THERE AREN'T THAT MANY MORAVA E–THEORIES

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ABSTRACT. Let k be a perfect field of characteristic p. Associated to any (1-dimensional, commutative) formal group law of finite height n over k there is a complex oriented cohomology theory represented by a spectrum denoted $E(n)$ and commonly referred to as Morava E-theory. These spectra are known to admit E_∞-structures, and the dependence of the E_∞-structure on the choice of formal group law has been well studied (cf. [GH], [R], [L], Section 5, [PV]). In this note we show that the underlying homotopy type of $E(n)$ is independent of the choice of formal group law.

1. Introduction

In this section we collect some standard results in chromatic homotopy theory that will be used in the proof. A combined reference for (almost) all of them is [P]. All rings in this note are commutative, graded, and concentrated in even degree. Rings that are commonly ungraded (e.g. a perfect characteristic p field k and its p-typical Witt vectors $W(k)$) are viewed graded rings concentrated in degree zero.

Definition 1. In light of the above, define a *formal group law* over a ring R to be a (graded) ring map $MU_* \to R$. Recall that MU_* is the coefficient ring of the complex bordism spectrum MU, and $MU_* \simeq \mathbb{Z}[a_1, a_2, ...]$, $|a_i| = 2i$. Define an *ungraded formal group law* over R to be a (graded) ring map $\mathrm{Laz} \to R$, $\mathrm{Laz} \simeq \mathbb{Z}[b_1, b_2, ...]$, $|b_i| = 0$.

Theorem 2. *([Lan], cf. also [H]) Let $f\colon MU_* \to R$ be a formal group law over a ring R. Let $\mathcal{M}_{MU}$ be the stack associated to the the Hopf algebroid (MU_*, MU_*MU). It admits a canonical map $\operatorname{Spec} MU_* \to \mathcal{M}_{MU}$. Then the assignment*

$$X \mapsto MU_*X \otimes_{MU_*} R$$

defines a homology theory if and only if the composite

$$\operatorname{Spec} R \xrightarrow{f} \operatorname{Spec} MU_* \to \mathcal{M}_{MU}$$

is flat.

Theorem 3. *([A]) Every homology theory is represented by a spectrum, which is unique up to homotopy equivalence. Every morphism of homology theories is represented by some morphism of representing spectra. This morphism of spectra is well-defined up to phantom maps.*

Theorem 4. *([HS]) Let $MU_* \to E_*$ and $MU_* \to F_*$ be two Landweber exact formal group laws and let E and F be spectra representing the homology theories $E_*X = MU_*X \otimes_{MU_*} E_*$ and $F_*X = MU_*X \otimes_{MU_*} F_*$. There are no phantom maps $E \to F$.*

Theorem 5. *([LT]) Let k be a perfect field of characteristic p. Let $f\colon \mathrm{Laz} \to k$ be an ungraded formal group law of height n. Define the ring*

$$LT(k) = W(k)[[u_1, u_2, \ldots, u_{n-1}]][\beta^{\pm}], \quad |\beta| = 2.$$

There is a Landweber exact formal group law

$$LT(f)\colon MU_* \to LT(k)$$

satisfying the following:

(1) *$LT(f)$ is a universal deformation of f up to $\star$–isomorphism.*
(2) *If $g\colon \mathrm{Laz} \to k$ is an ungraded formal group law that is isomorphic to f, then $LT(g)$ can be chosen to be isomorphic to $LT(f)$.*
(3) *If $t\colon k \to k'$ is a map of perfect characteristic p fields and $T\colon LT(k) \to LT(k')$ denotes the map which is $W(t)\colon W(k) \to W(k')$ on coefficients and sends u_i to u_i and β to β, then $LT(t \circ f)$ can be chosen such that $T \circ LT(f) = LT(t \circ f)$.*

Theorem 6. *([Laz] Theorem IV) All ungraded formal group laws of a fixed height n over an algebraically closed field of positive characteristic are isomorphic.*

2. The Proof

Lemma 7. *Let R be a ring with two Landweber exact formal group laws $e, f\colon MU_* \to R$, and let E and F be the spectra corresponding to e and f under the Landweber exact functor theorem. If there is a ring extension $u\colon R \to S$ which is split as an R-module map and over which the formal group laws $u \circ e$ and $u \circ f$ are isomorphic and Landweber exact, then E and F have the same homotopy type.*

Proof. First we show that $E \simeq F$ if and only if the map $\eta\colon E_* \to F_*E$ (induced by $1 \wedge id\colon \mathbb{S} \wedge E \to F \wedge E$) is split as a map of F_*-modules. Note that E_* is canonically an F_* module by the *equality* $E_* = R = F_*$. We claim that F_*E is flat over F_*. Indeed, interpret Landweber exactness as flatness of maps to $\mathcal{M}_{MU}$ (cf. Theorem 2). Consider the following pullback diagram

$$\begin{array}{ccc} \mathrm{Spec}(F_*E) & \longrightarrow & \mathrm{Spec}(E_*) \\ \downarrow & & \downarrow e \\ \mathrm{Spec}(F_*) & \xrightarrow{f} & \mathcal{M}_{MU}. \end{array}$$

The right vertical map is flat by the assumption that e is Landweber exact, hence so is the left vertical map as flat maps are preserved under pullback, i.e., F_*E is flat over F_*. It follows that $(F \wedge E)_*X = F_*X \otimes_{F_*} F_*E$. For an F_*-module map $\xi\colon F_*E \to F_*$ consider, for each spectrum X, the diagram

$$\begin{array}{ccc} E_*X & \xrightarrow{(1\wedge id)\circ(-)} & (F \wedge E)_*X = F_*X \otimes_{F_*} F_*E \\ & & \downarrow id\otimes\xi \\ & & F_*X \otimes_{F_*} F_*. \end{array}$$

The diagonal composite $t_\xi\colon E_*X \to F_*X$ defines a morphism of homology theories. Applying Brown representability for maps of homology theories (cf. Theorem 3) and the absence of phantom maps between Landweber exact homotopy types (cf. Theorem 4) produces a map of spectra $T_\xi\colon E \to F$ well-defined up to homotopy.

The assignment $\xi \mapsto T_\xi$ defines a map $\varphi\colon \mathsf{Modules}_{F_*}(F_*E, F_*) \to F^*E$. Now suppose a splitting of η exists and call it s. Then there is a map $\varphi(s)\colon E \to F$ which on homotopy groups (set $X = \mathrm{pt}$ above) is the composite of η and s, which is the identity map, so $E \simeq F$. In the other direction, if $E \simeq F$ then $F_*E \simeq E_*E$, so it suffices to split $(1 \wedge id)_*\colon E_* \to E_*E$, which is split by the multiplication map $E \wedge E \to E$.

Now we show that $\eta\colon E_* \to F_*E$ is split if there is a ring extension $u\colon R \to S$ which is split as an R-module map, and such that the formal group laws $u \circ e$ and $u \circ f$ are isomorphic and Landweber exact. Write $g := u \circ e$ and $h := u \circ f$. Assume they are isomorphic and Landweber exact. Let G and H be the spectra associated to g and h. Then we have the following commutative diagram of $F_*(= R)$-modules

$$\begin{array}{ccc} S = G_* & \xrightarrow{1\otimes 1\otimes id} & H_*G = S \otimes_{MU_*} MU_*MU \otimes_{MU_*} S \\ {\scriptstyle u}\uparrow & & \uparrow{\scriptstyle u\otimes id\otimes u} \\ R = E_* & \xrightarrow{1\otimes 1\otimes id} & F_*E = R \otimes_{MU_*} MU_*MU \otimes_{MU_*} R. \end{array}$$

Since the right vertical map is split (as a map of F_*-modules), splitting the bottom horizontal map (as a map of F_*-modules) is equivalent to splitting the diagonal map (as a map of F_*-modules). Moreover, the top horizontal map is split (as an $S = H_*$-algebra (and hence F_*-algebra) map in fact) by first choosing an isomorphism of g and h, which induces an H_*-algebra isomorphism $H_*G \simeq G_*G$ and then using the multiplication map $G \wedge G \to G$ to split $G_* \to G_*G$. Therefore splitting the diagonal map is equivalent to splitting the left vertical map (as a map of F_*-modules). □

Lemma 8. *At height $n < \infty$ the homotopy type of Morava E-theory depends only on the choice of a perfect characteristic p field. In other words, let k be a perfect characteristic p field and let $E(n)_1$ and $E(n)_2$ be two Morava E-theory spectra corresponding to two arbitrary height $n < \infty$ ungraded formal group laws e_1 and e_2 over k. Then as spectra, $E(n)_1 \simeq E(n)_2$.*

Proof. It suffices to check the conditions of the lemma. Let $\overline{k}$ be the algebraic closure of k. Let R and S be the (2-periodic) Lubin-Tate deformation rings

$$R := W(k)[[u_1, ..., u_{n-1}]][\beta^{\pm}]$$
$$S := W(\overline{k})[[u_1, ..., u_{n-1}]][\beta^{\pm}].$$

Let e and f be formal group laws over R universally deforming e_1 and e_2. Then e and f are Landweber exact (cf. Theorem 5) and the corresponding spectra are the Morava E-theory spectra $E(n)_1$ and $E(n)_2$ (cf. [P] Definition 3.5.4). Let $u\colon R \to S$ be the map that extends coefficients by $W(k) \to W(\overline{k})$, sends u_i to u_i, and β to β. Then the formal group laws $g := u \circ e$ and $h = u \circ f$ over S are isomorphic and Landweber exact (again, cf. Theorem 5), since they universally deform the isomorphic formal group laws $u \circ e$ and $u \circ f$ over $\overline{k}$ (cf. Theorem 6). Finally, the map $u\colon R \to S$ is split as an R-module map, because the map $W(k) \to W(\overline{k})$ is split as a $W(k)$-module map (cf. [S]). □

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4. References

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