, \ldots, x_j + x_j', \ldots, x_k), \\ &\mu_j \left((\beta\alpha \otimes \beta\alpha)(g \times \mathbf{x}) \otimes (g' \times \mathbf{x}') \right) = \mu_j \left((ge(\mathbf{x}) \times \mathbf{x}) \otimes (g'e(\mathbf{x}') \times \mathbf{x}') \right) \\ &\quad = (gg'u(x_1, \ldots, px_i, \ldots, \widehat{x_j}, \ldots, x_k)(x_j, x_j')e(\mathbf{x})e(\mathbf{x}')) \times (x_1, \ldots, x_j + x_j', \ldots, x_k). \end{aligned}$$

Because $\beta\alpha$ is a map of multiextensions, the G-coordinates of these expressions must be equal, and hence

$$\begin{aligned} u(x_1, \ldots, \widehat{x_j}, \ldots, x_k)(x_j, x_j')^p &= \frac{e(x_1, \ldots, x_j, \ldots, x_k)e(x_1, \ldots, x_j', \ldots, x_k)}{e(x_1, \ldots, x_j + x_j', \ldots, x_k)} \\ &= \delta_1 e(x_1, \ldots, x_k). \quad \square \end{aligned}$$

Remark 6.5. The classical Weil pairing associated to a cubical structure arises as the composite isomorphism

$$(p \times 1)^* \mathscr{L} \xrightarrow{(\beta_1\alpha)^{-1}} p_* \mathscr{L} \xrightarrow{\beta_2\alpha} (1 \times p)^* \mathscr{L},$$

which in the fiber over a point $(x_1, x_2) \in \hat{\mathbb{G}}_a^2$ acts by multiplication by

$$\prod_{i=1}^{p-1} \frac{u(x_1, ix_1, x_2)}{u(x_1, ix_2, x_2)},$$

where u is the 2-cocycle associated to $\mathscr{L}$. This was the object used by Mumford [11] to compare Weil pairings and cubical structures.

Lemma 6.6. *There is an additive version of the half-Weil form. To a multiplicative* 2*-cocycle u with additive part u_+, we associate an additive half-Weil form e_+, which is determined by*

$$e_+ = \sum_{i=1}^{p-1} u_+(ix_1, x_1, x_2, \dots, x_{k-1})$$

when this sum is nonzero. Again, $\delta_1 e_+ = pu_+ \equiv 0$.

Proof. Reuse the above argument for multiextensions with structure group $\hat{\mathbb{G}}_a$ rather than $\hat{\mathbb{G}}_m$ to produce an additive notion of Weil pairing. The statement about the interaction of u_+ and e_+ and u and e stems from studying the filtration on the tangent space used in Section 5. □

Lemma 6.7. *The sum given above determining the additive half-Weil form associated to ζ_2^n is $-\zeta_1^n$ when $n = p^i$ and* 0 *otherwise.*

Proof. Setting $u_+ = (x + y)^n - x^n - y^n$ as an additive 2-cocycle over $\mathbb{Z}$, we telescope and calculate

$$e_+ = \sum_{j=1}^{p-1} x^n \left((j + 1)^n - j^n - 1\right) = px^n \left(p^{n-1} - 1\right).$$

Then, $\zeta_2^{p^i} = p^{-1}((x + y)^{p^i} - x^{p^i} - y^{p^i})$, and hence the associated half-Weil form over $\mathbb{F}_p$ is $-\zeta_1^{p^i}$. But, when n is not of the form p^i, $\zeta_2^n = (x + y)^n - x^n - y^n$ exactly, and hence reducing modulo p gives 0. □

7. Obstructions and the calculation of $C^k(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_m) \times \operatorname{Spec} \mathbb{Z}_{(2)}$

We now have enough tools to investigate to what extent symmetric additive 2-cocycles can be lifted to multiplicative 2-cocycles. There are two special constructions concerning the cocycles ζ_k^n which we describe first, and the remaining additive cocycles are handled en masse by an obstruction result.

Theorem 7.1. *Write $\nu_p(n)$ for the order of p-divisibility of the integer n, and recall the function $\phi(n, k)$ from Definition 4.7. Let $E_p(t)$ be the Artin–Hasse exponential, a p-integral series defined by*

$$E_p(t) = \exp\left(\sum_{k=0}^{\infty} \frac{t^{p^k}}{p^k}\right).$$

Then, when $\nu_p\phi(n, k) < \nu_p(n)$, the power series

$$\tilde{\zeta}_k^n = (\delta^1)^{\circ(k-1)} E_p(cx^n)^{p^{-\nu_p\phi(n,k)}}$$

is a multiplicative extension of $c\zeta_k^n$ over an $\mathbb{F}_p$-algebra.

Proof. See Ando et al. [2, Corollary 3.22]. □

Remark 7.2. Ando et al. [2, Proposition A.10] also provide the equation

$$\nu_p\phi(n, k) = \max\left\{0, \left\lceil \frac{k - \sigma_p(n)}{p - 1} \right\rceil\right\}$$

to aid in computing facts about this power series, where $\sigma_p(n)$ is the $\mathbb{N}$-valued digital sum of n in base p. This is an immediate consequence of work of Kummer [7].

Theorem 7.3. *Every additive cocycle u_+ over a ring S of characteristic* 2 *can be written in the form*

$$u_+ = \sum_{\substack{n,m \\ \ell(I)=k-3}} r_{n,m,I}\zeta_2^n x_3^m (x_4, \dots, x_k)^I,$$

where $r_{n,m,I}$ is an element in S. If $r_{2^n,2^m,I} \neq r_{2^m,2^n,I}$ for any choice of m, n, and I, then any multiplicative 2*-cocycle $1+bu_+ +o(|u_+|)$ must satisfy $b^2 = 0$.*

	dim 2	3	4	5	6
deg 2	$\tau(1,1)$	0	0	0	0
3	$\tau(2,1)$	$\tau(1,1,1)$	0	0	0
4	$\tau(2,2)$	$\tau(2,1,1)$	$\tau(1,1,1,1)$	0	0
5	$\tau(4,1)$	$\tau(2,2,1)$	$\tau(2,1,1,1)$	$\tau(1,1,1,1,1)$	0
6	$\tau(4,2)$	$\tau(2,2,2)$, $\tau(4,1,1)$	$\tau(2,2,1,1)$	$\tau(2,1,1,1,1)$	$\tau(1,1,1,1,1,1)$
7	$\tau(6,1)+$ $\tau(5,2)+$ $\tau(4,3)$	$\tau(4,2,1)$	$\tau(2,2,2,1)$, $\tau(4,1,1,1)$	$\tau(2,2,1,1,1)$	$\tau(2,1,1,1,1,1)$
8	$\tau(4,4)$	$\tau(4,2,2)$	$\tau(2,2,2,2)$, $\tau(4,2,1,1)$	$\tau(2,2,2,1,1)$, $\tau(4,1,1,1,1)$	$\tau(2,2,1,1,1,1)$
9	$\tau(8,1)$	$\tau(4,4,1)$	$\tau(4,2,2,1)$	$\tau(2,2,2,2,1)$, $\tau(4,2,1,1,1)$	$\tau(2,2,2,1,1,1)$, $\tau(4,1,1,1,1,1)$
10	$\tau(8,2)$	$\tau(4,4,2)$, $\tau(8,1,1)$	$\tau(4,2,2,2)$, $\tau(4,4,1,1)$	$\tau(2,2,2,2,2)$, $\tau(4,2,2,1,1)$	$\tau(2,2,2,2,1,1)$, $\tau(4,2,1,1,1,1)$
11	$\tau(10,1)+$ $\tau(9,2)+$ $\tau(8,3)$	$\tau(8,2,1)$	$\tau(4,4,2,1)$, $\tau(8,1,1,1)$	$\tau(4,2,2,2,1)$, $\tau(4,4,1,1,1)$	$\tau(2,2,2,2,2,1)$, $\tau(4,2,2,1,1,1)$
12	$\tau(8,4)$	$\tau(4,4,4)$, $\tau(8,2,2)$	$\tau(4,4,2,2)$, $\tau(8,2,1,1)$	$\tau(4,2,2,2,2)$, $\tau(4,4,2,1,1)$, $\tau(8,1,1,1,1)$	$\tau(2,2,2,2,2,2)$, $\tau(4,2,2,2,1,1)$, $\tau(4,4,1,1,1,1)$
13	$\tau(12,1)+$ $\tau(9,4)+$ $\tau(8,5)$	$\tau(8,4,1)$	$\tau(4,4,4,1)$, $\tau(8,2,2,1)$	$\tau(4,4,2,2,1)$, $\tau(8,2,1,1,1)$	$\tau(4,2,2,2,2,1)$, $\tau(4,4,2,1,1,1)$, $\tau(8,1,1,1,1,1)$
14	$\tau(12,2)+$ $\tau(10,4)+$ $\tau(8,6)$	$\tau(8,4,2)$	$\tau(4,4,4,2)$, $\tau(8,2,2,2)$, $\tau(8,4,1,1)$	$\tau(4,4,2,2,2)$, $\tau(4,4,4,1,1)$, $\tau(8,2,2,1,1)$	$\tau(4,4,2,2,1,1)$, $\tau(8,2,1,1,1,1)$, $\tau(4,2,2,2,2,2)$
15	$\tau(14,1)+$ $\tau(13,2)+$ $\tau(12,3)+$ $\tau(11,4)+$ $\tau(10,5)+$ $\tau(9,6)+$ $\tau(8,7)$	$\tau(12,2,1)+$ $\tau(10,4,1)+$ $\tau(9,4,2)+$ $\tau(8,6,1)+$ $\tau(8,5,2)+$ $\tau(8,4,3)$	$\tau(8,4,2,1)$	$\tau(4,4,4,2,1)$, $\tau(8,2,2,2,1)$, $\tau(8,4,1,1,1)$	$\tau(4,4,2,2,2,1)$, $\tau(4,4,4,1,1,1)$, $\tau(8,2,2,1,1,1)$
16	$\tau(8,8)$	$\tau(8,4,4)$	$\tau(8,4,2,2)$, $\tau(4,4,4,4)$	$\tau(8,4,2,1,1)$, $\tau(8,2,2,2,2)$, $\tau(4,4,4,2,2)$	$\tau(8,4,1,1,1,1)$, $\tau(8,2,2,2,1,1)$, $\tau(4,4,4,2,1,1)$, $\tau(4,4,2,2,2,2)$
⋮	⋮	⋮	⋮	⋮	⋮

Fig. 3. A depiction of $C^k(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_a)(\mathbb{F}_2)$, reproduced from the previous paper [6, Appendix A.1]. The gray-shaded cells are those to which Theorem 7.1 applies. See Theorem 4.9 for a reminder about notation.

Proof. Select such a cocycle u and select indices n, m, and I so that $r_{2^n,2^m,I} \neq r_{2^m,2^n,I}$ and assume $r_{2^n,2^m,I} \neq 0$. Construct the associated half-Weil pairing e as in Definition 6.3; by assumption and Lemma 6.7 e_+ is nonzero and the projection of the cohomology class $[e_+] \in H^2(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_a)(S[x_3, \dots, x_{k-1}])$ onto the module factor $S\{a_n a_m\}$ is nonzero with coefficient $r_{2^n,2^m,I} - r_{2^m,2^n,I} \neq 0$. Hence, the tangent spectral sequence and Theorem 5.2 dictate that the leading coefficient of e must have vanishing square, as e is a multiplicative lift of e_+. By distributivity and the characterization of e_+ in Lemma 6.6, this coefficient is an integer multiple of b. □

We need a small lemma, reminiscent of arguments used to prove Theorem 4.9, to interrelate these obstructions before we can perform the promised 2-primary calculation.

Lemma 7.4. *Fix an integer n and construct a graph whose nodes are labeled by unordered tuples of powers of 2 whose sum is n, and insert an edge from a node of tuple length ℓ to a node of tuple length $\ell - 1$ if exactly two entries of ℓ can be summed together to produce the second tuple. Every subgraph consisting of all nodes of lengths ℓ and $\ell - 1$ is connected, i.e., for any two tuples of length ℓ, we can find a path in the graph connecting them.*

Proof. Associate to such a tuple λ the finite sequence of natural numbers c_n^λ so that c_n^λ counts the number of times 2^n appears in λ, and order the set of tuples λ by the dictionary order on the associated sequences c_n^λ. Given any two nonequal tuples

λ and λ'' of length ℓ, we can assume $\lambda > \lambda''$; we want to construct a tuple λ' with $\lambda > \lambda' > \lambda''$ by following edges in the graph. An edge from λ of length ℓ to μ of length $\ell - 1$ corresponds in sequences to

$$c_n^\mu = \begin{cases} c_n^\lambda & n \neq i, \\ c_i^\lambda - 2 & n = i, \\ c_{i+1}^\lambda + 1 & n = i + 1 \end{cases}$$

for some selected index i. So, as $\lambda < \lambda''$, we select the first differing index i and remove 2 from c_i^λ, add 1 to c_{i+1}^λ, then select any index $j > i$ and remove 1 from c_j^λ, adding 2 to c_{j-1}^λ. The resulting sequence describes a new tuple λ' satisfying $\lambda < \lambda' < \lambda''$. Induction on the imposed ordering gives the lemma. □

Corollary 7.5. *Let u_+ be an additive 2-cocycle over $\mathbb{F}_2$. If $u_+ \neq \zeta_2^n$ for some n, then u_+ is obstructed by Theorem 7.3.*

Proof. Every obstruction $a_i a_j$ stemming from an application of Theorem 7.3 to a term of the form $\zeta_2^{2^i} x_3^{2^j}$ can only be canceled by the appearance of a term of the form $\zeta_2^{2^j} x_3^{2^i}$, and hence the entire connected component of the graph in the above lemma containing any of the tuples appearing in u_+ must appear, lest we produce a nontrivial obstruction. The lemma says that the graph itself is connected; hence u_+ must be a scalar multiple of $\zeta_2^{|u_+|}$. □

Lemma 7.6. *For $\nu_2\phi(n, k) > \nu_2 n$, $u_+ = \zeta_2^n$ is obstructed from lifting to a multiplicative 2-cocycle.*

Proof. The case $\nu_2\phi(n, k) > \nu_2 n$ corresponds exactly to the appearance of summands of the form $\tau(2^{i_1}, \dots, 1)$ in ζ_k^n. Applying Theorem 7.3, we produce an obstruction of the form $a_{i_1} a_0$ with no mirror $a_0 a_{i_1}$, since $[a_0]$ is not in the image of the additive Weil pairing e_+. □

Theorem 7.7. *We compute*

$$\begin{aligned} \mathcal{O}(\operatorname{Spec} \mathbb{F}_2 \times C^k(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_m)) &= \mathbb{F}_2[z_n \mid \nu_2\phi(n, k) \leq \nu_2 n] \otimes \Gamma[b_{n,\gamma_2(n,k)} \mid \nu_2\phi(n, k) > \nu_2 n] \\ &\quad \otimes \mathbb{F}_2[b_{n,i} \mid \gamma_2(n, k) < i < D_{n,k}]/\langle b_{n,i}^2 \rangle, \end{aligned}$$

where $n \geq k$ ranges over integers, $D_{n,k}$ is the coefficient of the generating function

$$\prod_{i=0}^{\infty} \frac{1}{1 - tx^{2^i}} = \sum_{n,k} D_{n,k} x^n t^k,$$

and $\gamma_p(n, k) = \max\{0, \min\{k - \sigma_p(n), \nu_p(n)\}\}$ counts the number of divided power classes introduced already.

Proof. These tensor factors correspond, in order, to the additive cocycles ζ_k^n which lift without additional restrictions to multiplicative cocycles, via Theorem 7.1, the additive cocycles ζ_k^n which are obstructed by Lemma 7.6, lifted using the divided power exponential, and the remaining additive cocycles $\tau(\lambda)$ not already belonging to a divided power structure, which are also obstructed by Theorem 7.3 and which are lifted using the truncated exponential. □

Remark 7.8. We have remained deliberately vague about the description of the third tensor factor in Theorem 7.7 above, specifically in the choices of λ. It is clear that the polynomial ζ_k^n and its 2^j-powers play important roles in describing the space of multiplicative cocycles. However, there is too little structure present in the computation and in the resulting answer to produce a preferred basis of additive cocycles extending $\{(\zeta_k^n)^{2^j}\}$.

Corollary 7.9. *We compute the 2-primary component to be*

$$\begin{aligned} \mathcal{O}(\operatorname{Spec} \mathbb{Z}_{(2)} \times C^k(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_m)) &= \mathbb{Z}_{(2)}[z_n \mid \nu_2\phi(n, k) \leq \nu_2 n] \otimes \Gamma[b_{n,\gamma_2(n,k)} \mid \nu_2\phi(n, k) > \nu_2 n] \\ &\quad \otimes \mathbb{Z}_{(2)}[b_{n,i} \mid \gamma_2(n, k) < i < D_{n,k}]/\langle 2b_{n,i}, b_{n,i}^2 \rangle. \end{aligned}$$

Proof. This follows immediately from Theorem 7.7, which gives the general structure of the answer, and Theorem 4.9, which shows that the j in the $2^j b_{n,i}$ in the quotient must be a 1. □

8. Outro

We now reflect on this paper's relation to previous work, its relation to topology, and what is left to pin down internally, so that the thread may easily be picked up again in the future.

Remark 8.1 (*Odd Dimensional Topological Classes*)**.** The whole of Ando, Hopkins, and Strickland's argument [2] is to compare the functors $\operatorname{Spec} E_0\underline{kU}_{2k}$ and $C^k(\operatorname{Spf} E^0\mathbb{CP}^\infty; \hat{\mathbb{G}}_m)$ for certain (co)homology theories E, where $\underline{kU}_{2k}$ denotes the $(2k-1)$-connected cover of $BU \times \mathbb{Z}$ or equivalently the $2k$th space in the Ω-spectrum representing connected complex K-theory. The multiplicative structure on $E_0\underline{kU}_{2k}$ arises from the destabilization of the Whitney sum of stable virtual bundles, and in the case that $E_0\underline{kU}_{2k}$ is even-concentrated, $\operatorname{Spec} E_0\underline{kU}_{2k}$ makes sense. They reduce to the cases $E = H\mathbb{Q}$ and $E = H\mathbb{F}_p$, where $\operatorname{Spf} E^0\mathbb{CP}^\infty \cong \hat{\mathbb{G}}_a$, and they complete the proof by explicitly calculating $\mathcal{O}(C^k(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_m))$ for $k \leq 3$ and recalling the previously known calculation of $H^*(\underline{kU}_{2k}; \mathbb{F}_p)$ due to Singer [12] and at $p = 2$ to Stong [14].

The computation presented in this paper is part of an attempt to compare $\mathcal{O}(C^k(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_m))$ and $H_*\underline{kU}_{2k}$ for $k > 3$, where we find an immediate obstruction on the topological side. Singer's calculation describes $H^*\underline{kU}_{2k}$ as a quotient of H^*BU tensored together with a certain subalgebra of $H^*K(\mathbb{Z}, 2k-3)$. At $p = 2$, this subalgebra contains the class $\mathrm{Sq}^7\,\mathrm{Sq}^3\,\iota_{2k-3}$, which is nonvanishing for $k > 3$ and of odd cohomological degree. The usual supercommutativity present in algebraic topology thus presents an obstacle to the immersion of the ring $H_*\underline{kU}_{2k}$ into algebraic geometry, which traditionally takes as input only commutative rings, and so we must modify what ring we expect to compare to $\mathcal{O}(C^k(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_m))$.

Calculational experiments with Mathematica have shown that the graded ranks of indecomposables in $H^*(\underline{kU}_{2k}; \mathbb{F}_2)$ match those of the indecomposables in $\mathcal{O}(\operatorname{Spec} \mathbb{F}_2 \times C^k(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_m))$ through some 240 bidegrees, ranging in both n and k, after we delete the closure of the odd dimensional classes under the action of the Steenrod algebra. This is strong evidence that these two objects can yet be successfully compared.

Remark 8.2 (*Equivariance*)**.** The construction of the map

$$\operatorname{Spec} H_*(\underline{kU}_{2k}; \mathbb{F}_2) \to C^k(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_m) \times \operatorname{Spec} \mathbb{F}_2$$

described by Ando, Hopkins, and Strickland admits a certain compatibility with the Steenrod algebra suggested to be present by the above brute-force computation. The module $H_*(\underline{kU}_{2k}; \mathbb{F}_2)$ is a coalgebra over the dual Steenrod algebra almost by definition, and the scheme $C^k(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_m)$ carries an action of the scheme $\underline{\operatorname{Aut}}(\hat{\mathbb{G}}_a)$ of group automorphisms of the additive formal group. Long-standing work identifying the role of cohomology operations/homology cooperations in the context of chromatic homotopy theory has shown that the dual Steenrod algebra occurs as the ring of functions on $\underline{\operatorname{Aut}}(\hat{\mathbb{G}}_a)$, and hence the coaction of the dual Steenrod algebra on $H_*(\underline{kU}_{2k}; \mathbb{F}_2)$ can be seen as an action of $\underline{\operatorname{Aut}}(\hat{\mathbb{G}}_a)$ on $\operatorname{Spec} H_*(\underline{kU}_{2k}; \mathbb{F}_2)$. Moreover, the map $\operatorname{Spec} H_*(\underline{kU}_{2k}; \mathbb{F}_2) \to C^k(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_m) \times \operatorname{Spec} \mathbb{F}_2$ is seen to be $\underline{\operatorname{Aut}}\,(\hat{\mathbb{G}}_a)$-equivariant.

To use their map to form the comparison of $H_*(\underline{kU}_{2k}; \mathbb{F}_2)$ and $C^k(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_m) \times \operatorname{Spec} \mathbb{F}_2$ for $k > 3$, we need to be able to describe it in some detail. Its equivariance with respect to this action greatly rigidifies this problem, provided we can calculate the $\underline{\operatorname{Aut}}(\hat{\mathbb{G}}_a)$-action on both of these objects. The description of the action on the nilpotent part of $C^k(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_m) \times \operatorname{Spec} \mathbb{F}_2$ is quite easy to calculate, but the action on the free part is not known at this time.

Understanding the Steenrod action on $C^k(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_m)$ may also allow us to employ an argument similar to that of Adams and Priddy [1], where any spectrum with a particular sort of homology is shown to be equivalent to kSO. Their argument rests upon investigating the Adams spectral sequence, and so requires as input the Steenrod action on the homology as well. We are in a similar sort of position: an understanding of the dual Steenrod coaction on $\mathcal{O}(C^k(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_m)) \otimes \mathbb{F}_2$ would allow us to construct at least the E_2-page of a fantasy Adams spectral sequence. Using that, we could potentially recover a variety of partial information about the homotopy theory of spectra whose homology is as desired.

Remark 8.3 (*Hopf Rings and Ring Schemes*)**.** One idea unexploited in this paper is Cartier duality. For an even-concentrated H-space X and even periodic ring spectrum E, both the homology E_0X and cohomology E^0X are Hopf algebras, and the duality between their multiplications and diagonals is encoded in the algebro-geometric formula $\underline{\operatorname{Hom}}\,(\operatorname{Spf} E^0X, \hat{\mathbb{G}}_m) \cong \operatorname{Spec} E_0X$. In general, the object $\underline{\operatorname{Hom}}\,(\operatorname{Spf} E^0X, \hat{\mathbb{G}}_m)$ is called the Cartier dual of the group scheme $\operatorname{Spf} E^0X$. Our calculation in Theorem 7.7 demonstrates that $\operatorname{Spec} \mathbb{Z}_{(2)} \times C^k(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_m)$ has a well-behaved Cartier dual $C_{k,(2)}(\hat{\mathbb{G}}_a)$ satisfying $\underline{\operatorname{Hom}}\,(C_{k,(2)}(\hat{\mathbb{G}}_a), \hat{\mathbb{G}}_m) \cong C^k(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_m) \times \operatorname{Spec} \mathbb{Z}_{(2)}$, and we expect these congruences to match up in the sense that the following diagram should commute:

$$\begin{array}{ccc} \operatorname{Spec} H_*(\underline{kU}_{2k}; \mathbb{F}_2) & \longrightarrow & \operatorname{Spec} \mathbb{F}_2 \times C^k(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_m) \\ \| & & \| \\ \underline{\operatorname{Hom}}\,(\operatorname{Spf} H^*(\underline{kU}_{2k}; \mathbb{F}_2), \hat{\mathbb{G}}_m) & \longrightarrow & \underline{\operatorname{Hom}}\,(C_{k,(2)}(\hat{\mathbb{G}}_a), \hat{\mathbb{G}}_m). \end{array}$$

Then, because ku is a ring spectrum and $H\mathbb{F}_2$ has Künneth isomorphisms, we should expect that $H_*(\underline{kU}_{2k}; \mathbb{F}_2)$ assemble into a Hopf ring as k varies. The induced structure on formal schemes is harder to understand; that Spf and Spec are arrow-reversing indicates that $\operatorname{Spec} H_*(\underline{kU}_{2*}; \mathbb{F}_2)$ should assemble into a "coring scheme", a somewhat unfamiliar object. However, the dual schemes $\operatorname{Spf} H^*(\underline{kU}_{2*}; \mathbb{F}_2)$ assemble into a graded ring scheme, and using Cartier duality, understanding these objects should in turn give descriptions of the original homological objects of interest. This program is outlined in part by Ando et al. [2, Remark 2.32] in their original paper, and there are known computations of the Hopf ring structure for $H_*\underline{kO}_*$, due to Cowen Morton [10, Section 5], and for $H_*\underline{kU}_*$, due to Hara [5, Section 4].

Remark 8.4 (*Odd Primary Information*)**.** We note that these same techniques give partial results at odd primes. The entire program goes through for building the obstruction theory, but the resulting theorem at the end does not cover all possible cases any longer. For the interested reader, we recount statements of the relevant computations. First, the Lubin–Tate cohomology is computed to be

$$H^*(\hat{\mathbb{G}}_a; \hat{\mathbb{G}}_a)(\mathbb{F}_p) \cong \left(\bigotimes_i \Lambda[a_i]\right) \otimes \left(\bigotimes_i \mathbb{F}_p[b_i]\right),$$

where a_i represents x^{p^i} and b_i represents $\zeta_2^{p^i}$. Then, using truncated exponentials to partially lift $x^{p^i}y^{p^j}$ to a multiplicative 2-cocycle, one computes the following differential in the tangent spectral sequence, pictured in Fig. 4:

$$d_{(p-1)(p^i+p^j)}(ca_ia_j) = c^p(a_{i+1}b_{j+1} - a_{j+1}b_{i+1}).$$

E_1	1	2	3	4	5	6
9	a_2	b_2			$a_1b_1^2$	b_1^3
8						$a_0a_1b_0b_1$
7				$a_0a_1b_1$	$a_1b_0b_1, a_0b_1^2$	$b_0b_1^2$
6			a_1b_1	b_1^2		$a_0a_1b_0^2$
5				$a_0a_1b_0$	$a_0b_0b_1$	$b_0^2b_1$
4		a_0a_1	a_1b_0, a_0b_1	b_0b_1		
3	a_1	b_1			$a_0b_0^2$	b_0^3
2			a_0b_0	b_0^2		
1	a_0	b_0				

Fig. 4. The E_1-page of the tangent spectral sequence over $R = \mathbb{F}_3$.

By combining this differential with Lemma 6.7, one produces an obstruction theorem which produces a large family of obstructions to multiplicative lifts. However, this is only sometimes useful, with frequency decreasing as p grows. For a useful instance, one can compute all the obstructions necessary to describe the multiplicative lifts of linear combinations of the additive cocycles $\tau(9, 2, 1) - \tau(10, 1, 1)$ and $\tau(6, 3, 3)$ over $\mathbb{F}_3$, precisely because $2 + 1$ in $(9, 2, 1)$ is a power of 3 and $6 + 3$ in $(6, 3, 3)$ is a power of 3, so the Weil pairing calculation of Lemma 6.7 applies. On the other hand, we have no information about the adjacent cocycles $\tau(9, 1, 1, 1)$ and $\tau(3, 3, 3, 3)$, since no pair of elements in these multi-indices sum to a power of 3. It is not clear how to get around this problem — our suspicion is that the methods in this paper cannot be made to extend to give a complete solution at odd primes.

Remark 8.5 (*The Adams Splitting*)**.** The observations in Remark 8.4 are somewhat reflected in the topological situation by the Adams splitting of the connective K-theory spectrum kU. There is a spectrum BP occurring as the minimal summand in the p-localization of the complex bordism spectrum MU, and Stephen Wilson [17] describes a sequence of approximating spectra

$$BP\langle\infty\rangle \to \cdots \to BP\langle n\rangle \to \cdots \to BP\langle 1\rangle \to BP\langle 0\rangle,$$

with $BP\langle\infty\rangle \simeq BP$, $BP\langle 0\rangle \simeq H\mathbb{Z}_{(p)}$, and $\pi_*BP\langle n\rangle \cong \mathbb{Z}_{(p)}[v_1, \ldots, v_n]$ with $|v_n| = 2(p^n - 1)$. The folk theorem states that as ring spectra we have a splitting

$$L_pkU \simeq \bigvee_{i=0}^{p-2} \Sigma^{2i}BP\langle 1\rangle.$$

This is to say that the data in connective K-theory falls neatly into bands described by these truncated Brown–Peterson summands. In the previous paper [6], we witnessed a similar banding in the data, described in the 0th stratum by power-of-p multi-indices and in the nth stratum by distance leftward (i.e., in decreasing dimension) from the power-of-p band. Something similar happens in our Theorem 7.3: it is a necessary hypothesis that we be working in the band one step leftward of the power-of-p band. If not, the obstruction produced by the half-Weil pairing is always 0, as illustrated in the above example with $\tau(3, 3, 3, 3)$.

It would be interesting (and likely important) to understand what subfunctor Spec $H_*\underline{BP\langle 1\rangle}_k$ represents and what of our methods are more appropriately cast in that language. This splitting is made use of by Hara [5, Proposition 4.9] in his study of the Hopf ring structure of $H_*\underline{kU}_{2k}$, so this can be made to tie in with Remark 8.3. Moreover, it is an interesting question what Spec $H_*\underline{BP\langle k'\rangle}_k$ represents in general, and how these are assembled from the even further split objects Spec H_*Y_k described using methods highly relevant to Remark 8.1 by Zabrodsky [18, Section 5.2]. This question for $\underline{BP\langle k'\rangle}_*$ has been addressed in a point-wise fashion by Sinkinson [13].

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