FORMALLY SMOOTH HOMOTOPICAL COALGEBRAS AND THE ANNULAR TOWER

ERIC PETERSON

ABSTRACT. We import into homotopy theory the algebro-geometric construction of the cotangent space of a geometric point on a scheme. Specializing to the category of spectra local to a Morava K–theory of height d, we show that this can be used to produce a choice-free model of the determinantal sphere as well as an efficient Picard-graded cellular decomposition of $K(\mathbb{Z}_p, d+1)$. Coupling these ideas to work of Westerland, we use these to give a "Snaith's theorem" for the Iwasawa extension of the $K(d)$–local sphere.

1. INTRODUCTION

Much of modern chromatic homotopy theory is underpinned by a process that converts spaces to formal schemes: given an even–periodic cohomology theory E and a space X, if E^*X is suitably nice we can define a formal scheme X_E by the formula

$$X_E := \operatorname{Spf} E^0 X,$$

where the formal topology can be taken to come from the subskeleta of a cellular structure on X. The very definition of complex–orientability is designed so that this construction carries $\mathbb{C}\mathrm{P}^\infty$ to the formal affine line:

$$\operatorname{Spf} E^0\mathbb{C}\mathrm{P}^\infty = \{\operatorname{Spec} E^0\mathbb{C}\mathrm{P}^n\} \cong \{\operatorname{Spec} E^0[x]/x^{n+1}\} =: \widehat{\mathbb{A}}^1_{/\operatorname{Spec} E^0}.$$

In fact, many familiar values of X are carried to other familiar schemes—for instance, $BU(n)_E$ models the scheme of effective horizontal Weil divisors on $\mathbb{C}\mathrm{P}^\infty_E$ of rank n, $\mathbb{H}\mathrm{P}^\infty_E$ is sent to a formal curve, and the desymplectification map is sent to a degree 2 isogeny $\mathbb{H}\mathrm{P}^\infty_E \to \mathbb{C}\mathrm{P}^\infty_E$. The reach of this construction is maximized when $E = E_\Gamma$ is taken to be the Morava E–theory associated to a formal group Γ of finite height d over a perfect field k of positive characteristic p. In this setting, a great many theorems have been proven about this assignment from spaces to formal schemes, as partially tabulated in Figure 1.

The most remarkable feature of this table is that when Γ varies and the space X is fixed, the same family of formal schemes appears as output. This inspires us to consider X itself as playing some scheme–like role at the level of homotopy theory and to study algebro-geometric operations on X intrinsically. The goal of the present work is to use a kind of Hochschild homology construction to define the "tangent spectrum" of certain extremely nice spaces X, then to couple it to E_Γ–theoretic computations to produce some interesting objects in Γ–local spectra. The main result of this paper is thus:

Theorem. *For X any space and Γ any finite height formal group over a field k, there is a tower of Γ–local spectra, functorially in X:*

X	X_{E_Γ}
one–point space	Lubin–Tate space for Γ
$\mathbb{C}\mathrm{P}^\infty$	universal deformation $\tilde{\Gamma}$ of Γ
$BU(n)$	effective horizontal Weil divisors of rank n on $\tilde{\Gamma}$
$BU \times \mathbb{Z}$	stable Weil divisors on $\tilde{\Gamma}$
BU	stable Weil divisors on $\tilde{\Gamma}$ of virtual rank 0
$BS^1[p^j]$	the p^j–torsion subgroup $\tilde{\Gamma}[p^j]$
$K(\mathbb{Z}_p, q+1)$	the q^{th} exterior power of $\tilde{\Gamma}$
$K(\mathbb{Z}_p/p^j, q)$	the p^j–torsion subgroup of $\tilde{\Gamma}^{\wedge q}$
$B\Sigma_n$ (mod transfers)	subgroup divisors of rank n on $\tilde{\Gamma}$
BSU	special stable Weil divisors on $\tilde{\Gamma}$ of virtual rank 0
$BU[6,\infty)$	$\mathrm{Sym}^3_{\mathrm{Div}\,\tilde{\Gamma}} \mathrm{Div}_0\,\tilde{\Gamma}$
$\mathbb{H}\mathrm{P}^\infty$	a ramified double cover of $\tilde{\Gamma}$
$B\mathrm{Sp}$	stable Weil divisors on $\mathbb{H}\mathrm{P}^\infty_{E_\Gamma}$
$\vdots$	$\vdots$

FIGURE 1. Some of the known association between spaces and formal schemes

$$\begin{tikzcd}
C_0^\infty \arrow[r] & C_1^\infty \arrow[r] & C_2^\infty \arrow[r] & \cdots \\
C_0^0 \arrow[u, "\mathrm{fib}"] & C_1^1 \arrow[u, "\mathrm{fib}"] & C_2^2 \arrow[u, "\mathrm{fib}"] & \cdots,
\end{tikzcd}$$

with natural equivalences

$$C_0^0 \simeq \mathbb{S}, \qquad C_0^\infty \simeq \Sigma_+^\infty X, \qquad C_1^\infty \simeq \Sigma^\infty X.$$

If X_{E_Γ} is a formal curve (i.e., $E_\Gamma^0 X$ is noncanonically isomorphic to a univariate formal power series E_Γ^0–algebra) and $p \gg \operatorname{ht}\Gamma$, then

$$T_0^*(X_{E_\Gamma}) \cong E_\Gamma^0(C_1^1)$$

and $C_k^k = (C_1^1)^{\wedge k}$.

In the case that X_{E_Γ} is a formal curve, we are thus motivated to take C_1^1 as a definition of T_+X, the tangent space of X at its natural pointing $\mathbb{S}^0 \to \Sigma_+^\infty X$. This construction is most interesting when X is chosen so as to have some special relevance to the Γ–local category—for instance, we will show in Theorem 3.3.5 that $X = \Sigma_+^\infty \underline{H\mathbb{Z}}_{d+1}$ has X_{E_Γ} a formal curve, which is not at all typical of other complex-oriented cohomology theories.

Corollary. *For $p \gg \operatorname{ht}\Gamma$, the Γ–local spectrum $T_+\Sigma_+^\infty \underline{H\mathbb{Z}}_{d+1}$ gives a choice–free model of $\mathbb{S}^{\det}$, the determinantal sphere of Gross and Hopkins (cf.* [13, Remark 2.5], [17, Theorem 6 and Corollary 3]*).*

Corollary. *For $p \gg \operatorname{ht}\Gamma$, the spectrum $\Sigma_+^\infty \underline{H\mathbb{Z}}_{d+1}$ acquires an efficient Γ–local Picard–graded cellular decomposition, with one cell of the form $C_k^k \simeq (\mathbb{S}^{\det})^{\wedge k}$ for each $k \geq 0$.*

This is meant to be analogous to the motivic cellular decomposition of $\mathbb{C}\mathrm{P}^\infty$ into even spheres, attached along homotopy classes graded by $\Sigma^{-1}(\mathbb{C}\mathrm{P}^1)^{\wedge k}$. Inspired by the algebro-geometric interpretation, we refer to this as the *annular tower*.

Alternatively, the cellular decomposition of $\mathbb{C}\mathrm{P}^\infty$ can also be interpreted as the skeletal filtration for the bar construction $B(\mathbb{C}^\times)$. This, too, admits generalization:

Corollary. *There is an A_∞–ring structure on $(\Sigma^{-1}\mathbb{S}^{\det})_+$ and a Γ–local equivalence*

$$B((\Sigma^{-1}\mathbb{S}^{\det})_+) \xrightarrow{\simeq} \Sigma^\infty_+ \underline{H\mathbb{Z}}_{d+1}.$$

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2. Homotopical coalgebras and comodules

2.1. Motivation: coalgebraic formal schemes.

To set the stage, we expand on our discussion from the introduction of treating a space as a scheme–like object, starting with a more careful description of "X_E".

Definition 2.1.1. Suppose that X is a CW–space, so that $X = \operatorname{colim}_\alpha\{X_\alpha\}$ can be written as the forward limit over the lattice of inclusions of its compact subspaces X_α. Suppose further that the system of finite E^*–algebras $\{E^*_\Gamma X_\alpha\}$ is equivalent to the pro-system of even–concentrated finite E^*–algebras $\{E^0 X_\alpha \otimes_{E^0} E^*\}$. We then set

$$X_{E_\Gamma} := \operatorname{Spf} E^0 X.$$

Remark 2.1.2 ([33, Section 8.2, Definition 8.15]). A sufficient condition on X to ensure that X_{E_Γ} exists is for $H_* X$ to be torsion–free and even, in which case there is a skeletal structure on X whose subskeleta form a cofinal subsystem of $\{X_\alpha\}$ and which have the desired evenness property. For instance, this covers the cases of $X = \mathbb{C}\mathrm{P}^\infty$ and $X = BU$. It does not, however, cover $X = BS^1[p^j]$, to which the definition given here nonetheless applies.

Remembering our goal of studying X directly, we would now like to identify a topological system underlying this construction. Using the rewriting

$$E^*_\Gamma X_\alpha = \pi_{-*} F(\Sigma^\infty_+ X_\alpha, E_\Gamma) = \pi_{-*}\left(F(\Sigma^\infty_+ X_\alpha, \mathbb{S}) \wedge E_\Gamma\right),$$

we are inspired to consider the E_∞ pro-spectrum

$$\widehat{D}X = \{F(\Sigma^\infty_+ X_\alpha, \mathbb{S})\}.$$

This pro-spectrum depends only on X and not on E_Γ, and the system appearing in Definition 2.1.1 then arises as the homotopy of the base change

$$\eta^* \widehat{D}X \simeq \{E_\Gamma \wedge F(\Sigma^\infty_+ X_\alpha, \mathbb{S})\},$$

where $\eta\colon \mathbb{S} \to E_\Gamma$ is the unit map for the ring spectrum E_Γ. Hence, $\widehat{D}X$ plays something of a role of a universal object for these constructions.

We now immediately set our eyes on the main Theorem. Recall the defining diagram of A–modules, presented in Figure 2, for the cotangent space for a geometric point $\operatorname{Spec} A/\mathfrak{m}$ including into an affine scheme $\operatorname{Spec} A$. Each angle in the diagram is a cokernel sequence of A–modules. Our goal is to lift this diagram of cokernel sequences of modules to a diagram of cofiber sequences of spectra, such that the original diagram is recovered upon applying E_Γ–cohomology. A candidate replacement for A itself is the pro-spectrum $\widehat{D}X$, and it comes equipped with an augmentation map $\widehat{D}X \to \mathbb{S}$ to the constant pro-spectrum $\mathbb{S}$.

$$\begin{tikzcd}
A \arrow[d] & \mathfrak{m} \arrow[l] \arrow[d] & \mathfrak{m}^2 \arrow[l] \\
A/\mathfrak{m} & \mathfrak{m}/\mathfrak{m}^2 = T_0^* \operatorname{Spec} A, &
\end{tikzcd}$$

FIGURE 2. Definition of the cotangent space for a pointed affine scheme

However, computing with $\widehat{D}X$ is prohibitively complicated because of the Spanier–Whitehead dual and inverse limit in its definition. To skirt this problem, we take inspiration from the algebraic geometers: for a field k, pro-finite k–algebras are equivalent to ind-finite k–coalgebras by k–linear duality, and *all* k-coalgebras are equivalent to ind-finite ones. Hence, the theory of formal schemes (over $\operatorname{Spec} k$) can be taken to be underpinned by coalgebras instead.[1] At the level of spectra, pro-finite spectra are also equivalent to ind-finite spectra by Spanier–Whitehead duality, and spectra themselves are equivalent to ind-finite spectra. Hence, the analogous coalgebraic spectral object to the "profinite ring spectrum" $\widehat{D}X$ is $C = \Sigma^\infty_+ X$ itself. We need only record the sense in which C is a coalgebra before we can start doing coalgebraic geometry with it.

2.2. **Definitions and constructions.**

We borrow the following definition from Haugseng and Lurie:

Definition 2.2.1 ([14, Definition 4.12], [24, Proposition 4.1.2.15])**.** An *associative coalgebra spectrum* in a monoidal ∞–category C is a Δ^{op}–algebra in C^{op}.

Lemma 2.2.2. *A space X determines an associative algebra object in the monoidal ∞–categories* $\mathsf{Spectra}^{\mathrm{op}}$ *and* $\mathsf{Spectra}_\Gamma^{\mathrm{op}}$.

Proof. Spaces (using the underlying diagonal map of sets) already determine strictly coassociative algebra objects on the 1–categorical level. By localizing away from the weak equivalences and passing to the associated ∞–category, we recover an associative algebra object in $\mathsf{Spaces}^{\mathrm{op}}$. Because the stabilization functor $\Sigma^\infty_+ : \mathsf{Spaces} \to \mathsf{Spectra}$ respects the Cartesian monoidal structure on Spaces and the smash monoidal structure on $\mathsf{Spectra}$ [24,

[1] This perspective is recorded by by Demazure [10] in the case of a field and by Strickland [33, Section 4.8] more generally.

Propositions 4.8.2.9 and 4.8.2.18] and because the monoidal structure on Γ–local spectra is defined so that the localization functor L_Γ is monoidal, it follows that $L_\Gamma \Sigma^\infty_+ X$ is a coaugmented coassociative coalgebra spectrum. $\square$

Our spectral analogue of the first fiber sequence in Figure 2 is thus given by the following diagram:

$$\begin{tikzcd} C \arrow[r] & M \\ \mathbb{S}, \arrow[u, "\eta"] \end{tikzcd}$$

where $C = \Sigma^\infty_+ X$ is a coalgebraic spectrum and $\eta\colon \mathbb{S} \to C$ is the coaugmentation map. In order to construct the second fiber sequence, we would like to interpret this diagram as a sequence of C–comodules, and we would like access to a coalgebraic analogue of the product of ideals. In general, this is too much to ask: the missing theory of ideals for (structured) ring spectra remains, to date, a thorn in homotopy theorists's sides. However, in the classical case that two A–ideals I and J are *monogenic*, there is a natural isomorphism $I \otimes_A J \cong I \cdot J$. This includes our primary case of interest, where $A = k\llbracket x \rrbracket$ and $I = J = \mathfrak{m} = (x)$ is the ideal of functions vanishing to first order at $x = 0$.[2] Motivated by this observation, we, again, borrow a definition from Haugseng and Lurie:

Definition 2.2.3 ([14, Definition 4.12], cf. [24, Definition 4.3.1.6, Definition 4.4.1.1, Proposition 4.4.1.11])**.** A *cobimodule object* in a monoidal ∞–category C is a $\Delta^{\mathrm{op}}_{/[1]}$–algebra in C^{op}. (Restriction to either endpoint of $[1]$ gives two coalgebra objects C and D, in which case we call the cobimodule a C–D–cobimodule.) A *length n chain of compatible comodules* is a $\Lambda^{\mathrm{op}}_{/[n]}$–algebra in C^{op}. A *length n cotensor witness* is a $\Delta^{\mathrm{op}}_{/[n]}$–algebra in C^{op}.

Lemma 2.2.4 ([14, Lemma 4.19, Corollary 4.20], cf. [24, Theorem 4.4.2.8])**.** *Let $P\colon \Lambda^{\mathrm{op}}_{/[n]} \to \mathsf{C}^{\mathrm{op}}$ be a length n chain of compatible cobimodules in C, and let $F\colon \Delta^{\mathrm{op}}_{/[n]} \to \mathsf{C}^{\mathrm{op}}$ be a filling of P to a length n tensor witness. The spectrum underlying the cotensor product comodule witnessed by F is necessarily weakly equivalent to the limit of the cobar construction on the compatible cobimodules. Conversely, suppose that $n = 2$ and only P is given. If the natural C^{op}–map*

$$|X \otimes B(M; S; N) \otimes Y| \to X \otimes |B(M; S; N)| \otimes Y$$

is an equivalence for all objects X and Y, then there must exist a filler F. $\square$

Remark 2.2.5. Haugseng's filling result is not specific to cotensor products of length 2, and it is functorial as the $\Lambda^{\mathrm{op}}_{/[n]}$–algebra varies. However, we will be content with the weakened result stated here.

The primary obstacle for us now is the natural equivalence hypothesis in the Lemma. In our setting of $\mathsf{C} = \mathsf{Spectra}$ or $\mathsf{C} = \mathsf{Spectra}_\Gamma$, totalizations of cosimplicial objects *do not* typically commute with the (local) smash product of (local) spectra. So, while we can use the cobar construction to define a cotensor product and various maps concerning it, essentially no good properties can be deduced for the maps without first manually checking this condition about commuting limits and smash products. For the moment, we content ourselves with constructing the maps, leaving the question of properties for the next section.

[2] However the augmentation ideal of a higher–dimensional power series algebra is *not* monogenic.

Lemma 2.2.6. *For a right C-comodule M, there is a natural retraction*

$$M \to M \square_C C \to M,$$

and dually for a left C-comodule N. □

Lemma 2.2.7. *Let M be a C-C-cobimodule equipped with a map $\pi\colon C \to M$ of cobimodules. There is then a diagram*

$$\begin{tikzcd}
C \arrow[r, "\pi"] \arrow[d, "\Delta_C"] & M \arrow[d, "\psi_R"] \arrow[dr, "\widetilde{\Delta}_R"] & \\
C \square_C C \arrow[r, "\pi \square_C C"] \arrow[d] & M \square_C C \arrow[r, "M \square_C \pi"] \arrow[d] & M \square_C M \\
C \arrow[r] & M &
\end{tikzcd}$$

where the vertical sequences are retractions. Alternatively, there is also a map $\widetilde{\Delta}_L$ using the left C-comodule structure of M.

Proof. The arrow Δ_C could just as well be called ψ_C, and by this name the top square commutes because π is a homomorphism of cobimodules. The arrow $\widetilde{\Delta}_R$ is then defined by either composite, and it receives its name from the relabeling of ψ_C as Δ_C. □

Classically, if π is surjective, then Δ_R and Δ_L will agree on points exactly if M is taken to be a coideal. The word "coideal" is too bold in our homotopical context, since it suggests that the quotient is again a coalgebra. We instead adopt the following terminology:

Definition 2.2.8. Let M be a C-C-cobimodule spectrum equipped with a cobimodule map $\pi\colon C \to M$. Such a cobimodule spectrum is a *symmetric cobimodule* (for C) when $\widetilde{\Delta}_R$ and $\widetilde{\Delta}_L$ are equivalent.

Lemma 2.2.9. *Let $f : D \to C$ be a map of coalgebras, and suppose that there is a splitting of the cofiber:*

$$\begin{tikzcd}
D \arrow[r, "f"] & C \arrow[r, "\pi"] & \operatorname{cofib} f. \arrow[l, dotted, bend right]
\end{tikzcd}$$

The cofiber of f, considered as a C-C-cobimodule, is then a symmetric cobimodule.

Proof. We enlarge the diagram above to include $\widetilde{\Delta}_L$ and $\widetilde{\Delta}_R$:

$$\begin{tikzcd}
D \arrow[r, "f"] & C \arrow[rr, "\pi"] \arrow[dd, "\Delta"] & & \operatorname{cofib} f \arrow[ll, dotted, bend right] \arrow[dl, "\psi_R"'] \arrow[d, bend right, "\widetilde{\Delta}_R"'] \arrow[d, bend left, "\widetilde{\Delta}_L"] \arrow[dr, "\psi_L"] & \\
& & \operatorname{cofib} f \wedge C \arrow[r, "1\wedge\pi"] & \operatorname{cofib} f \wedge \operatorname{cofib} f & C \wedge \operatorname{cofib} f \arrow[l, "\pi\wedge 1"] \\
& C \wedge C. \arrow[ur, "\pi\wedge 1"] \arrow[urr, "\pi\wedge\pi"] \arrow[urrr, bend right, "1\wedge\pi"] & & &
\end{tikzcd}$$

We see that the two maps $\tilde{\Delta}_L$ and $\tilde{\Delta}_R$ agree upon precomposition to C. Since $\operatorname{cofib} f$ splits off of C, they agree on $\operatorname{cofib} f$ as well. □

Lemma 2.2.10. *For M a symmetric cobimodule admitting cotensor powers $M^{\square_C k}$ and $M^{\square_C (k+1)}$, all of the following maps coincide:*

$$M^{\square_C k} \xrightarrow{1\square\cdots\square 1\square\tilde{\Delta}\square 1\square\cdots\square 1} M^{\square_C (k+1)}.$$

Proof. The definition of the cobar construction gives a homotopy between the following two maps:

$$M\square_C M \underset{1\square_C \psi_L}{\overset{\psi_R \square_C 1}{\rightrightarrows}} M\square_C C\square_C M.$$

Using the symmetric cobimodule property of M, we can trade $\tilde{\Delta}_L$ for $\tilde{\Delta}_R$, and hence we can transfer the $\tilde{\Delta}$ to any coordinate. $\square$

We are now poised to properly state our construction:

Definition 2.2.11. Given a pointed coalgebra spectrum $\eta\colon \mathbb{S} \to C$, we define C_1^1 by the following chain of fiber sequences:

$$\begin{array}{ccccc} C & \longrightarrow & M & \xrightarrow{\tilde{\Delta}} & M\square_C M \\ {\scriptstyle\eta}\uparrow & & \uparrow & & \\ \mathbb{S} & & C_1^1. & & \end{array}$$

Remark 2.2.12. If C is merely a pointed coalgebra spectrum, this is as much of the diagram as we can form at this time. After all, we have not yet shown that $M\square_C M$ is a C–C–cobimodule, and hence we cannot guarantee the existence of "$M\square_C(M\square_C M)$". However, if we start with a space X and follow Lemma 2.2.2, then we can form the tower of iterated cobar diagrams in the 1–category of spaces:

$$X_+ \longrightarrow X \longrightarrow \Omega(X;X_+;X) \longrightarrow \Omega(X;X_+;X;X_+;X) \longrightarrow \cdots.$$

Pushing these diagrams forward along Σ^∞ and along the localizer L_Γ then produces inverse diagrams of (Γ–local) spectra. Thus, we are at least assured that, for $C = \Sigma^\infty_+ X$, the diagrams *determining* the higher cotensor powers (and hence the annular tower) are well-defined. We are still obligated to justify their utility by computing the value of E_Γ on their limits.

2.3. Computational tools.

We would like to justify Definition 2.2.11 above by calculating $E^0_\Gamma C^1_1$ and checking that it gives the desired cotangent module, and then we would also like to extend it to the full tower presented in the main Theorem. In view of Lemma 2.2.4, this rests directly on having tools available to compute the homology and cohomology of a cotensor product of comodule spectra. With a computational task ahead, it is now convenient to introduce Morava K–theory.

Definition 2.3.1. The coefficient ring $\pi_0 E_\Gamma$ is a power series ring over $\mathbb{W}_p(k)$, itself a complete local ring with maximal ideal generated by p. The *Morava K–theory spectrum*, K_Γ, is the quotient of E_Γ in E_Γ–modules by a generating regular sequence of $\pi_0 E_\Gamma$.

As a result, $\pi_* K_\Gamma = k[u^\pm]$ is a graded field. This makes it extremely valuable for computations—especially ones such as ours, where large smash products arise, since K_Γ has Künneth isomorphisms. Remarkably, despite being defined as a quotient, "K_Γ carries the same information as E_Γ" in the precise sense that K_Γ–acyclics, homological or cohomological, agree with cohomological E_Γ–acyclics [22, Proposition 2.5]. Hence, any of E_Γ–cohomology,

K_Γ–cohomology, or K_Γ–homology are equally good for testing Γ–local equivalences. We will favor the last option, both for technical reasons (cf. [2] and Appendix A) and because it covariantly converts the coalgebraic spectrum C into the $(K_\Gamma)_*$–coalgebra $(K_\Gamma)_* C$. To save on notational overhead, we will often write "K" alone when some fixed Γ is understood.

We now turn back to the task at hand: given C–C–bimodules M and N, we want to study the natural map

$$X \wedge \lim \Omega(M; C; N) \wedge Y \to \lim (X \wedge \Omega(M; C; N) \wedge Y)$$

and to find conditions under which it becomes a K–homology equivalence. The presentation of $M \square N$ as the totalization of a cosimplicial object equips it with a coskeletal filtration, but the homology of an inverse limit of spectra typically compares poorly with the inverse limit of the homologies of the individual spectra in an infinite system. Remarkably, a result of Sadofsky shows that K–homology does not suffer from this.

Theorem 2.3.2 (cf. Theorem A.0.1). *Take $p \gg \operatorname{ht}\Gamma$, and let $\{X_\alpha\}_\alpha$ be a sequential inverse system of Γ–local spectra such that $\{K_* X_\alpha\}_\alpha$ is a Mittag-Leffler system of K_*–modules. There is then a convergent spectral sequence of signature*

$$R^* \lim_\alpha \{K_* X_\alpha\}_\alpha \Rightarrow K_* \lim_\alpha \{X_\alpha\}_\alpha,$$

where the right–derived inverse limit making up the input to the spectral sequence is taken in the category of K_*K–comodules. □

In order to make use of this spectral sequence, we need to compute some of its inputs. The homologies of the finite stages of the filtration are accessible because homology does pass through *finite* limits, and these admit the following uniform description.

Theorem 2.3.3. *Let C be a coalgebra spectrum, M a right C–comodule spectrum, and N a left C–comodule spectrum. Writing F for the fiber of the counit map $\varepsilon \colon C \to \mathbb{S}$, there is an n–indexed system of spectral sequences:*

$$E^1_{*,*} \cong K_* M \otimes_{K_*} (K_* F)^{\otimes (* \le n)} \otimes_{K_*} K_* N \Rightarrow K_* \operatorname{Tot} \operatorname{cosk}^n \Omega(M; C; N),$$

whose inverse limit in the category of K_*–modules is the spectral sequence

$$\left.\begin{array}{r} E^1_{*,*} \cong K_* M \otimes_{K_*} (K_* F)^{\otimes *} \otimes_{K_*} K_* N, \\ E^2_{*,*} \cong \operatorname{Cotor}^{K_* C}_{*,*}(K_* M, K_* N) \end{array}\right\} \Rightarrow \lim_n K_* \operatorname{Tot} \operatorname{cosk}^n \Omega(M; C; N).$$

The spectral sequences in the system are strongly convergent, and the full spectral sequence is conditionally convergent.

Proof. This is an aggregation of several standard results in the literature. Ravenel and Wilson [30, Section 2] provide a convenient summary of the bar spectral sequence, and these spectral sequences arise as its dual. Ravenel [28, Appendix A1] also provides a collection of results on the homological algebra of comodules and in particular gives a definition and lists properties for "Cotor". Finally, Boardman [6, Theorem 7.1] provides tools for analyzing the convergence of the spectral sequences. □

We now sew these together to analyze Sadofsky's inverse limit spectral sequence in the case at hand. This is mostly an exercise in homological algebra, so requires substantial bookkeeping but is otherwise fairly straightforward.

Theorem 2.3.4. *Continue to assume $p \gg \operatorname{ht}\Gamma$. When M, N, and C have even-concentrated K-homology, there is a further spectral sequence*

$$R^* \lim_t \{K_* \operatorname{Tot} \operatorname{cosk}^t \Omega(M;C;N)\}_t \Rightarrow K_*(M \square_C N),$$

where the derived inverse limit is taken in the category of K_*K-comodules.

Proof. Throughout, we will consider the degree $s+t$ part of the t-cochains C^t_{s+t}, coboundaries B^t_{s+t}, and cocycles Z^t_{s+t} of the normalized cobar complex

$$\Omega(K_*M; K_*C; K_*N).$$

With an induction beginning at $t = 0$ and $t = 1$, we claim the K-homology of the t^{th} level of the tower (i.e., the target of the t^{th} partial spectral sequence) is

$$K_s \operatorname{Tot} \operatorname{cosk}^t \Omega(M;C;N) = D^1_{s,t} = \begin{cases} D^1_{s,t-1} \oplus \dfrac{C^t_{\frac{s+t}{2}}}{B^t_{\frac{s+t}{2}}} & \text{if } s+t \text{ is even,} \\ D^1_{s,t-2} \oplus H^{t-1}_{\frac{s+t-1}{2}} & \text{if } s+t \text{ is odd,} \end{cases} \tag{2.1}$$

where $D^1_{*,*}$ denotes the rear of the exact couple of the full spectral sequence. In particular, this formula shows that the inverse limit tower is Mittag-Leffler, so that Sadofsky's hypotheses are satisfied.

Induction proceeds by considering one triangle in that exact couple:

$$\begin{tikzcd} K_s \operatorname{Tot}\operatorname{cosk}^{t-1}\Omega(M;C;N) \arrow[dr, "{[-1]}"'] & K_s \operatorname{Tot}\operatorname{cosk}^{t}\Omega(M;C;N) \arrow[l] & & D^1_{s,t-1} \arrow[dr, "{[-1]}"'] & D^1_{s,t} \arrow[l] \\ & K_s\Omega^t(M \wedge C^{\wedge t} \wedge N) \arrow[u] & = & & E^1_{s,t}. \arrow[u] \end{tikzcd}$$

The bottom vertex in the triangle is the K_*-module of cochains, and we take s to be a degree in which $K_{s+t}(M \wedge C \wedge \cdots \wedge C \wedge N)$ is nonvanishing, i.e., $s+t$ is even. Then, the triangle unrolls into an exact sequence which, using the inductive hypothesis, takes the form

$$\begin{tikzcd} 0 \arrow[r] & D^1_{s+1,t} \arrow[r] \arrow[d, equal] & D^1_{s+1,t-1} \arrow[r] \arrow[d, equal] & E^1_{s,t} \arrow[r] \arrow[d, equal] & D^1_{s,t} \arrow[r] \arrow[d, equal] & D^1_{s,t-1} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & H^{t-1}_{\frac{s+t}{2}} \oplus X \arrow[r, "i\oplus 1"] & \frac{C^{t-1}_{\frac{s+t}{2}}}{B^{t-1}_{\frac{s+t}{2}}} \oplus X \arrow[r, "\partial\oplus 0"] & C^t_{\frac{s+t}{2}} \arrow[r, "\pi\oplus 0"] & \frac{C^t_{\frac{s+t}{2}}}{B^t_{\frac{s+t}{2}}} \oplus Y \arrow[r, "0\oplus 1"] & Y \arrow[r] & 0 \end{tikzcd}$$

for some modules X and Y to be determined. We then splice three of these long sequences together to form Figure 3, which is labeled in terms of the cobar cohomology groups in Figure 4. The action lies in the zig-zag containing $D^1_{s,t+1}$ through $D^1_{s,t-2}$: $D^1_{s,t-1}$ and $D^1_{s,t-2}$ are assumed to be as claimed, then the observation involving Y shows that $D^1_{s,t-1}$ is as well, which in turn determines $D^1_{s,t+1}$ as the kernel of the map $D^1_{s,t} \to E^1_{s-1,t+1}$, i.e., as the cocycle subgroup of the cochain group, taken modulo the coboundaries. This proves the inductive claim. $\square$

$$\begin{tikzcd}
 & & & & 0 \arrow[d] & & \\
 & & & & D^1_{s,t+1} \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & D^1_{s+1,t} \arrow[r] & D^1_{s+1,t-1} \arrow[r] & E^1_{s,t} \arrow[r] & D^1_{s,t} \arrow[r] \arrow[dl, bend left] & D^1_{s,t-1} \arrow[r] \arrow[d] & 0 \\
0 & D^1_{s-1,t} \arrow[l] & D^1_{s-1,t+1} \arrow[l] & E^1_{s-1,t+1} \arrow[l] & & D^1_{s,t-2} \arrow[dl, bend left] & \\
 & 0 & D^1_{s-1,t-2} \arrow[l] & D^1_{s-1,t-1} \arrow[l] & E^1_{s-1,t-1} \arrow[l] & &
\end{tikzcd}$$

FIGURE 3. Three interacting exact sequences.

$$\begin{tikzcd}
 & & & & 0 \arrow[d] & & \\
 & & & & H^t_{\frac{s+t}{2}} \oplus D^1_{s,t-1} \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & D^1_{s+1,t} \arrow[r] & D^1_{s+1,t-1} \arrow[r] & C^t_{\frac{s+t}{2}} \arrow[r] & \frac{C^t_{\frac{s+t}{2}}}{B^t_{\frac{s+t}{2}}} \oplus D^1_{s,t-1} \arrow[r] \arrow[dl, bend left] & D^1_{s,t-1} \arrow[r] \arrow[d] & 0. \\
0 & D^1_{s-1,t} \arrow[l] & D^1_{s-1,t+1} \arrow[l] & C^{t+1}_{\frac{s+t}{2}} \arrow[l] & & D^1_{s,t-2} \arrow[dl, bend left] & \\
 & 0 & D^1_{s-1,t-2} \arrow[l] & D^1_{s-1,t-1} \arrow[l] & E^1_{s-1,t-1} \arrow[l] & &
\end{tikzcd}$$

FIGURE 4. The interacting sequences with cobar complex names.

2.4. **Computation in the case of a formal variety.** In general, it is very hard to control the spectral sequence of Theorem 2.3.4. Even computing these derived functors is prohibitively complicated in almost any nondegenerate case—for instance, Hopkins [25, Section 14] has recommended them in an approach to analyzing the chromatic splitting conjecture.[3] This moves us to consider progressively more specialized situations where we can fully determine this spectral sequence, beginning with the case that the formal scheme $\operatorname{Sch} K_* C$ is a formal variety.

Definition 2.4.1. A k–coalgebra C over a field k will be called a *formal variety (of dimension n)* when there is a natural isomorphism between the group–like elements of $C \otimes A$ and some fixed power $\mathfrak{m}^{\times n}$ of the maximal ideal $\mathfrak{m}$ of the augmented nilpotent k–algebra A. (Equivalently, the dual profinite algebra C^* is isomorphic to a power series ring.) A spectral coalgebra C will be called a *formal coalgebraic variety spectrum (for the formal group* Γ*)* when $K_* C$ is even–concentrated and $K_0 C$ is a formal variety in coalgebras. In both settings, we use *formal curve* as a synonym for a formal variety of dimension 1.

[3] Computing these derived functors can be compared to computing certain local cohomology groups and to certain "noncontinuous" forms of the group cohomology of the Morava stabilizer group. These are both very difficult invariants.

For the rest of this section, we take C to be a formal coalgebraic variety spectrum for Γ, together with a pointing $\eta\colon \mathbb{S} \to C$ with cofiber M. Supposing that $M^{\square_C j}$ has been shown to exist as a C–C–cobimodule spectrum, we will inductively pursue a C–C–cobimodule structure on $M^{\square_C (j+1)}$.

Theorem 2.4.2. *The system* $\{K_* \operatorname{cosk}^n \Omega(M; C; M^{\square_C j})\}$ *is proconstant.*

Proof. This system appears as the rear of the exact couple in Theorem 2.3.4, where we showed that it was presented levelwise as direct sums of two sorts of groups: Cotor groups and cochain groups modulo coboundaries. These groups are then linked together by appropriate projections (away from a direct sum factor) and inclusions (of cocycles modulo coboundaries—i.e., Cotor groups—into cochains modulo coboundaries). Our conclusion will follow from a direct calculation of these Cotor groups.

This problem is easily addressed by standard tools in homological algebra, which is perhaps more visible after dualizing to profinite algebras:

$$\operatorname{Cotor}^{*,*}_{K_* C}(K_* M, K_* N) \cong \operatorname{Tor}^{*,*}_{K_* C^\vee}(K_* M^\vee, K_* N^\vee)^\vee.$$

In our case, we can choose isomorphisms

$$K_* C^\vee \cong K_* [\![x_1, \ldots, x_n]\!], \quad K_* M^\vee = \langle x_1, \ldots, x_n \rangle$$

which gives a choice of regular sequence $(x_1, \ldots, x_n)$ on $K_* C^\vee$ and an associated Koszul complex [36, Corollary 4.5.5]:

$$K^{K_* C^\vee}(x_j : 1 \le j \le n) \cong \bigotimes_{1 \le j \le n} \left(K_* C^\vee \xrightarrow{\cdot x_j} K_* C^\vee \right).$$

This complex is a free resolution of the ground field K_*, and hence the shifted subcomplex $K^{K_* C^\vee}(x_j : 1 \le j \le n)^{\ge 1}[-1]$ models a free resolution of the kernel of the pointing $K_* C^\vee \to K_*$, i.e., of $K_* M^\vee$. However, the higher Tor groups

$$\operatorname{Tor}^{K_* C^\vee}_{>n,*}(K^{K_* C^\vee}(x_1, \ldots, x_n), (K_* M^\vee)^{\otimes j})$$

vanish, so this must also be true of

$$\operatorname{Tor}^{K_* C^\vee}_{>n,*}(K^{K_* C^\vee}(x_1, \ldots, x_n)^{\ge 1}[-1]; (K_* M^\vee)^{\otimes j}).$$

We now return to the original problem of pro-constancy. Our calculation shows that all but finitely many of the Cotor groups vanish, and hence the maps in the pro-system are zero on those nonvanishing factors, where the maps are the respective identity morphisms. It follows that the each tower $\{D^1_{s,t}\}_t$ is pro-constant. □

Corollary 2.4.3. *There is an isomorphism*

$$K_*(M \square_C M^{\square_C j}) \cong K_* M^{\square_{K_* C} (j+1)}.$$

Proof. Because the tower is pro-constant, the higher derived inverse limit functors of Theorem 2.3.4 vanish. □

Corollary 2.4.4. *The spectrum* $M^{\square_C (j+1)}$ *is a* C–C*–cobimodule spectrum.*

Proof. Tensoring the pro-constant towers of Theorem 2.4.2 with $K_* X$ and $K_* Y$ does not disturb their pro-constancy. It follows that the natural map

$$K_* X \otimes K_* M^{\square_{K_* C} (j+1)} \otimes K_* Y \to K_* \lim_n \operatorname{cosk}^n (X \wedge \Omega(M; C; M^{\square_C j}) \wedge Y)$$

is an isomorphism, and hence the conditions of Lemma 2.2.4 are satisfied. □

This completes the induction and shows that $M^{\square_C j}$ is a C–C–cobimodule spectrum for all values of j. We use this to justify the following definition.

Definition 2.4.5. Writing C_n^∞ for $M^{\square_C n}$, there are natural projection maps $C_n^\infty \to C_{n'+1}^\infty$ for $n' \geq n$. We define spectra $C_n^{n'}$ by the fiber sequence

$$C_n^{n'} \to C_n^\infty \to C_{n'+1}^\infty$$

and the *annular tower* is the sequential system

$$\begin{tikzcd}
C_0^\infty \arrow[r] & C_1^\infty \arrow[r] & \cdots \arrow[r] & C_n^\infty \arrow[r] & C_{n+1}^\infty \arrow[r] & \cdots \\
C_0^0 \arrow[u] & C_1^1 \arrow[u] & \cdots & C_n^n \arrow[u] & C_{n+1}^{n+1} \arrow[u] & \cdots .
\end{tikzcd}$$

In fact, our computational tools have been explicit enough that we can now also deduce that K–cohomology takes the correct value on C_1^1.

Corollary 2.4.6. *If C is a coalgebraic formal curve spectrum, the K–cohomology of the spectrum C_1^1 computes the cotangent space of* $\operatorname{Spf} K^0 C$.

Proof. In the case of a 1–dimensional coalgebraic formal variety spectrum, the Cotor groups computed by the Koszul complex in the proof of Theorem 2.4.2 vanish above homological degree 0, and hence

$$K_*(M \square_C M) \cong K_* M \square_{K_* C} K_* M.$$

The calculation follows directly. □

Definition 2.4.7. In the case where C is a coalgebraic formal curve spectrum, we write $T_\eta C$ for the spectrum C_1^1, so that the isomorphism of Corollary 2.4.6 reads as an interchange law

$$K^* T_\eta C = T^*_{\eta_K} C_K.$$

2.5. **Koszul duality.** In this section, we give an alternative interpretation of the annular tower of Definition 2.4.5. In Section 3.2 below, we will show that $T_\eta C$ for $C = \Sigma^\infty_+ \mathbb{C}\mathrm{P}^\infty$ is $\mathbb{S}^2 \simeq \Sigma^\infty \mathbb{C}\mathrm{P}^1$ and that the annular decomposition of C is into one sphere of each even dimension. This familiar cellular filtration of $\mathbb{C}\mathrm{P}^\infty$ also arises from another avenue: recognizing $\mathbb{C}\mathrm{P}^\infty \simeq BU(1)$, the skeletal filtration on $BU(1)$ has filtration quotients $(\Sigma U(1))^{\wedge n} \simeq \mathbb{S}^{2n}$, and $T_+ \Sigma^\infty_+ BU(1) \simeq \Sigma U(1)$. This phenomenon is generic: we will show that any coalgebraic formal curve spectrum C can be written as "BG" for an A_∞ ring spectrum G extracted from the bottom annular layer $T_\eta C$. The starting point for this observation is a second presentation of $T_\eta C$, arising as follows:

Lemma 2.5.1. *Given a fiber sequence of C–C–cobimodule spectra $M'' \to M \to M'$ and a fourth cobimodule spectrum N, the sequence*

$$M'' \square_C N \to M \square_C N \to M' \square_C N$$

is also a fiber sequence.

Proof. Because $-\square_C N$ is defined by taking the inverse limit of a diagram constructed from smashing the input with the fixed spectra $C^{\wedge n} \wedge N$, this functor preserves fiber sequences. □

Corollary 2.5.2. *There is a natural equivalence* $\mathbb{S} \square_C M \simeq C_1^1$.

Proof. By applying Lemma 2.5.1, one can construct the annular tower by iteratively cotensoring the fiber sequence

$$\begin{tikzcd} C \arrow[r] & M \\ \mathbb{S} \arrow[u, "\eta"] \end{tikzcd}$$

with cotensor powers of M and sewing together the overlapping nodes. In particular, C_1^1 appears as $\mathbb{S}\square_C M$. $\square$

Theorem 2.5.3. *When $\mathbb{S} \to C$ is a pointed coalgebra spectrum so that $\mathrm{Sch} K_* C$ is a formal variety, the spectrum $\mathbb{S}\square_C \mathbb{S}$ has a split filtration of the form*

$$\Sigma^{-1} C_1^1 \to \mathbb{S}\square_C \mathbb{S} \to \mathbb{S}.$$

Proof. Using Lemma 2.5.1, we cotensor the resolution sequence $\mathbb{S} \to C \to M$ with the C–comodule $\mathbb{S}$ to get the new fiber sequence

$$\cdots \to \Sigma^{-1}\mathbb{S}\square_C M \to \mathbb{S}\square_C \mathbb{S} \to \mathbb{S}\square_C C \to \mathbb{S}\square_C M \to \cdots .$$

We can use the method of Theorem 2.3.4 to identify $\mathbb{S}\square_C C$ with $\mathbb{S}$, and we can use Corollary 2.5.2 to identify $\mathbb{S}\square_C M$ with C_1^1. This yields the split fiber sequence

$$\Sigma^{-1} C_1^1 \to \mathbb{S}\square_C \mathbb{S} \to \mathbb{S}. \qquad \square$$

The object $\mathbb{S}\square_C \mathbb{S}$ is the subject of Koszul duality. There is the following general result:

Theorem 2.5.4 ([9, Proposition 7.26])**.** *Let $\mathcal{O}$ be an operad in k–module spectra, where k is some algebra spectrum, and let A be a left–module for the operad $\mathcal{O}$ (i.e., an $\mathcal{O}$–algebra). It follows that the arboreal bar construction $B(k;\mathcal{O};k)$ provides a co-operad $\mathcal{O}^\vee$ for which $B(A;\mathcal{O};k)$ is a left–co-module, and dually. Additionally, the operad produced in this way from the coassociative co-operad is the associative operad.* $\square$

Corollary 2.5.5. *The spectrum $\mathbb{S}\square_C \mathbb{S}$ is an associative algebra spectrum.* $\square$

Remark 2.5.6. Theorem 2.3.4 can be used to show that

$$\mathrm{Cotor}^{*,*}_{K_* C}(K_*, K_*) \Rightarrow K_*(\mathbb{S}\square_C \mathbb{S})$$

collapses and that the input is the Hochschild homology of $K^* C$. It follows that $K_*(\mathbb{S}\square_C \mathbb{S})$ is isomorphic to the exterior algebra generated by the tangent space of $\mathrm{Sch} K_* C$ [26, Example 2.2(2)]. Using Theorem 2.5.3, we also see that $K_* C_1^1$ computes the augmentation ideal inside of this Hochschild homology algebra—i.e., it is populated by nonconstant polynomials in $K^* C$ where each variable appears to at most first order. This observation motivates the unusual examples studied in the remainder of this paper, especially in the discussion at the beginning of Section 3.3.

Theorem 2.5.7 ([9, Proposition 6.4])**.** *When $\mathcal{O}$ and A are suitably of finite type (e.g., the associative operad and a polynomial algebra satisfy this), this operation is involutive: there is a natural equivalence of algebras*

$$\mathrm{Tot}\,\Omega(k; |B(k;A;k)|; k) \to A. \quad \square$$

Corollary 2.5.8. *For C a Γ–local coalgebraic formal variety spectrum, the natural map*

$$|B(\mathbb{S}; \mathbb{S}\,\square_C\,\mathbb{S}; \mathbb{S})| \to C$$

is a Γ–local equivalence.

Proof. Though Theorem 2.5.7 is sufficient to deduce the equivalence of co-operads, the equivalence of co-operadic left–modules (i.e., algebras for the co-operads) is not immediate since C does not have good finiteness properties. Nonetheless, the two coalgebraic formal schemes

$$\operatorname{Sch} K_* C, \quad \operatorname{Sch} K_* |B(\mathbb{S}; \mathbb{S}\square_C \mathbb{S}; \mathbb{S})|$$

are both formal varieties, and the natural map is an isomorphism on tangent spaces. The inverse function theorem for formal varieties thus applies. □

Remark 2.5.9. Finally, the natural map $|B(\mathbb{S}; \mathbb{S}\square_C \mathbb{S}; \mathbb{S})| \to C$ carries the bar filtration into the annular filtration. In the case that C is a coalgebraic formal curve spectrum, this map induces an isomorphism at the level of the associated graded, and hence the filtrations are themselves equivalent.

3. Projective spaces and formal lines

3.1. **General features.** In this section, we specialize still further to the case that C is a coalgebraic formal line spectrum, i.e., $K^0 C$ is noncanonically isomorphic to a 1–dimensional power series ring. Our main reason for making this specialization are the following definition and theorem:

Definition 3.1.1. A spectrum L is said to be Γ–*locally invertible* if there is some other spectrum L^{-1} with $L \wedge L^{-1} \simeq \mathbb{S}$ in the Γ–local category. The collection of isomorphism classes of such spectra forms an abelian group, called the *Picard group* of the Γ–local stable category.

Theorem 3.1.2 ([19, Theorem 1.3])**.** *A Γ–local spectrum L is Γ–locally invertible if and only if $K_* L$ is a K_*–line (i.e., a one–dimensional K_*–vector space, or a $\otimes$–invertible object in the category of K_*–vector spaces).* □

Corollary 3.1.3. *When C is a pointed coalgebraic formal line spectrum, $T_\eta C$ (and, indeed, any stratum C_n^n) gives an element of the Picard group of the Γ–local stable category.* □

This Corollary can be viewed in two ways. First, it can be used to construct elements of the Picard group of the Γ–local stable category by finding examples of pointed coalgebraic formal line spectra. Second, given such a coalgebraic formal line spectrum, the annular tower can be used to give a kind of cellular decomposition of C, by considering the dual tower

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & C_0^0 \arrow[r] \arrow[d] & C_0^1 \arrow[r] \arrow[d] & C_0^2 \arrow[r] \arrow[d] & \cdots \\
C_0^\infty \arrow[r, equal] \arrow[d] & C_0^\infty \arrow[r, equal] \arrow[d] & C_0^\infty \arrow[r, equal] \arrow[d] & C_0^\infty \arrow[r, equal] \arrow[d] & \cdots \\
C_0^\infty \arrow[r] & C_1^\infty \arrow[r] & C_2^\infty \arrow[r] & C_3^\infty \arrow[r] & \cdots
\end{tikzcd}$$

where each vertical sequence is a fiber sequence. Using Corollary 3.1.3, and thinking of Picard elements as "generalized spheres", this gives a Γ–local cellular decomposition of C, possibly in terms of Γ–local Picard elements which are *not* standard spheres.

We are thus moved to analyze the homotopy types of C_n^n as n varies, in order to understand this cellular decomposition of C. The algebraic analogue of C_n^n is the k–module of the terms of homogeneous degree n in the power series ring $K^0 C$. Because $K^0 C$ is 1–dimensional, these modules enjoy a further special property: the module of $(n+1)^{\text{st}}$

powers is naturally identified with the tensor product (over k) of the n^{th} powers and the cotangent space. The spectral version of this fact is as follows:

Theorem 3.1.4. *When* $\operatorname{Sch} K_* C$ *is* 1*-dimensional, there is a* Γ*-local equivalence* $C_n^n \simeq (T_\eta^* C)^{\wedge n}$.

Proof. Consider the following diagram of short exact sequences of k-modules, where the dashed line is filled in by the universal property of the quotient:

$$\begin{tikzcd}
0 \arrow[r, tail] & \langle x\rangle \otimes_k \langle x^{n+1}\rangle \oplus \langle x^2\rangle \otimes_k \langle x^n\rangle \arrow[r] \arrow[d] & \langle x\rangle \otimes_k \langle x^n\rangle \arrow[r] \arrow[d] & \frac{\langle x\rangle}{\langle x^2\rangle} \otimes_k \frac{\langle x^n\rangle}{\langle x^{n+1}\rangle} \arrow[r, tail] \arrow[d, dashed] & 0 \\
0 \arrow[r] & \langle x^{n+2}\rangle \arrow[r] & \langle x^{n+1}\rangle = \langle x\rangle \otimes_A \langle x^n\rangle \arrow[r] & \frac{\langle x^{n+1}\rangle}{\langle x^{n+2}\rangle} \arrow[r] & 0.
\end{tikzcd}$$

Accordingly, we construct a dual diagram of fiber sequences of spectra, modeling the situation for coalgebras and comodules:

$$\begin{tikzcd}
(M \wedge M^{\square_C (n+1)}) \times (M^{\square_C 2} \wedge M^{\square_C n}) & M \wedge M^{\square_C n} \arrow[l] & T_\eta^* C \wedge C_n^n \arrow[l] \\
M^{\square_C (n+2)} \arrow[u] & M^{\square_C (n+1)} \arrow[l] \arrow[u] & C_{n+1}^{n+1}. \arrow[l] \arrow[u, dashed]
\end{tikzcd}$$

The solid vertical maps are induced by the natural map

$$M \square_C N = \operatorname{Tot} \Omega(M; C; N) \to \operatorname{Tot}^0 \Omega(M; C; N) = M \wedge N,$$

and the dashed map exists (but is perhaps not unique) by the extension property for fiber sequences. Applying K_* to the diagram and chasing the left-hand square shows the dashed map to be a Γ-local equivalence. $\square$

Remark 3.1.5. We can use Corollary 2.5.8 to think of the attaching map in the sequence

$$\begin{tikzcd}
\Sigma^{-1} C_2^2 \arrow[r] \arrow[d, equal] & C_1^1 \arrow[r] \arrow[d, equal] & C_1^2 \arrow[d, equal] \\
\Sigma^{-1} (T_\eta C)^{\wedge 2} \arrow[r] & T_\eta C \arrow[r] & C_1^2
\end{tikzcd}$$

as a kind of "Hopf map for C". This agrees with the multiplicative definition of the Hopf map, using Remark 2.5.9.

Remark 3.1.6. Following on from Remark 2.5.6, we see that in the case where C is a pointed coalgebraic formal line spectrum, $K_* C_1^1$ computes exactly the tangent space of $\operatorname{Sch} K_* C$.

3.2. **Classical projective spaces.** We now work through the examples where C is taken to be a classical projective space. In light of Theorem 3.1.4, we will be especially interested in determining the homotopy type of $T_\eta C$.

To begin, let $C = \Sigma_+^\infty \mathbb{C}\mathrm{P}^\infty$, with $\eta \colon \mathbb{S} \to \Sigma_+^\infty \mathbb{C}\mathrm{P}^\infty$ its natural pointing by the disjoint basepoint. In this case, we apply the version of Definition 2.2.11 in the global stable category. However, the global stable category is not local for a field spectrum, so we appeal to the auxiliary spectra $K = H\mathbb{Q}$ and $K = H\mathbb{F}_p$ in order to analyze the coskeletal tower. Here, we find

$$H\mathbb{Q}_* T_+ \Sigma^\infty_+ \mathbb{C}P^\infty \cong \Sigma^2 \mathbb{Q},$$
$$(H\mathbb{F}_p)_* T_+ \Sigma^\infty_+ \mathbb{C}P^\infty \cong \Sigma^2 \mathbb{F}_p.$$

Since $T_+ \Sigma^\infty_+ \mathbb{C}P^\infty$ is a *connective* spectrum, it follows from Sullivan's adèlic reconstruction [35, Proposition 3.20] that the integral homology is

$$H\mathbb{Z}_* T_+ \Sigma^\infty_+ \mathbb{C}P^\infty \cong \Sigma^2 \mathbb{Z},$$

and hence that its homotopy type is

$$T_+ \Sigma^\infty_+ \mathbb{C}P^\infty \simeq \mathbb{S}^2.$$

To verify that the annular tower records the cellular decomposition of $\mathbb{C}P^\infty$, we transfer to the dual of the annular tower as in Section 3.1. By connectivity, it again follows that $C^n_m \simeq \mathbb{C}P^n_m$. The analysis of $C = \Sigma^\infty_+ \mathbb{H}P^\infty$ proceeds similarly.

We can also address $C = \Sigma^\infty_+ \mathbb{R}P^\infty$ and $K = K(\infty) = K_{\widehat{\mathbb{G}}_a} = H\mathbb{F}_2$, but we must first note that the proof of Theorem 2.3.4 does not work as given: $(H\mathbb{F}_2)_* \mathbb{R}P^\infty$ is not even–concentrated. However, using the fact that the groups

$$\operatorname{Cotor}^{(H\mathbb{F}_2)_* \mathbb{R}P^\infty}_{*,*>0}\left(\widetilde{(H\mathbb{F}_2)_*}\mathbb{R}P^\infty ; \widetilde{(H\mathbb{F}_2)_*}\mathbb{R}P^\infty\right)$$

all vanish, we can conclude that the truncated spectral sequences of Theorem 2.3.3 have no differentials beyond after the first page. It thus follows that there are still exact sequences of the form

$$0 \to D^1_{s+1,t} \to D^1_{s+t,t-1} \to E^1_{s,t} \to D^1_{s,t} \to D^1_{s,t-1} \to 0,$$

and hence the rest of the proof of Theorem 2.3.4 goes through. By similar reasoning to the complex projective case, performing the tangent space construction in the 2–adic stable category (in fact, because our objects are connective, it suffices to consider the $\widehat{\mathbb{G}}_a$–local category over $\mathbb{F}_2$) yields

$$T_+ \Sigma^\infty_+ \mathbb{R}P^\infty \simeq (\mathbb{S}^1)^\wedge_2,$$

and the annular tower again recovers the cellular decomposition of $\mathbb{R}P^\infty$.

Finally, it's worth remarking that the ambient category chosen to perform the tangent space construction is very important. It can neither be too localized nor too delocalized:

- Passing to the $\widehat{\mathbb{G}}_m$–local category (or, generally, the Γ–local category for Γ defined over k of characteristic 2) factors through 2–completion.[4] By work of Ravenel [27, Theorem 9.1], there is an equivalence

  $$L_{\widehat{\mathbb{G}}_m} \mathbb{R}P^\infty_{8k+1} \simeq L_{\widehat{\mathbb{G}}_m} \mathbb{S}^{-1}$$

  for any k. Taking $k = 0$, one sees $L_{\widehat{\mathbb{G}}_m} \mathbb{R}P^\infty \simeq L_{\widehat{\mathbb{G}}_m} \mathbb{S}^{-1}$, so its $K(1)$–cohomology is not a power series ring, and it is furthermore too small to have the correct Cotor groups. Letting k range, it's also plain that the bar filtration looks wildly different from the behavior of any sort of expected annular filtration.
- On the other hand, $\mathbb{R}P^\infty$ is p–locally acyclic for $p \neq 2$. It follows that the integral homology of $T_+ \Sigma^\infty_+ \mathbb{R}P^\infty$, as computed in the global stable category, will not be a $\mathbb{Z}$–line.

[4]...but not through $H\mathbb{F}_2$–localization.

3.3. **Determinantal projective space.** We dedicate this subsection to a particularly interesting set of examples, stemming from a calculation of Ravenel and Wilson:

Theorem 3.3.1 (Ravenel–Wilson [30], see also Johnson–Wilson [23, Appendix] and Hopkins–Lurie [18, Section 2])**.** *Take the ambient prime p to satisfy $p \geq 3$. There is a Hopf ring isomorphism*

$$\bigoplus_{t=0}^{\infty} K_* \underline{H\mathbb{Z}/p^j}_t \cong \bigoplus_{t=0}^{\infty} (K_* \underline{H\mathbb{Z}/p^j}_1)^{\wedge t},$$

where the target is the free alternating Hopf ring on $K_* K(\mathbb{Z}/p^j, 1)$. Moreover, as j tends to ∞, for fixed $t \geq 1$ the system $\{(\underline{H\mathbb{Z}/p^j}_t)_K\}_j$ has the structure of a connected p-divisible group of dimension $\binom{d-1}{t-1}$. The corresponding formal group is given by $(\underline{H\mathbb{Z}}_{t+1})_K$. $\square$

In particular, each of these spaces $\underline{H\mathbb{Z}}_{t+1}$ gives rise to a formal variety spectrum, and so a wealth of examples—but these examples take an unusual form. The space $\underline{H\mathbb{Z}}_t$ is the unstable loopspace of $\underline{H\mathbb{Z}}_{t+1}$ (cf. [3]), and one can view the Hochschild object $\mathbb{S}\square_{\Sigma^\infty_+ \underline{H\mathbb{Z}}_{t+1}} \mathbb{S}$ as also attempting to recover a loopspace of $\underline{H\mathbb{Z}}_{t+1}$ in the Γ–local stable category. These are visibly different: the K–homology of the former is infinite and complicated, but the K–homology of the latter is a finite exterior algebra. This is to be compared with the example of $BU(n)$, where the Hochschild object $K_* \mathbb{S}\square_{\Sigma^\infty_+ BU(n)} \mathbb{S}$ is isomorphic to $K_* U(n)$.

Of these examples, the value of $t = 1$ corresponds to the interesting spectrum $\Sigma^\infty_+ \mathbb{C}\mathrm{P}^\infty$, and the symmetry of Pascal's triangle suggests that the value $t = d$ may also be especially interesting. In this other extreme case, $T_+ \Sigma^\infty_+ \underline{H\mathbb{Z}}_{d+1}$ has 1-dimensional K–homology, and hence Corollary 3.1.3 and Theorem 3.1.4 can be applied. In order to determine the Picard element given by Corollary 3.1.3, we need a finer invariant than the mere detection property provided by Theorem 3.1.2. We find what we seek in the $K(n)$–local version of Morava E–homology:

Definition 3.3.2 ([22, Definition 8.3], [34, Theorem 12])**.** The *continuous Morava E–homology* or *Γ–local Morava E–homology* functor is defined by

$$E^\vee_\Gamma(X) := \pi_* L_\Gamma(E_\Gamma \wedge X).$$

It is valued in topologized modules over $E_{\Gamma *}$ equipped with a continuous coaction of $E^\vee_\Gamma E_\Gamma$. Equivalently, it can be taken to have values in sheaves on the Lubin–Tate stack for Γ.

Remark 3.3.3 ([22, Proposition 8.4.e-f]). The acyclics for $E^\vee_\Gamma$–homology agree with the acyclics for K_Γ–homology, completing the collection mentioned in Section 2.3 [22, Proposition 2.5]. The main point is that there is a Bockstein spectral sequence taking K_Γ–homology as input and converging to $E^\vee_\Gamma$–homology. If the K_Γ–homology is even–concentrated, this spectral sequence collapses and the $E^\vee_\Gamma$–homology is pro-free on any basis lifting a basis of the K_Γ–homology.

Theorem 3.3.4 ([19, Proposition 7.5])**.** *There is a factorization*

$$\begin{tikzcd}
\mathsf{Spectra}_\Gamma \arrow[r, "E^\vee_\Gamma"] \arrow[rr, bend left, "K_\Gamma"] & \mathsf{Modules}_{E_{\Gamma *}, \mathrm{Aut}(\Gamma)} & \mathsf{Modules}_{K_{\Gamma *}} \\
\mathrm{Pic}(\mathsf{Spectra}_\Gamma) \arrow[u] \arrow[r, "E^\vee_\Gamma"] & \mathsf{Lines}_{E_{\Gamma *}, \mathrm{Aut}(\Gamma)} \arrow[u] \arrow[r, "/\mathfrak{m}"] & \mathsf{Lines}_{K_{\Gamma *}}. \arrow[u]
\end{tikzcd}$$

$$\begin{array}{ccccccccc} & \bigoplus_{\substack{t_1,\ldots,t_n\geq 0\\ t_+=r}}(K_*\underline{H\mathbb{Z}/p^j}_1)^{\wedge t} & \longrightarrow & (E_*\underline{H\mathbb{Z}/p^j}_1)^{\wedge t}\otimes_{E_*}E_*/\mathfrak{m}^{r+1} & \longrightarrow & (E_*\underline{H\mathbb{Z}/p^j}_1)^{\wedge t}\otimes_{E_*}E_*/\mathfrak{m}^{r} & \longrightarrow & 0 \\ & \downarrow & & \downarrow & & \downarrow & & \\ 0 \longrightarrow & \bigoplus_{\substack{t_1,\ldots,t_n\geq 0\\ t_+=r}}K_*\underline{H\mathbb{Z}/p^j}_t & \longrightarrow & E_*\underline{H\mathbb{Z}/p^j}_t\otimes_{E_*}E_*/\mathfrak{m}^{r+1} & \longrightarrow & E_*\underline{H\mathbb{Z}/p^j}_t\otimes_{E_*}E_*/\mathfrak{m}^{r} & \longrightarrow & 0. \end{array}$$

FIGURE 5. Diagram of short exact sequences in Theorem 3.3.5.

Both squares are pullback squares: a Γ–local spectrum is invertible if and only if its continuous E–homology is a line, which is true if and only if its K–homology is a line. Additionally, for $p \gg d$ (specifically: $2p-2 \geq d^2$ and $p \neq 2$) the map labeled $E_\Gamma^\vee$ in the lower-left is an injection on objects. □

We are moved by this theorem to understand the Morava E–homology of Eilenberg–Mac Lane spaces, together with the $\operatorname{Aut}\Gamma$ action. Theorem 3.3.1 shows that $K_*\underline{H\mathbb{Z}/p^j}_q$ is even-concentrated for any choice of q, and hence Remark 3.3.3 shows that $E_*^\vee\underline{H\mathbb{Z}/p^j}_q$ is a pro-free module. We are left with determining the rest of the Hopf ring structure. Taking a cue from the proof of Remark 3.3.3, we begin by considering the short exact sequence of coefficients:

$$0 \to \bigoplus_{\substack{t_1,\ldots,t_n\geq 0\\ t_+=r}} K_* \to E_*/\mathfrak{m}^{r+1} \to E_*/\mathfrak{m}^r \to 0.$$

Because we know that $E_*^\vee\underline{H\mathbb{Z}/p^j}_t$ is a pro-free (and thus flat [22, Theorem A.9]) $E_*^\vee$–module deforming the original $K_*\underline{H\mathbb{Z}/p^j}_t$, we tensor against $E_*^\vee\underline{H\mathbb{Z}/p^j}_t$ to get a new short exact sequence appearing as the top row in Figure 5. We can also build the free alternating Hopf ring $(E_*^\vee\underline{H\mathbb{Z}/p^j}_1)^{\wedge *}$. Tensoring the above short exact sequence of coefficients with any graded piece of this ring gives the exact sequence (which is not, a priori, left exact) on the bottom row of Figure 5. Finally, the cup product map $(\underline{H\mathbb{Z}/p^j}_1)^{\wedge t} \to \underline{H\mathbb{Z}/p^j}_t$ induces a map on homology

$$(E_*^\vee\underline{H\mathbb{Z}/p^j}_1)^{\otimes t} \to (E_*^\vee\underline{H\mathbb{Z}/p^j}_1)^{\wedge t} \xrightarrow{\circ} E_*^\vee\underline{H\mathbb{Z}/p^j}_t.$$

Bifunctoriality of the tensor product induces the map between these rows by $\circ$–product.

Theorem 3.3.5 (cf. [18, Section 3.4]). *There is an isomorphism of Hopf rings*

$$\bigoplus_{t=0}^{\infty}(E_*\underline{H\mathbb{Z}/p^j}_1)^{\wedge t} \cong \bigoplus_{t=0}^{\infty}E_*\underline{H\mathbb{Z}/p^j}_t.$$

Proof. We perform an induction on r. In the base case of $r=1$, Theorem 3.3.1 gives a chain of isomorphisms

$$\begin{aligned} (E_*\underline{H\mathbb{Z}/p^j}_1)^{\wedge t}\otimes_{E_*}E_*/\mathfrak{m}^1 &= (K_*\underline{H\mathbb{Z}/p^j}_1)^{\wedge t} && \text{(Remark 3.3.3)} \\ &\xrightarrow{\cong} K_*\underline{H\mathbb{Z}/p^j}_t && \text{(Theorem 3.3.1)} \\ &= E_*\underline{H\mathbb{Z}/p^j}_t/\mathfrak{m}^1. && \text{(Remark 3.3.3)} \end{aligned}$$

This also tells us that the left-hand vertical map in Figure 5 is always an isomorphism. In particular, this map is injective, as is the first nontrivial horizontal map on the second row, so their composite is injective. It follows that the first horizontal map on the first row

$$\bigoplus_{\substack{t_1,\dots,t_n\geq 0\\ t_+=r}} (K_*\underline{H\mathbb{Z}/p^j}_1)^{\wedge t} \to (E_*\underline{H\mathbb{Z}/p^j}_1/\mathfrak{m}^{r+1})^{\wedge t}$$

is also injective, and thus that the top sequence is short exact.

Then, assume that $\circ$–multiplication induces an isomorphism modulo $\mathfrak{m}^r$ for some fixed r, i.e., that the right-hand vertical map in the above diagram is an isomorphism of modules. As the left-hand and right-hand vertical maps are isomorphisms, the center map must be as well. As t varies, the center maps additionally assemble into a map of graded Hopf rings, and so furthermore induce an isomorphism of such. Induction provides isomorphisms for all r, and the Milnor sequence finishes the argument:

$$\begin{aligned} E_*\underline{H\mathbb{Z}/p^j}_t &= \lim_r E_*\underline{H\mathbb{Z}/p^j}_t/\mathfrak{m}^r \\ &= \lim_r (E_*\underline{H\mathbb{Z}/p^j}_1/\mathfrak{m}^r)^{\wedge t} \\ &= (E_*\underline{H\mathbb{Z}/p^j}_1)^{\wedge t}. \end{aligned}$$

□

Remark 3.3.6. Given this calculation of the Hopf algebra structure on $E_*\underline{HS^1[p^j]}_t$, one might be led to wonder whether this translates into a presentation of $(\underline{HS^1[p^j]}_t)_E$ as the t^{th} exterior power of the p^j–torsion of $\tilde{\Gamma}$. This turns out to be true, but making sense of the statement is quite complicated: Buchstaber–Lazarev [8] and Goerss [12] showed that Dieudonné theory admits a notion of exterior power, Hedayatzadeh [16, 15] showed that alternating powers of 1–dimensional p–divisible groups can be constructed intrinsically in special cases (and are compatible with the Dieudonné theory model), and Hopkins and Lurie [18, Section 3.5] give another manual (but coordinate-free!) construction relevant to the case at hand.

We are now in a position to identify the $\operatorname{Aut}\Gamma$–representation structure of the E_*–line $E_*^\vee T_+\Sigma_+^\infty \underline{H\mathbb{Z}/p^\infty}_d$, using the $\operatorname{Aut}\Gamma$–equivariant map

$$E_*^\vee \underline{H\mathbb{Z}/p}_1^{\times d} \xrightarrow{\circ} E_*^\vee \underline{H\mathbb{Z}/p}_d.$$

It will be helpful to have access to a presentation of the group $\operatorname{Aut}\Gamma$, which arises as the group of units of the algebra $\operatorname{End}\Gamma$. Writing S for the Frobenius endomorphism of Γ, one can take the set $\{1, S, \dots, S^{d-1}\}$ as a $\mathbb{W}(k)$–basis for $\operatorname{End}\Gamma$ [28, Theorem A2.2.17].

Lemma 3.3.7. *Let $\beta_1 \in E_*\underline{H\mathbb{Z}/p^\infty}_1$ be an element dual to a coordinate on $\tilde{\Gamma}$. The algebro-geometric cotangent space of $(\underline{H\mathbb{Z}/p^\infty}_d)_E$ is spanned by the dual to the element*

$$\beta_* = S^0\beta_1 \otimes S^1\beta_1 \otimes \cdots \otimes S^{d-1}\beta_1 \in E_*\underline{H\mathbb{Z}/p^\infty}_1^{\times d}$$

pushed forward along the $\circ$–product map.

Proof. This is a rephrasing of results of Ravenel and Wilson [30, Theorems 5.7.(d) and 9.2.(a)]. □

Definition 3.3.8. Left–multiplication gives a map

$$\operatorname{Aut}\Gamma \to GL_{\mathbb{W}(k)}(\operatorname{End}\Gamma)$$

and postcomposing with the determinant map gives definition of the *determinant representation*:

$$\operatorname{Aut}\Gamma \to GL_{\mathbb{W}(k)}(\operatorname{End}\Gamma) \xrightarrow{\det} \mathbb{W}(k)^\times.$$

Theorem 3.3.9 (cf. [37, Proposition 3.21]). *The* $\operatorname{Aut}\Gamma$*–representation structure of* $E_*^\vee T_+\Sigma_+^\infty \underline{H\mathbb{Z}/p^\infty}_d$ *is the determinant representation.*

Proof. The cup product map

$$E_* \underline{H\mathbb{Z}/p}_1^{\times d} \xrightarrow{\circ} E_* \underline{H\mathbb{Z}/p}_d$$

is surjective and respects the $\mathbb{S}_d$-action, as it is induced by a map of spaces. There is a Künneth formula

$$(E_\Gamma)_* \underline{H\mathbb{Z}/p}_1^{\times d} \cong ((E_\Gamma)_* \underline{H\mathbb{Z}/p}_1)^{\otimes d},$$

and the stabilizer group intertwines with Künneth maps

$$E_* A \otimes_{E_*} E_* B \xrightarrow{\phi} E_*(A \times B)$$

as

$$g \cdot \phi(a \otimes b) = \phi((g \cdot a) \otimes (g \cdot b)).$$

In turn, an element $g \in \mathbb{S}_d$ acts on β_* by the formula

$$\begin{aligned} g \cdot (\circ \beta_*) &= \circ(g \cdot \beta_*) \\ &= \circ(g \cdot (S^0 \beta_1 \otimes \cdots \otimes S^{d-1} \beta_1)) \\ &= \circ(g \cdot (S^0 \otimes \cdots \otimes S^{d-1}) \beta_1) \\ &= \circ(\det g (S^0 \otimes \cdots \otimes S^{d-1}) \beta_1) \\ &= (\det g)(\circ \beta_*). \end{aligned}$$

□

Corollary 3.3.10. *For* $p \gg n$*, the spectrum* $T_+\Sigma_+^\infty \underline{H\mathbb{Z}/p^\infty}_d$ *models the determinantal sphere* $\mathbb{S}^{\det}$*.*

Proof. Couple the above with the last part of Theorem 3.3.4. □

Remark 3.3.11. The object $\mathbb{S}^{\det}$ is fairly familiar to chromatic homotopy theorists. Its first appearance was in work of Hopkins and Gross on describing the Γ–local homotopy type of the Brown–Comenetz dualizing spectrum [17, Theorem 6]. It has subsequently played a prominent role in the study of chromatic homotopy theory at the height 2. For instance, it has been shown to span the rest of the torsion–free part of the $\widehat{C_{ss}}$–local Picard group [5, Theorem 8.1]. It also has been used to study duality phenomena relating to topological modular forms; see for example work of Behrens [4, Proposition 2.4.1] and of Stojanoska [32, Corollary 13.3].

Remark 3.3.12. It follows from Theorem 3.3.9 and Theorem 3.1.4 that there is *no* global finite complex with a map to $\underline{H\mathbb{Z}/p^\infty}_d$ which in E–cohomology projects to precisely the 1–jets. The reader should compare with the situation with $\mathbb{CP}^\infty$ and ordinary homology, where $\mathbb{CP}^1 \simeq \mathbb{S}^2$ performs this selection of the 1–jets, but selecting the second annulus is obstructed by $\eta \in \pi_1\mathbb{S}$, available only after coning off $\mathbb{CP}^1$ or inverting 2. By enlarging the variety of spheres available to us, we have given ourselves more tools by which we can carefully select certain individual homology classes in Γ–local spectra.

The rest of the annular tower gives a remarkable filtration of the Γ–local homotopy type of $\underline{H\mathbb{Z}/p^\infty}_d$, as though it were a cell complex built out of Picard–graded cells with a simple inductive structure. The global homotopy type $\underline{H\mathbb{Z}/p^\infty}_d$ of course also comes with a cellular decomposition by global cells—after all, it is presented simplicially by an iterated bar construction—but this information is dramatically more complex. Morally, passing to the Γ–local category has simultaneously enlarged our notion of "cell" and simplified the homotopy type of a complicated space, at once resulting in a simple pattern not globally visible.

3.4. **Comparison of $K^{\det}$ with Westerland's R_d.** The previous section and Section 2.5 together suggest the presence of an interesting Γ–local spectrum $K^{\det}$, which we now describe. First, Corollary 2.5.8 grants us access to Γ–local A_∞ ring spectrum, which we notate by

$$\mathbb{X}_+^\times := \mathbb{S}\square_{\Sigma_+^\infty \underline{H\mathbb{Z}/p^\infty}_d}\mathbb{S}.$$

Theorem 3.1.4 then suggests that $\Sigma_+^\infty \underline{H\mathbb{Z}/p^\infty}_d$ itself could be given the name $\mathbb{X}\mathrm{P}_+^\infty$, and the annular filtration gives rise to an infinite sequence of intermediate spectra

$$* \longrightarrow \mathbb{S}^0 \longrightarrow \mathbb{X}\mathrm{P}_+^1 \longrightarrow \mathbb{X}\mathrm{P}_+^2 \longrightarrow \cdots \longrightarrow \mathbb{X}\mathrm{P}_+^\infty.$$

In the analogous decomposition of $\mathbb{C}\mathrm{P}_+^\infty$, the composite

$$\beta\colon \mathbb{S}^2 \simeq T_+\Sigma_+^\infty \mathbb{C}\mathrm{P}^\infty \to \Sigma_+^\infty \mathbb{C}\mathrm{P}^1 \to \Sigma_+^\infty \mathbb{C}\mathrm{P}^\infty$$

is known as the Bott class.

Definition 3.4.1. We define the *determinantal Bott class* by the composite

$$\beta^{\det}\colon \mathbb{S}^{\det} \simeq T_+\mathbb{X}\mathrm{P}_+^\infty \to \mathbb{X}\mathrm{P}_+^1 \to \mathbb{X}\mathrm{P}_+^\infty.$$

Accordingly, we set *determinantal K–theory* to be

$$K^{\det} := \left(L_\Gamma \Sigma_+^\infty \underline{H\mathbb{Z}/p^\infty}_d\right)[(\beta^{\det})^{-1}].$$

Remark 3.4.2. When $d = 1$, $\underline{H\mathbb{Z}/p^\infty}_1$ is p–adically (and hence $\widehat{\mathbb{G}}_m$–locally) indistinguishable from $\underline{H\mathbb{Z}}_2 \simeq \mathbb{C}\mathrm{P}^\infty$. It follows that β agrees with the usual Bott class for $\mathbb{C}\mathrm{P}^\infty$, and hence by Snaith's theorem [31] that $K^{\det}$ agrees with p–adic K–theory $KU_p^\wedge$.

Craig Westerland has recently considered a spectrum related to $K^{\det}$:

Definition 3.4.3 ([37, Definition 3.11])**.** Take $p > 2$ and consider the action of $\mathbb{Z}/p^\times$ on $\underline{H\mathbb{Z}/p}_d$ by field multiplication. Averaging this action gives rise to an idempotent in K–homology which splits the suspension spectrum as follows:

$$L_\Gamma \Sigma_+^\infty \underline{H\mathbb{Z}/p}_d \simeq \bigvee_{k=0}^{p-1} Z^{\wedge k},$$

where Z is a (non-uniquely specified) spectrum with the property $Z^{\wedge p} \simeq Z$. The spectrum R_d is defined by inverting the element of Picard–graded homotopy determined by the composite

$$\alpha\colon Z \to \bigvee_{k=0}^{p-1} Z^{\otimes k} \simeq L_\Gamma \Sigma_+^\infty \underline{H\mathbb{Z}/p}_d \to L_\Gamma \Sigma_+^\infty \underline{H\mathbb{Z}/p^\infty}_d \xrightarrow{\text{Bockstein}} L_\Gamma \Sigma_+^\infty \underline{H(\mathbb{Z}_p)}_{d+1}.$$

Upon picking a coordinate on $\mathbb{C}\mathrm{P}_K^\infty$ and applying K–homology to this composite, one sees that it selects the dual of the induced coordinate on $(\underline{H(\mathbb{Z}_p)}_{d+1})_K$.

Theorem 3.4.4. *There is an equivalence $K^{\det} \simeq R_d$.*

Proof. We mimic the style of one of Westerland's proofs [37, Corollary 3.18], where he shows that inverting the class α above is equivalent to inverting another class ρ, which participates in a Γ–local diagram

$$\begin{tikzcd} & \Sigma^\infty_+ \underline{H\mathbb{Z}}_{d+1} \arrow[r, "\psi^g - g"] \arrow[d] & \Sigma^\infty_+ \underline{H\mathbb{Z}}_{d+1} \arrow[d] \\ \mathbb{S}^{\det} \arrow[ur, "\rho"] \arrow[r, "\delta"] & \Sigma^\infty_+ \underline{H\mathbb{Z}}_{d+1}[\alpha^{-1}] \arrow[r, "\psi^g - g"] & \Sigma^\infty_+ \underline{H\mathbb{Z}}_{d+1}[\alpha^{-1}]. \end{tikzcd}$$

Here g is a generator of $\mathbb{Z}_p^\times$ and ψ^g encodes the natural action of $\mathbb{Z}_p^\times$ on $L_\Gamma \Sigma^\infty_+ \underline{H\mathbb{Z}}_{d+1}$ (via the multiplicative action of $\mathbb{Z}_p^\times$ on $\mathbb{Z}_p$). He constructs isomorphisms [37, Propositions 3.10 and 3.12]

$$\begin{tikzcd} K_* \underline{H\mathbb{Z}}_{d+1} \arrow[r, "\simeq"] \arrow[d] & C(\mathbb{Z}_p, K_*) \arrow[d, "\text{restrict}"] \\ K_*(\Sigma^\infty_+ \underline{H\mathbb{Z}}_{d+1}[\alpha^{-1}]) \arrow[r, "\simeq"] & C(\mathbb{Z}_p^\times, K_*), \end{tikzcd}$$

which he uses to calculate that the two maps

$$\Sigma^\infty_+ \underline{H\mathbb{Z}}_{d+1}[\alpha^{-1}] \xrightarrow{\simeq} \Sigma^\infty_+ \underline{H\mathbb{Z}}_{d+1}[\alpha^{-1}][\delta^{-1}] \xleftarrow{\simeq} \Sigma^\infty_+ \underline{H\mathbb{Z}}_{d+1}[\rho^{-1}]$$

are both equivalences, essentially by calculating the images of δ and ρ in the ring $C(\mathbb{Z}_p^\times, K_*)$.

Similarly, we will investigate the two maps

$$\Sigma^\infty_+ \underline{H\mathbb{Z}}_{d+1}[(\beta^{\det})^{-1}] \to \Sigma^\infty_+ \underline{H\mathbb{Z}}_{d+1}[\rho^{-1}, (\beta^{\det})^{-1}] \leftarrow \Sigma^\infty_+ \underline{H\mathbb{Z}}_{d+1}[\rho^{-1}].$$

Using the identification above of $K_* \underline{H\mathbb{Z}}_{d+1}$ as a ring of functions, $\beta^{\det}$ has been handicrafted so that a generator of $K_* \mathbb{S}^{\det}$ is sent by $\beta^{\det}$ according to

$$\begin{aligned} K_* \underline{H\mathbb{Z}}_{d+1} &\xrightarrow{\simeq} C(\mathbb{Z}_p, K_*) \\ K_* \beta^{\det} &\longmapsto (w \mapsto w \pmod p). \end{aligned}$$

Westerland denotes this element of $C(\mathbb{Z}_p, K_*)$ as f_0, and he shows that inverting either of α or ρ has the effect of inverting exactly this element. It follows that both of our two maps are equivalences. $\square$

Remark 3.4.5 ([37, Section 3.9]). Westerland exposes a variety of remarkable features of $K^{\det}$, the grandest of which is the E_∞-equivalence

$$K^{\det} \simeq E_\Gamma^{hS\mathbb{G}_\Gamma^\pm}.$$

Here Γ is specifically taken over $k = \mathbb{F}_{p^d}$, the group $\mathbb{G}_\Gamma$ is the extension of $\operatorname{Aut}\Gamma$ by $\operatorname{Gal}(\mathbb{F}_{p^d}/\mathbb{F}_p)$, and $S\mathbb{G}_\Gamma^\pm$ is the subgroup of "special" elements [37, Section 2.2]—i.e., it lies in the fiber sequence

$$1 \to S\mathbb{G}_\Gamma^\pm \to \mathbb{G}_\Gamma \xrightarrow{\det^\pm} \mathbb{Z}_p^\times \to 1,$$

where $\det^\pm$ acts by the determinant on $\operatorname{Aut}\Gamma$ and by $\mathrm{Frob} \mapsto (-1)^{d-1}$ on the Galois component. It follows that there is a short resolution:

$$L_\Gamma \mathbb{S} \to K^{\det} \xrightarrow{\psi - 1} K^{\det},$$

where ψ is a certain Adams-type operation inherited from the action of $\mathbb{Z}_p^\times$ on $\underline{H(\mathbb{Z})}_{d+1}$ (or, equivalently, the lingering action of $\mathbb{Z}_p^\times$ on the fixed point spectrum) [37, Proposition

3.14], [11, Theorem 2]. This spectrum has been investigated before, under the names "Iwasawa extension of the $K(d)$–local sphere" and "Mahowald's half-sphere".

Remark 3.4.6 ([37, Corollary 4.21]). Westerland also describes a cellular filtration of $\Sigma^\infty_+ \underline{H\mathbb{Z}}_{d+1}$ which bears resemblance to the annular filtration described here, but which is accessed by wholly different means. He constructs an analogue for $K^{\det}$ of the classical complex J–homomorphism, and he then checks that the Thom spectrum of the canonical bundle on $\underline{H\mathbb{Z}}_{d+1}$ has the homotopy type of $(\mathbb{S}^{\det})^{-1} \wedge \Sigma^\infty \underline{H\mathbb{Z}}_{d+1}$. This mirrors the following classical fact about projective spaces [1, Proposition 4.3]:

$$\mathrm{Thom}(\mathscr{L} - 1 \downarrow \mathbb{C}\mathrm{P}^\infty) \simeq \Sigma^{-2}\Sigma^\infty \mathbb{C}\mathrm{P}^\infty,$$

and more generally

$$\mathrm{Thom}(m(\mathscr{L} - 1) \downarrow \mathbb{C}\mathrm{P}^{n-m}) \simeq (\mathbb{C}\mathrm{P}^1)^{\wedge(-m)} \wedge \mathbb{C}\mathrm{P}^n_m.$$

It would be interesting to know if these filtrations coincide.

4. Some open questions

Before closing the paper, we record some avenues of research left open by the present work.

4.1. **Gross–Hopkins dualizing object.** As remarked on in the introduction, the determinantal sphere has previously arisen in connection with the Gross–Hopkins dualizing object $\widehat{\mathbb{I}}_{\mathbb{Q}/\mathbb{Z}}$ in the $K(d)$–local category. Our identification of $\mathbb{X}\mathrm{P}^1$ with $\widehat{\mathbb{I}}_{\mathbb{Q}/\mathbb{Z}}$ rests on working at a prime p satisfying $p \gg d$, so that we can avail ourselves of Theorem 3.3.4. At small primes, the picture is murkier: we conjecture that $\mathbb{X}\mathrm{P}^1$ disagrees with $\widehat{\mathbb{I}}_{\mathbb{Q}/\mathbb{Z}}$, just as the Goerss–Henn–Mahowald–Rezk $\mathbb{S}[\det]$ does, but we do not have a proof that this is so—and we furthermore have neither a proof that $\mathbb{X}\mathrm{P}^1$ agrees with $\mathbb{S}[\det]$ in the small prime regime nor one that it disagrees. Understanding the relationship between these spectra has the opportunity either to shed light on Gross–Hopkins duality or to shed light on exotic Picard groups at greater heights, both of which are interesting prospects.

4.2. **Chromatic splitting.** The chromatic splitting conjecture describes the homotopy group of such objects as $\mathbb{Q} \otimes L_{K(d)}\mathbb{S}$, and the Gross–Hopkins dualizing object induces a kind of Poincaré duality among these groups. Westerland has noticed that there is a factorization α of the fundamental class as in

$$\begin{tikzcd} \mathbb{S}^{d+1} \arrow[r, "\iota_{d+1}"] \arrow[dr, dotted, "\alpha"'] & L_{K(d)}\Sigma^\infty_+ K(\mathbb{Z}, d+1) \\ & \mathbb{X}\mathrm{P}^1, \arrow[u, "\beta^{\det}"'] \end{tikzcd}$$

which at large primes can be interpreted as follows:

$$\begin{aligned} \alpha \in [\mathbb{S}^{d+1}, \mathbb{X}\mathrm{P}^1] &= [\mathbb{S}^{d+1}, \Sigma^{d-d^2}\widehat{\mathbb{I}}_{\mathbb{Q}/\mathbb{Z}}] \\ &= [\mathbb{S}^{1+d^2}, \widehat{\mathbb{I}}_{\mathbb{Q}/\mathbb{Z}}] \\ &= \mathrm{Hom}(\pi_{-1-d^2} M_d \mathbb{S}^0, \mathbb{Q}_p/\mathbb{Z}_p). \end{aligned}$$

Conjecture 4.2.1. The class α is nonzero. It is dual to the top-dimensional class (i.e., the Poincaré dualizing class) predicted by the rational chromatic splitting conjecture.

A positive resolution of this conjecture could give a foothold on a topological explanation for the chromatic splitting conjecture. (The reader should beware that Beaudry has shown the splitting conjecture to be *false* in the small prime region, and an adjusted version has not yet been set out.)

4.3. **Validity at small primes.** Several of the utility theorems in this paper required the large prime assumption in the proofs presented here, but it is not clear that this assumption is necessary. In particular, it would be extremely desirable to have a version of Theorem 2.3.2 (cf. Theorem A.0.1) that does not distinguish between the large and small prime cases.

4.4. **Analogues of $BU(n)$.** The construction of $\mathbb{X}P^\infty$ and $\mathbb{X}P^1$ given here, as well as Westerland's construction of determinantal analogues of $L_{K(1)}BU$, of $L_{K(1)}MU$, and of a kind of "J-map", may well be a premonition of a much more extensive story of "determinantal homotopy theory". The most enticing facet of this would be to find determinantal analogues of the intermediate spaces $BU(n)$, to understand these as classifying some manner of geometric object, and to be able to interpret them as *spaces* in some kind of unstable determinantal context. Several more questions long these lines can be found at the end of Westerland's work [37, Section 5.2 and Section 7].

Appendix A. Homological algebra of comodules for a Hopf k-algebra

In this section we describe a result, attributed to Hal Sadofsky and expressed in a talk by Mike Hopkins [25, Section 14], concerning the homology of inverse limits of certain local systems.[5] None of the material in this section is original; all of it was known to (at least) Hal Sadofsky and Mike Hopkins, and is of "folk lore" status among the experts who might be interested. Sadofsky's theorem is stated as follows:

Theorem A.0.1 (Sadofsky)**.** *Let k be a field spectrum, i.e., let k be a ring spectrum with k_* a graded field. (In the case that k is a Morava K-theory for Γ, we require $p \gg \operatorname{ht}\Gamma$.) Furthermore let $\{X_\alpha\}_\alpha$ be a sequential inverse system of k-local spectra such that $\{k_*X_\alpha\}_\alpha$ is Mittag-Leffler as a system of k_*-modules. There is then a spectral sequence of signature*

$$R^* \lim_\alpha \{k_* X_\alpha\}_\alpha \Rightarrow k_* \lim_\alpha \{X_\alpha\}_\alpha,$$

where the derived inverse limit on the left is taken in an appropriate category of comodules. The spectral sequence is at least *conditionally convergent.*

Remark A.0.2. There is room for improvement in this result: the author sees no reason to require $p \gg d$ except for his own ineptness. It is quite likely that this Theorem holds without this assumption, and a proof of this would strengthen many of the results in this paper.

Remark A.0.3. The locality is *the* essential assumption. For instance, set $k = H\mathbb{Q}$ and consider the system $\{\mathbb{S}^0/p^j\}_j$, with maps the natural projections. The constituent spaces in this system are all rationally acyclic, but the inverse limit is given by the p-adic sphere $(\mathbb{S}^0)^\wedge_p$. Its rational homology is $H\mathbb{Q}_*(\mathbb{S}^0)^\wedge_p \cong \mathbb{Q}_p$, and hence no such convergent spectral sequence can exist. On the other hand, first rationalizing this system produces the trivial system of zero spectra, and thus the rational homology of the system—which is empty—compares well to the rational homology of the inverse limit—which is also empty. Noting

[5] The interested reader can also find related results in a paper of Hovey [21].

that $\mathbb{S}^0/p^j$ is also known as the Moore spectrum $M_0(p^j)$, similar systems can also be constructed for any Morava K–theory by employing the generalized Moore spectra of Hopkins and Smith [20, Proposition 5.14].

Remark A.0.4. In the case $k = H\mathbb{Q}$, rational homology can be exchanged for rational homotopy, and the categories of $H\mathbb{Q}_* H\mathbb{Q}$–comodules and $\mathbb{Q}$–modules agree. In turn, the inverse limit of rational homotopy groups participates in a Milnor exact sequence. The Mittag-Leffler hypothesis forces this exact sequence to degenerate, and the easiest case of Sadofsky's theorem follows. In general, the game is to gain topological control over the derived inverse limits of *comodules*, while retaining hypotheses so that the derived inverse limits of *modules* continue to vanish.

Remark A.0.5. For any sequence of spectra (X_α), the inverse system $\{Y_\alpha := \prod_{\beta \le \alpha} X_\beta\}$ with maps given by projections is Mittag-Leffler. Then, using the fiber sequence $\lim_\alpha X_\alpha \to \prod_\alpha X_\alpha \to \prod_\alpha X_\alpha$, applying k–homology gives

$$\cdots \to k_* \lim_\alpha X_\alpha \to k_* \left(\prod_\alpha X_\alpha \right) \to k_* \left(\prod_\alpha X_\alpha \right) \to \cdots .$$

The middle and right–hand terms of this sequence are calculable by Sadofsky's theorem, even if the inverse system $\{X_\alpha\}$ is itself not Mittag-Leffler, giving some foothold on that case as well.

Remark A.0.6. We remark for a second time that computations in the E_2–page of this spectral sequence are *exceedingly* complex and promise to shed light on some of the most long-standing conjectures in chromatic homotopy theory. Meanwhile, we will only work with this spectral sequence in a maximally degenerate case.

A.1. **Homological algebra for inverse systems of comodules.** Before engaging in any algebraic topology, we will first make sense of the algebraic derived inverse limits of sequential systems of comodules. The homological algebra of comodules is well documented elsewhere [28, Appendix 1], but the homological algebra of *diagrams* of comodules is more scarce. Though we are interested in the case of a Hopf algebroid $(E_*, E_* E)$, in this section we will refer to this pair opaquely as (A, Γ) with Γ flat over A. To begin, we recall some classical results.

Lemma A.1.1 ([28, Lemma A1.2.1-2])**.** *A comodule Y of the form $Y = \Gamma \otimes_A Y'$ for an A–module Y' is said to be an extended comodule. This construction gives an adjunction*

$$\mathsf{Modules}_A(M, Y') \cong \mathsf{Comodules}_{(A,\Gamma)}(M, Y).$$

Furthermore, if I is an injective A–module, then $\Gamma \otimes_A I$ is an injective Γ–comodule, and hence $\mathsf{Comodules}_{(A,\Gamma)}$ *has enough injectives.* □

Corollary A.1.2. *If M is a sequential inverse system of A–modules and $\Gamma \otimes_A M$ is the induced system of extended Γ–comodules, then there is an isomorphism*

$$\lim M \cong \lim(M \otimes_A \Gamma),$$

where the left– and right–hand limits are taken in $\mathsf{Modules}_A$ *and* $\mathsf{Comodules}_{(A,\Gamma)}$ *respectively.* □

Remark A.1.3. The adjunction in Lemma A.1.1 should be thought of as geometrically analogous to the adjunction

$$\mathsf{Sets}(X, Y) \cong G\text{-}\mathsf{Sets}(G \times X, Y).$$

Now consider the category $\mathsf{Comodules}^{\mathbb{N}}_{(A,\Gamma)}$ of sequential inverse systems of comodules, where $\mathbb{N}$ denotes the category associated to the natural numbers with their partial ordering.

Lemma A.1.4 (Sadofsky)**.** *The category* $\mathsf{Comodules}^{\mathbb{N}}_{(A,\Gamma)}$ *has enough injectives. That is, for* X *a sequential inverse system of* (A,Γ)*–comodules, there is a levelwise injection to an injective system* J*.*

Proof. Begin by, for each $n \in \mathbb{N}$, choosing an injection $j'_n : X(n) \to J'_n$ with J'_n an injective comodule. We form a diagram J equipped with a levelwise injection $j : X \to J$ by setting $J(n) = \prod_{i=1}^n J'_i$, with the map $J(n+1) \to J(n)$ specified by

$$\prod_{i=1}^{n+1} J'_i \xrightarrow{\left(\prod_{i=1}^n 1_{J'_i}\right)\times 0_{J'_{n+1}}} \prod_{i=1}^{n} J'_i$$

and the structure map by

$$X(n) \xrightarrow{\prod_{i=1}^n (j'_i \circ x_i^n)} \prod_{i=1}^{n} J'_i,$$

where x_i^n is the morphism specified by the diagram of signature

$$x_i^n : X(n) \to X(i).$$

We now check that this diagram has the relevant lifting property:

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d] & J \\ Y \arrow[ur, dotted, "\exists k"] & \end{tikzcd}$$

whenever the vertical arrow is a levelwise injection. We argue inductively, beginning with the case $n = 1$. In that case, the diagram reduces to

$$\begin{tikzcd} X(1) \arrow[r, "j'_1"] \arrow[d] & J(1) \\ Y, \arrow[ur, dotted, "\exists k'_1"] & \end{tikzcd}$$

which is precisely the diagram describing the classical injectivity condition. Because $J(1)$ was selected to be an injective comodule under $X(1)$, such an extension exists. In the case of a general n, we have the following diagram:

$$\begin{tikzcd} X(n-1) \arrow[r, "\prod_{i=1}^{n-1}(j'_i \circ x_i^{n-1})"] \arrow[dd] & \prod_{i=1}^{n-1} J'_i & & \\ & & X(n) \arrow[ull] \arrow[r, "\prod_{i=1}^{n}(j'_i \circ x_i^{n})"] \arrow[dd] & \prod_{i=1}^{n} J'_i \arrow[ull, "(\prod_{i=1}^{n-1} 1_{J'_i})\times 0_{J'_n}"'] \\ Y(n-1) \arrow[uur, "k(n-1)"'] & & & \\ & & Y(n). \arrow[ull] \arrow[uur, dotted] & \end{tikzcd}$$

The dashed map is specified by a pair of morphisms $Y(n) \to \prod_{i=1}^{n-1} J'_i$ and $Y(n) \to J'_n$. The former arrow is specified by restriction. We also have the following diagram of injective comodules

$$\begin{tikzcd} X(n) \arrow[r, "j'_n"] \arrow[d] & J'_n \\ Y(n), \arrow[ur, dotted, "\exists"] & \end{tikzcd}$$

which allows us to select the remaining arrow. □

Remark A.1.5. It is not true that a diagram whose objects consist of levelwise injective comodules is always an injective object in sequential inverse systems of comodules. The construction above is designed to skirt past questions of compatibility of levelwise lifts. Relatedly, the above proof generalizes to "well–founded" inverse systems, i.e., inverse systems indexed on diagram categores where each object is the source of finitely many morphisms, so that induction still applies.

Because we have enough injectives, the general machinery of homological applies to produce right–derived functors of the left–exact inverse limit functor

$$\lim : \mathsf{Comodules}_{(A,\Gamma)}^{\mathbb{N}} \to \mathsf{Comodules}_{(A,\Gamma)}.$$

Additionally, there is a cobar complex which performs this computation. To see this, we first remark on a consequence of the adjunction of Lemma A.1.1.

Lemma A.1.6 (Sadofsky)**.** *If M is a Mittag-Leffler sequential inverse system of A–modules, then the system $M \otimes_A \Gamma$ of extended Γ–comodules has no higher derived limits, i.e., it is flasque.*

Proof. In the usual way, M can be resolved by a double complex J such that $J(n,*)$ gives an injective resolution of $M(n)$ and $J(*,m)$ is itself a Mittag-Leffler sequential inverse system. Each system $J_m = J(*,m)$ has the desired property, which we can verify by constructing the following resolution:

$$\begin{tikzcd} 0 \arrow[r] & J_m(n) \arrow[r] \arrow[d] & 0 & \\ 0 \arrow[r] & \prod_{n' \leq n} J_m(n') \arrow[r] & \prod_{n' < n} J_m(n') \arrow[r] & 0. \end{tikzcd}$$

This is an exact resolution of J_m by injectives in $\mathsf{Modules}_A^{\mathbb{N}}$, hence tensoring up with the flat module Γ gives an exact resolution of $J_m \otimes A$ by injectives in $\mathsf{Comodules}_{(A,\Gamma)}^{\mathbb{N}}$. Applying lim to this resolution of comodules and using Corollary A.1.2, we have

$$\lim\left(\left(\prod_{n'\leq n} J_m(n')\right)\otimes_A \Gamma \to \left(\prod_{n'<n} J_m(n')\right)\otimes_A \Gamma \to 0\right) =$$

$$\left((\lim J_m) \oplus \prod_n J_m(n)\right)\otimes_A \Gamma \xrightarrow{\text{project onto second factor}} \left(\prod_n J_m(n)\right)\otimes_A \Gamma \to 0,$$

which is visibly flasque.

To compute the derived inverse limits of $M \otimes_A \Gamma$ itself, we use the resolution J_* of M:

$$R^* \lim(M \otimes_A \Gamma) = H^*(\lim(J_* \otimes_A \Gamma)) = H^*((\lim J_*) \otimes_A \Gamma).$$

Because M is Mittag-Leffer and J_* is a resolution of M in A–modules, $H^* \lim J_*$ is concentrated in degree zero. Finally, because Γ is a flat A–module, $- \otimes_A \Gamma$ preserves exact sequences, and so the complex of comodules $(\lim J_*) \otimes_A \Gamma$ also has cohomology concentrated in degree zero, i.e., $M \otimes_A \Gamma$ is flasque. □

Corollary A.1.7 (Sadofsky)**.** *Write $\Omega(\Gamma;\overline{\Gamma};-)$ for the usual one–sided cobar cochain complex with*

$$\Omega(\Gamma;\overline{\Gamma};-)[n] = \Gamma \otimes_A \overline{\Gamma}^{\otimes_A n} \otimes_A -.$$

If X is a sequential inverse system of comodules satisfying the Mittag-Leffler condition, then

$$R^s \lim X = H^s \lim \Omega(\Gamma;\overline{\Gamma};X),$$

where the right–hand object is interpreted as the total complex of the obvious double complex stemming from Ω and X. □

A.2. **Sadofsky's theorem for finite height Morava K–theories.** Now we will discuss a similar theorem communicated to the author by Mike Hopkins [25, Section 14] as a stepping stone toward Sadofsky's theorem for Morava K–theories.

Theorem A.2.1 (Hopkins)**.** *Let $E(d)$ be a Johnson–Wilson spectrum and $\{X_\alpha\}_\alpha$ be a system of $E(d)$–local spectra such that $\{E(d)_* X_\alpha\}_\alpha$ is Mittag-Leffler. There is then a convergent spectral sequence of signature*

$$R^* \lim_\alpha \{E(d)_* X_\alpha\}_\alpha \Rightarrow E(d)_* \lim_\alpha \{X_\alpha\}_\alpha,$$

where the derived inverse limit on the left is taken in the category of $E(d)_*E(d)$–comodules.

The proof of this theorem relies on some shorter results, useful in their own right. Our first subgoal is to show that the $E(d)$–homology of $E(d)$–modules results in an extended comodule, which gives us access to the limit trick in Corollary A.1.2.

Lemma A.2.2. *Let M be an $E(d)$–module spectrum; then there is a natural isomorphism*

$$E(d)_* M \cong E(d)_* E(d) \otimes_{E(d)_*} \pi_* M.$$

Proof. Since $E(d)$ is an A_∞–ring spectrum, there is a strongly convergent spectral sequence describing the tensor of a right $E(d)$–module spectrum N and left $E(d)$–module spectrum M:

$$\operatorname{Tor}^{E(d)_*}_{*,*}(N, M) \Rightarrow \pi_*(N \wedge_{E(d)} M).$$

Taking $N = E(d) \wedge E(d)$ and noting that $\pi_* N = E(d)_* E(d)$ is a flat right $E(d)_*$–module, the specialized spectral sequence

$$\operatorname{Tor}^{E(d)_*}_{*,*}(E(d)_* E(d), E(d)_* M) \Rightarrow \pi_*((E(d) \wedge E(d)) \wedge_{E(d)} M)$$

is concentrated on the 0–line and collapses at E^2. Using the freeness of N, this collapse gives

$$E(d)_* E(d) \otimes_{E(d)_*} \pi_* M \cong \pi_*(E(d) \wedge E(d) \wedge_{E(d)} M) \cong \pi_*(E(d) \wedge M) = E(d)_* M. \quad \square$$

Corollary A.2.3. *If $\{M_\alpha\}_\alpha$ is a system of $E(d)$–module spectra which is Mittag-Leffer on homotopy, then there is an isomorphism*

$$E(d)_* \lim_\alpha \{M_\alpha\}_\alpha \cong \lim_\alpha \{E(d)_* M_\alpha\},$$

where the right-hand limit occurs in the category of $E(d)_* E(d)$–comodules.

Proof.

$$
\begin{aligned}
E(d)_* \lim_\alpha \{X_\alpha\}_\alpha &\cong E(d)_* E(d) \otimes_{E(d)_*} \pi_* \lim_\alpha \{M_\alpha\}_\alpha && \text{(Lemma A.2.2)} \\
&\cong E(d)_* E(d) \otimes_{E(d)_*} \lim_\alpha \{\pi_* M_\alpha\}_\alpha && \text{(Mittag-Leffler assumption)} \\
&\cong \lim_\alpha \{E(d)_* E(d) \otimes_{E(d)_*} \pi_* M_\alpha\}_\alpha && \text{(Corollary A.1.2)} \\
&\cong \lim_\alpha \{E(d)_* M_\alpha\}_\alpha. && \text{(Lemma A.2.2)} \quad \square
\end{aligned}
$$

Our next goal is to find a topological object to which we can apply Corollary A.1.7. This will be a certain Adams–type spectral sequence, and because we have not really used that tool in this paper, we quickly remind the reader of its construction.

Definition A.2.4. For a ring spectrum E and spectrum X, the following diagram describes the E–Adams tower for X:

$$
\begin{tikzcd}
X \arrow[d] & \overline{E} \wedge X \arrow[l] \arrow[d] & \overline{E}^{\wedge 2} \wedge X \arrow[l] \arrow[d] & \overline{E}^{\wedge 3} \wedge X \arrow[l] \arrow[d] & \cdots \arrow[l] \\
E \wedge X & E \wedge \overline{E} \wedge X & E \wedge \overline{E}^{\wedge 2} \wedge X & E \wedge \overline{E}^{\wedge 3} \wedge X & \cdots,
\end{tikzcd}
$$

where $\overline{E} \to S \xrightarrow{\eta_E} E$ describes the fiber of the unit map.

Remark A.2.5. In good cases, this spectral sequence converges to homotopy of the E–nilpotent completion of X [7, Proposition 6.3]. In better cases, the homotopy of the E–nilpotent completion of X agrees with that of the Bousfield E–localization of X [7, Corollary 6.13]. In better cases still, the E^2–page of the spectral sequence can be identified ([28, Theorem 2.2.11]) as

$$E^2_{*,*} \cong \operatorname{Cotor}^{E_*E}_{*,*}(E_*, E_*X).$$

We are now in a position to construct the spectral sequence in Hopkins's inverse limit theorem.

Definition A.2.6. Suppose that $\{X_\alpha\}_\alpha$ is an inverse system of $E(d)$–local spectra with the induced system $\{E(d)_*X_\alpha\}_\alpha$ Mittag-Leffler. We then have the interlocking fiber sequences

$$
\begin{tikzcd}
\lim X_\alpha \arrow[d] & \lim \overline{E(d)} \wedge X_\alpha \arrow[l] \arrow[d] & \lim \overline{E(d)}^{\wedge 2} \wedge X_\alpha \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\lim E(d) \wedge X_\alpha & \lim E(d) \wedge \overline{E(d)} \wedge X_\alpha & \lim E(d) \wedge \overline{E(d)}^{\wedge 2} \wedge X_\alpha & \cdots
\end{tikzcd}
$$

which upon applying $E(d)$–homology gives a spectral sequence with target $E(d)_* \lim X_\alpha$ and E^1–page

$$E^1_{*,t} = E(d)_* \lim\left(E(d) \wedge \overline{E(d)}^{\wedge t} \wedge X_\alpha\right).$$

We combine the preceding corollaries to compute

$$E(d)_* \lim\left(E(d) \wedge \overline{E(d)}^{\wedge t} \wedge X_\alpha\right) \cong$$

$$E(d)_*E(d) \otimes_{E(d)_*} \left(E(d)_*\overline{E(d)}\right)^{\otimes_{E(d)_*} t} \otimes_{E(d)_*} \lim E(d)_*X_\alpha$$

and hence by Corollary A.1.7

$$E^2_{*,*} \cong R^* \lim E(d)_* X_\alpha.$$

Lemma A.2.7. *The $E(d)$–Adams resolution for the $E(d)$–local sphere $L_d\mathbb{S}^0$ is equivalent to a finite-dimensional cosimplicial resolution by $E(d)$–module spectra. (In particular, the $E(d)$–Adams spectral sequence has a horizontal vanishing line.)*

Proof. Work of Ravenel [29, Lemmas 8.3.7 and 8.3.1] gives an L_dBP–prenilpotent finite spectrum F whose ordinary homology is torsion-free. Since prenilpotent spectra form a thick subcategory, it follows from their classification [20, Theorem 9] that $\mathbb{S}^0$ is L_dBP–prenilpotent. Since L_dBP and $E(d)$ share a Bousfield class [29, Theorem 7.3.2b and Lemma 8.1.4], it follows that $L_d\mathbb{S}^0$ is thus $E(d)$–nilpotent. Also definitionally [29, Definition 7.1.6], this means that $L_d\mathbb{S}^0$ has a finite $E(d)$–Adams resolution. □

Corollary A.2.8. *Every $E(d)$–local spectrum X has a finite-dimensional cosimplicial resolution by $E(d)$–module spectra. The length of the resolution is independent of X and dependent only on the prime p and height n.*

Proof. Since L_d is a smashing localization [29, Theorem 7.5.6], we can smash the finite resolution for $L_d\mathbb{S}^0$ guaranteed by Lemma A.2.7 with X. □

Proof of Theorem A.2.1. Having constructed the relevant spectral sequence in Definition A.2.6, we need only address convergence. Corollary A.2.8 shows that the homotopy inverse system in Definition A.2.6 is weakly equivalent to a finite inverse system. It follows that, upon applying $E(d)$–homology, the resulting spectral sequence is concentrated in a finite horizontal band and hence is strongly convergent to $E(d)_* \lim_\alpha X_\alpha$. □

From here, we can conclude Sadofsky's theorem for $K(d)$.

Proof of Sadofsky's theorem for $k = K(d)$. For $p \gg d$, there exists an $E(d)$–local Smith–Toda complex $V(d-1)$, which is a finite complex with the property $E(d) \wedge V(d-1) \simeq K(d)$. Replacing the system $\{X_\alpha\}$ by $\{X_\alpha \wedge V(d-1)\}$ and applying Sadofsky's result for $E(d)$–homology yields the desired spectral sequence.[6] □

Remark A.2.9. An odd wrinkle of this construction is that the inverse limit of $\{K(d)_* X_\alpha\}_\alpha$ is still taken in the category of $E(d)_*E(d)$–comodules, *not* of $K(d)_*K(d)$–comodules. However, the $E(d)_*E(d)$–comodule structure of $K(d)_*X_\alpha$ factors through the Hopf algebroid $(K(d)_*, \Gamma')$, where Γ' is given by

$$\begin{aligned}\Gamma' = K(d)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} K(d)_* &= K(d)_*[t_1, t_2, \ldots]/(v_d t_j^{p^d} - v_d^{p^j} t_j \mid j > 0) \\ &\subsetneq K(d)_*K(d) = \Gamma' \otimes \Lambda[\tau_1, \ldots, \tau_{d-1}].\end{aligned}$$

The reader should compare these stray τ_* cooperations with the Bockstein operations appeaing in the proof of [22, Proposition 8.4.e-f].

A.3. **Sadofsky's theorem for ordinary homology with field coefficients.** In the case $k = HK$ for a field K of positive characteristic, all of the above constructions can be redone to produce a derived inverse limit spectral sequence for HK–homology. However, our convergence argument fails badly, as the p–complete sphere is no longer finitely resolvable by HK–module spectra—after all, the HK–Adams spectral sequence has both an infinite tower and a vanishing line of slope 1, rather than the horizontal vanishing line present in the $E(d)$–Adams spectral sequence. It follows that the resultant spectral sequence is merely conditionally convergent, and additional hypotheses on the system

[6] For a while, the author thought that $K(d)$ admitting a finite resolution in $E(d)$–module spectra was enough to carry this proof through at all primes, but it seems to be a dead end.

$\{X_\alpha\}_\alpha$ are required to do any better. In spite of the lack of topological finiteness, Sadofsky has proven the following theorem whose proof we will not recount:

Theorem A.3.1 (Sadofsky). *The HK–based inverse limit spectral sequence converges strongly in the case that $R^s \lim_\alpha \{H_*(X_\alpha)\}_\alpha = 0$ for $s \gg 0$.* □

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1 Oxford Street, Cambridge, MA 02143
Email address: `ecp@math.harvard.edu`