

ON CERTAIN GROUPS OF UNITARY OPERATORS

ANDRÉ WEIL

TRANSLATED BY ERIC PETERSON

This is a translation of the paper *Sur Certains Groupes D'Opérateurs Unitaires* by André Weil, published in *Acta Mathematica* in 1964. In a broader historical context, this paper was part of the migration from the consideration of modular forms directly as functions to their reincarnation as modular representations—Cartan said later that Weil's goal in this paper was to “remove θ -functions from the picture entirely”, by wrapping them up in this representation. Amusingly, the results of this paper would later induce Mumford to produce θ -functions for abelian varieties in positive characteristic by *extracting* them from similarly-defined representations. I embarked on reading this paper as part of an effort to understand more of Mumford's hand in this story, without much regard for (or, indeed, background in) the Langlands philosophy that drove the interest in automorphic representations.

While this translation is a good-faith effort to retain the mathematics of the original document, I have modified the language considerably—in some instances to improve the typesetting, and in some instances to eliminate French grammatical constructions that English-speakers will find hard to bear (e.g., long chains of semicolons). Corrections or clarifications are warmly welcomed.

———Eric Peterson



Simply through repeated exposure, that theta-series define modular functions has ceased to surprise us, but the appearance of the symplectic group as a *deus ex machina* in the celebrated work of Siegel on quadratic forms has lost none of its mystery or excitement. The goal of this memoir and its sequel papers is to begin to shed some light on these aspects of the theory of automorphic forms. The key player in the part of the story which we will focus on is a certain unitary representation, not of the symplectic group itself but of a certain central extension of it, which also appears in the analogous theorems in the local and adelic settings. In the real setting, this representation has previously been studied by D. Shale [Sha62], and its existence was initially discerned by I. Segal [Seg59, Seg63] in his work on quantum mechanics. I am grateful to Segal for sharing with me his manuscript and allowing me to reproduce his proof of the existence theorem.¹

We give a rough description of the contents of this memoir and its layout.

¹This is enunciated here as Theorem 1.6.3. For a different proof, see G. Mackey [Mac65].

- (1) We also borrow from Segal the idea of using the theory of locally compact abelian groups as our jumping-off point. In Section 1, the fundamental theorems concerning the unitary representation in question are stated for a locally compact abelian group without further restrictions and hypotheses.²
- (2) In Section 2, we apply the theory developed in Section 1 to the special case of finite-dimensional vector spaces over local fields and over adelic rings to give a proof of the law of quadratic reciprocity, similar to the one found in the final chapter of Hecke's classic text on algebraic number fields.³
- (3) In Section 3, we concern ourselves with the tame behavior of the unitary representation constructed in Section 1: by again specializing to the local and adelic case, we give a detailed analysis of the continuity of this representation which cannot be accessed in the general setting. Using this, we obtain a further unitary representation of the "metaplectic group", which except in characteristic 2 is a central extension of the symplectic group by a torus T .
- (4) In Section 4, we will see that this can be reduced to a (typically nontrivial) extension of the symplectic group by $\{\pm 1\}$ (i.e., the cohomology class which determines the extension for the metaplectic group is non-null and of order 2). Though this will not be of further use, it is an interesting enough fact in its own right that we have included it here anyway.
- (5) Finally, in Section 5 we apply our results to the setting of involutive algebras, which in turn yields applications to classical groups. We close by announcing a formula which generalizes the classical results of Siegel and which will be the principal focus of the sequel memoir to this one.

Table of notations.

Chapter 1	Section 1.8: $\pi_0, \mathbf{d}_0, \mathbf{t}_0, \mathbf{d}'_0, \mathbf{r}_0, \gamma(f)$.
Section 1.1: $T, G^*, \langle x, x^* \rangle, \alpha^*, X_2(G), \mathcal{F}, \Phi^*, dx^*, \alpha $.	Section 1.9: Γ_s .
Section 1.2: $Sp(G)$.	Section 1.10: $B_0(G, \Gamma), \mathbf{r}_\Gamma, \mathbf{B}_0(G, \Gamma), \Omega_0(G, \Gamma)$.
Section 1.3: $U(w), A(G), \mathbf{A}(G), B(G), B_0(G), (\sigma, f)$.	Section 1.11: $s_1 \otimes s_2, \mathbf{s}_1 \otimes \mathbf{s}_2$.
Section 1.4: $d_0(\alpha), d_0(\gamma), t_0(f), t'_0(f'), f^\alpha, f^-$.	Chapter II
Section 1.5: $\gamma(s), \Omega_0(G)$.	Section 2.1: $X^*, [x, x^*], \alpha^*, Q(X), Q_a(X), \mathfrak{o}, \mathfrak{p}, \chi, \gamma(f)$.
Section 1.6: $\mathbf{B}_0(G), \mathbf{T}$.	Section 2.2: q_m, L_* .
Section 1.7: $\mathcal{S}(G), \mathcal{S}(H, H')$.	Section 2.3: $k_v, \mathfrak{o}_v, A_k, X_k, X_A, X_v, X_v^\circ, S, X_S^\circ, \chi, \chi_v, \gamma_v(f), \gamma(f)$.

²This perspective encourages the further study of the *finite* case, which may also be interesting: this situation superficially resembles problems studied by H. D. Kloosterman in *The behavior of general theta functions under the modular group and the characters of binary modular congruence groups* [Cite this Kloosterman article](#).

³A very simple and direct proof of Theorem 1.8.2 and Theorem 1.10.6, independent of the discussion in Section 1, was recently obtained by P. Cartier (cf. [Car64]); together with the considerations of Section 2 of the present memoir, and notably Proposition 2.1.2 and Proposition 2.2.2, it constitutes in certain regards the more satisfactory method for establishing the law of quadratic reciprocity in its more general form.

Chapter III

Section 3.1: $B(z_1, z_2)$, $Sp(X)$, $\mathfrak{A}(X)$,
 (σ, f) , $Ps(X)$.

Section 3.2: $\text{Aut}(X)$, $d(\alpha)$, $\text{Is}(X^*, X)$,
 $d'(\gamma)$, $t(f)$, $t'(f')$, $\Omega(X)$, $Ps^+(X)$,
 $Ps^-(X)$, μ , $Mp(X)$, π , \mathbf{T} , $\mathbf{d}(\alpha)$,
 $\mathbf{d}'(\gamma)$, $\mathbf{t}(f)$, $\mathbf{t}'(f')$, $\mathbf{r}(s)$, $Ps(X, L)$,
 \mathbf{r}_L , \mathbf{r}'_L .

Section 3.3: $Ps(X)_k$, $Ps(X)_v$, $Ps(X)_A$,
 $Ps(X)_v^\circ$, $Ps(X)_S^\circ$, S_∞ , μ_A , $Mp(X)_A$,
 π , \mathbf{T} , Ω_v , Ω_S , \mathbf{r}_v , \mathbf{r}_S .

Section 3.4: \mathbf{r}_k .

Chapter IV

Section 4.2: $Sp_1(X)$, $Sp_2(X)$,
 $Mp^+(X)$, $Ps_2^+(X)$.

Section 4.4: π_v , $Mp(X)_v^\circ$, $M(S)$,
 $Mp(X)_S^\circ$.

Chapter V

Section 5.1: $P(X)$.

Section 5.3: \mathcal{A} , ι , τ , X^* , $\{x, x^*\}$,
 α^* , $Q_{\mathcal{A}}(X)$, $Q_{\mathcal{A}, \mathcal{A}}(X)$, $\text{Aut}_{\mathcal{A}}(X)$,
 $\text{Is}_{\mathcal{A}}(X, Y)$, $Ps_k(X)$, $Ps_{\mathcal{A}}(X)$.

Section 5.4: $P_k(X)$, $P_{\mathcal{A}}(X)$, $Mp_k(X)$,
 $Mp_{\mathcal{A}}(X)$, $Mp_k(X)_A$, $Mp_{\mathcal{A}}(X)_k$.

1. LOCALLY COMPACT ABELIAN GROUPS

In this chapter, we will study certain phenomena concerning generic locally compact abelian groups G , and we will place no further restrictions on G save for a passage where the results would be without interest unless G were also isomorphic to its dual. Once the general theory is established, all of our intended applications will fall into one of the following cases:

Local case: G is a vector space X of finite dimension over a field k , which is itself locally compact and nondiscrete.

Adelic case: G is of the form $X_A = X_k \otimes A_k$, where A_k is the ring of adeles of a global field k (i.e., k is either an algebraic number field or the field of algebraic functions of dimension 1 over a finite field) and where X_k is a finite dimensional vector space over k .

We note that in either of these two cases, G is isomorphic to its own dual. Throughout, we will denote the group law on such a locally compact abelian group additively.

1.1. Basic definitions. We begin by recalling some basic definitions. Let T be the multiplicative group of those numbers t such that $t\bar{t} = 1$. A *character* of G is then a morphism $\chi: G \rightarrow T$. If G and H are locally compact abelian groups, then a *bicharacter* of $G \times H$ is a continuous function $f: G \times H \rightarrow T$ such that for fixed $y \in H$ the formula $f(-, y)$ gives a character of G , and for fixed $x \in G$ the formula $f(x, -)$ gives a character of H .

Definition 1.1.1. A continuous function $f: G \rightarrow T$ will be called a character of G of *second degree* if the function

$$(x, y) \mapsto f(x+y)f(x)^{-1}f(y)^{-1}$$

is a bicharacter of $G \times G$, or equivalently if for any $x, y, z \in G$ the function f satisfies

$$1 = \frac{f(x+y)f(y+z)f(z+x)}{f(x+y+z)f(x)f(y)f(z)}.$$

We will always use G^* to denote the Pontryagin dual⁴ of G (whose group law we also denote additively), and for $x \in G$, $x^* \in G^*$, we denote by $\langle x, x^* \rangle$ the value on x of the

⁴This is also known as the character group of G .

character of G which corresponds to x^* . We will also identify the bidual $(G^*)^*$ of G with G itself such that we have⁵

$$\langle x, x^* \rangle = \langle x^*, x \rangle.$$

If $x \mapsto \alpha(x)$ is a morphism from G to H , its dual α^* is the morphism from H^* to G^* such that for every $x \in G$ and $y \in H^*$,

$$\langle \alpha(x), y^* \rangle = \langle x, \alpha^*(y^*) \rangle.$$

Every bicharacter of $G \times H$ can be written uniquely in the form

$$f(x, y) = \langle x, \alpha(y) \rangle_G = \langle y, \alpha^*(x) \rangle_H,$$

where α is a morphism $\alpha: H \rightarrow G^*$, and $\alpha^*: G \rightarrow H^*$ is its dual. If $G = H$, α is self-dual if and only if f is symmetric in x and y . In this case, we say that $\alpha: G \rightarrow G^*$ is *symmetric*. Additionally, one says that the character f of second degree is *nondegenerate* if the symmetric morphism $\rho: G \rightarrow G^*$ associated to f is an isomorphism.⁶

If f is a character of G of the second degree, one has

$$(1) \quad f(x+y)f(x)^{-1}f(y)^{-1} = \langle x, \rho(y) \rangle,$$

where $\rho: G \rightarrow G^*$ is a morphism determined by f .

Definition 1.1.2. We will say that Equation (1) holds by saying that f is *associated* to ρ or, equivalently, that ρ is *associated* to f .

Let $X_2(G)$ denote the multiplicative group of characters of G of second degree. The function $f \mapsto \rho$ is a homomorphism from $X_2(G)$ to the additive group of symmetric morphisms from G to G^* , whose kernel is the multiplicative group $X_1(G)$ of characters of G . One can say more in the case that multiplication by 2 is an automorphism of G (e.g., when G is of local or adelic type over a field k of characteristic not equal to 2): if $\rho: G \rightarrow G^*$ is a symmetric morphism, it is associated to the character of second degree $f_\rho(x) = \langle x, 2^{-1}\rho(x) \rangle$. If in this setting one denotes by $X_2^\circ(G)$ the subgroup of $X_2(G)$ formed by these f_ρ , there is then a splitting

$$X_2(G) = X_2^\circ(G) \times X_1(G),$$

and $X_2^\circ(G)$ is isomorphic to the additive group of symmetric morphisms from $G \rightarrow G^*$.

One of the main tools for working with locally compact abelian groups is the general theory of *Fourier transforms*. A *Haar measure* on G is a measure on G which is invariant under G -translations; such a measure always exists and is unique up to scale. Given such a Haar measure dx , the Fourier transform \mathcal{F} relative to this choice carries a function Φ on G to a function $\Phi^* = \mathcal{F}(\Phi)$ on G^* by the formula

$$\Phi^*(x^*) = \int \Phi(x) \cdot \langle x, x^* \rangle dx$$

⁵One can also make the identification so that $\langle x, x^* \rangle = \langle -x^*, x \rangle$, and while this would be convenient in many ways, it is ultimately too jarring.

⁶In order for such characters to exist, it is of course necessary that G is abstractly isomorphic to G^* , but it is not sufficient.

whenever this integral makes sense directly, and by a suitable continuous extension when it does not. There is also a unique measure dx^* on G^* , the *dual* of dx , such that the transformation \mathcal{F}^{-1} inverse to \mathcal{F} is given by the formula

$$\Phi(x) = \int \Phi^*(x^*) \cdot \langle x, -x^* \rangle dx^*.$$

Using this dual measure, we have Plancherel's formula

$$\int |\Phi(x)|^2 dx = \int |\Phi^*(x^*)|^2 dx^*.$$

It is clear that, for all $c > 0$, the Haar measure on G^* dual to $c \cdot dx$ is $c^{-1} dx^*$. This remark can also be expressed as follows. Recall first that if G and H are locally compact groups endowed with Haar measures, the modulus of an isomorphism $\alpha: G \rightarrow H$ is the number $|\alpha| = d(x\alpha)/dx$ as defined by the formula

$$\int F(y) dy = |\alpha| \int F(x\alpha) dx,$$

where $F \in L^1(H)$ is an arbitrary integrable function.

Lemma 1.1.3. *Let G and H be two locally compact abelian groups, respectively endowed with Haar measures dx and dy . Let G^* and H^* be their duals, respectively endowed with the Haar measures dx^* and dy^* each dual to dx and dy . If $\alpha: G \rightarrow H$ is an isomorphism, then $\alpha^*: H^* \rightarrow G^*$ is an isomorphism and $|\alpha^*| = |\alpha|$.⁷*

Proof. Set $m = |\alpha|$. By transport of structure, α transforms dx into a Haar measure $d'y$ on H , and one sees also that $d'y = m^{-1}dy$. It follows that α^* transforms $(d'y)^*$ into dx^* , which is mdy^* . Thus, α^* transforms dy^* into $m^{-1}dx^*$, from which the result follows. \square

1.2. Matrix presentations. Let σ be an automorphism of $G \times G^*$. Writing $z = (x, x^*)$, we can also write σ in "matrix form": 3.

$$\begin{aligned} (x, x^*) &\mapsto (x, x^*) \cdot \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \\ &= (\alpha(x) + \gamma(x^*), \beta(x) + \delta(x^*)), \end{aligned}$$

where

$$\alpha: G \rightarrow G, \quad \beta: G \rightarrow G^*, \quad \gamma: G^* \rightarrow G, \quad \delta: G^* \rightarrow G^*$$

are morphisms of the various indicated types. The dual automorphism σ^* of $G^* \times G$ is then presented as

$$\sigma^* = \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix}.$$

Let η be the twist isomorphism from $G \times G^*$ to $G^* \times G$, given explicitly by $\eta(x, x^*) = (-x^*, x)$ or by the matrix presentation⁸

$$\eta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

⁷If $G = H$, one generally takes $dx = dy$, and then this value $|\alpha|$ is independent of the choice of dx .

⁸We will denote by 1 the identity of a group, without reference to which group is meant.

The formula

$$(2) \quad \sigma^I = \eta \sigma^* \eta^{-1} = \begin{pmatrix} \delta^* & -\beta^* \\ -\gamma^* & \alpha^* \end{pmatrix}$$

defines an automorphism of $G \times G^*$, and Lemma 1.1.3 applied to Equation (2) gives $|\sigma^I| = |\sigma|$. The assignment $\sigma \mapsto \sigma^I$ gives an involutive anti-automorphism of the group of automorphisms of $G \times G^*$.

We will denote by \mathbf{F} the standard bicharacter of $(G \times G^*) \times (G \times G^*)$ given by

$$(3) \quad \mathbf{F}(z_1, z_2) = \langle x_1, x_2^* \rangle \quad \begin{pmatrix} z_1 = (x_1, x_1^*), \\ z_2 = (x_2, x_2^*). \end{pmatrix}$$

An automorphism σ of $G \times G^*$ is called *symplectic* if it preserves the bicharacter $\mathbf{F}(z_1, z_2)\mathbf{F}(z_2, z_1)^{-1}$ —i.e., if

$$\mathbf{F}(z_1\sigma, z_2\sigma) \cdot \mathbf{F}(z_2\sigma, z_1\sigma)^{-1} = \mathbf{F}(z_1, z_2) \cdot \mathbf{F}(z_2, z_1)^{-1},$$

where $z_1, z_2 \in (G \times G^*)$. We denote by $Sp(G)$ the group formed by these automorphisms.

One immediately sees that in order for σ to be symplectic, it is necessary and sufficient that $\sigma\sigma^I = 1$. As $|\sigma^I| = |\sigma|$, it follows that every symplectic automorphism is of modulus 1. The relation $\sigma\sigma^I = 1$ shows in particular that $\alpha\beta^* = \beta\alpha^*$ and $\gamma\delta^* = \delta\gamma^*$, so that $\alpha\beta^*: G \rightarrow G^*$ and $\gamma\delta^*: G^* \rightarrow G$ are both symmetric morphisms. Using $\sigma^I\sigma$, one deduces similar claims about $\beta^*\delta$ and $\gamma^*\alpha$.

4.

1.3. The standard unitary representation and its automorphisms. For every element $w = (u, u^*) \in G \times G^*$, we write $U(w)$ for the operator which amalgamates a function Φ on G with u and u^* in the only nontrivial linear manner:

$$(U(w)\Phi)(x) = \Phi(x + u) \cdot \langle x, u^* \rangle.$$

(We will often simply write $U(w)\Phi(x)$ for this quantity.) The operators $U(w)$ are unitary when considered as automorphisms of $L^2(G)$, and for $w_1, w_2 \in G \times G^*$ there is the relation

$$U(w_1)U(w_2) = \mathbf{F}(w_1, w_2)U(w_1 + w_2),$$

where \mathbf{F} is again the bicharacter from Equation (3). From this we see that the operators $t \cdot U(w)$ for $w \in G \times G^*$, $t \in T$ form a group, where the multiplication on the set $G \times G^* \times T$ is given by

$$(4) \quad (w_1, t_1) \cdot (w_2, t_2) = (w_1 + w_2, \mathbf{F}(w_1, w_2)t_1 t_2).$$

Let us denote this group by $A(G)$ (which is locally compact using the evident topology on this underlying set). The function $(w, t) \mapsto t \cdot U(w)$ thus defines a unitary representation of $A(G)$. Denoting this group of unitary operators by $\mathbf{A}(G)$, there is an isomorphism

$$\begin{aligned} A(G) &\rightarrow \mathbf{A}(G), \\ (w, t) &\mapsto t \cdot U(w) \end{aligned}$$

which is an isomorphism of topological groups if we use the “strong” topology on $\mathbf{A}(G)$ (cf. 35.). The center of the group $A(G)$ is evidently formed by the elements $(0, t)$, which we identify with the group T itself. It is also clear that $(w, t) \mapsto w$ is a homomorphism from $A(G)$ to $G \times G^*$, which permits us to identify $A(G)/T$ with $G \times G^*$.

5.

Let us now consider the group of automorphisms of $A(G)$, $B(G)$. An automorphism $s \in B(G)$ of $A(G)$ induces two further automorphisms: one of the center $T \leq A(G)$,

which can be either $t \mapsto t$ or $t \mapsto \bar{t}$; and one on the quotient $A(G)/T = G \times G^*$, which we denote by σ . Let $B_0(G) \leq B(G)$ be the subgroup of those automorphisms which induce the identity on the center T of $A(G)$.⁹ An element $s \in B_0(G)$ can be written

$$(5) \quad (\varpi, t)s = (\sigma(\varpi), f(\varpi)t),$$

where σ is the induced automorphism on $G \times G^*$ and $f: G \times G^* \rightarrow T$ is some continuous function. That f participates in an automorphism of $A(G)$ is equivalent to the equation

$$(6) \quad f(\varpi_1 + \varpi_2)f(\varpi_1)^{-1}f(\varpi_2)^{-1} = F(\sigma(\varpi_1), \sigma(\varpi_2))F(\varpi_1, \varpi_2)^{-1}.$$

In particular, f is a character of $G \times G^*$ of second degree. Moreover, since this formula is symmetric in ϖ_1 and ϖ_2 , it follows that σ is symplectic. Henceforth, we will denote this entire situation by $s = (\sigma, f)$ when s is an automorphism of $A(G)$ and f and σ are its component functions as defined by Equation (5).

In these terms, the group law on $B_0(G)$ is given by

$$(\sigma, f) \cdot (\sigma', f') = (\sigma' \circ \sigma, f''),$$

where f'' is defined for $\varpi \in G \times G^*$ by the formula

$$(7) \quad f''(\varpi) = f(\varpi)f'(\sigma(\varpi)).$$

The mapping $s \mapsto \sigma$ which extracts the component on the quotient by the center of $A(G)$ gives a homomorphism $B_0(G) \rightarrow Sp(G)$. Applying Equation (6) to a generic element $(1, f)$ of its kernel, we find that f is a character of G and that there exist $a \in G, a^* \in G^*$ with

$$f(u, u^*) = \langle u, a^* \rangle \cdot \langle a, u^* \rangle.$$

From this, we see that $(1, f)$ is the interior automorphism of $A(G)$ corresponding to the element $(-a, a^*, 1)$. The kernel of $s \mapsto \sigma$ thus consists of interior automorphisms of $A(G)$, hence the image is isomorphic to $A(G)/T \cong G \times G^*$.

We can make the right-hand side of Equation (6) still more explicit by turning to the matrix presentations from Section 1.2. Putting σ in its matrix form, we define

$$f'(u, u^*) = f(u, u^*) \cdot \langle \gamma(u^*), -\beta(u) \rangle,$$

and in terms of f' Equation (6) becomes

$$f'(u_1 + u_2, u_1^* + u_2^*) = f'(u_1, u_1^*)f'(u_2, u_2^*) \cdot \langle u_1, \beta^*(\alpha(u_2)) \rangle \cdot \langle \delta^*(\gamma(u_1^*)), u_2^* \rangle.$$

Set $g(u) = f'(u, 0)$ and $h(u^*) = f'(0, u^*)$, and by taking $u_2 = 0, u_1^* = 0$ in the above, we are led to the following conclusions:

- (1) $f'(u, u^*) = g(u) \cdot h(u^*)$.
- (2) The functions g and h satisfy the relations

$$\begin{aligned} g(u_1 + u_2) &= g(u_1)g(u_2) \cdot \langle u_1, \beta^*(\alpha(u_2)) \rangle, \\ h(u_1 + u_2) &= h(u_1^*)h(u_2^*) \cdot \langle \delta^*(\gamma(u_1^*)), u_2^* \rangle. \end{aligned}$$

Otherwise said, these are characters of G and of G^* of second degree, respectively associated to the symmetric morphisms $\beta^* \circ \alpha: G \rightarrow G^*$ and $\delta^* \circ \gamma: G^* \rightarrow G$.

⁹We will be primarily concerned with $B_0(G)$ from here on, although some of our results could be extended to all of $B(G)$.

(3) Finally, one then has

$$f(u, u^*) = g(u)h(u^*)\langle \gamma(u^*), \beta(u) \rangle.$$

There are more precise results when $x \mapsto 2x$ is an automorphism of G , which we recount for completeness. From the results of Section 1.1, it follows that to every symplectic automorphism σ there corresponds an element (σ, f) of $B_0(G)$ given by

$$g(u) = \langle u, 2^{-1}\beta^*(\alpha(u)) \rangle, \quad h(u) = \langle 2^{-1}\delta^*(\gamma(u^*)), u^* \rangle.$$

Moreover, these formulas define a monomorphism $Sp(G) \rightarrow B_0(G)$, and $B_0(G)$ is the semidirect product of this subgroup $Sp(G)$ and the group of interior automorphisms of $A(G)$, which we have in turn shown to be isomorphic to $G \times G^*$.

6.

1.4. Matrix presentations for symplectic automorphisms. Let $s = (\sigma, f)$ be an element of $B_0(G)$ as before, and write σ in the matrix form as in Section 1.2. We will now consider what can be deduced from imposing different conditions on the matrix components of σ .

Let us consider first the diagonal case, where $\beta = 0$ and $\gamma = 0$. The symplectic condition $\sigma\sigma^I = 1$ then gives $\delta = \alpha^{*-1}$, from which it follows that the right-hand side of Equation (6) is constant at 1. We are able to satisfy Equation (6) for this choice of σ by taking $f = 1$, and our generic s therefore differs from this standard automorphism $(\sigma, 1)$ by at most an interior automorphism. We record this construction by defining a monomorphism d_0 from automorphisms α of G to $B_0(G)$ by

$$d_0(\alpha) = \left(\left(\begin{array}{cc} \alpha & 0 \\ 0 & \alpha^{*-1} \end{array} \right), 1 \right).$$

Now consider the antidiagonal case, where $\alpha = 0$ and $\delta = 0$. Since σ is an automorphism of $G \times G^*$, we must have that $\beta: G \rightarrow G^*$ and $\gamma: G^* \rightarrow G$ are isomorphisms. Then $\sigma\sigma^I = 1$ gives $\beta = -\gamma^{*-1}$, and we may similarly verify that Equation (6) can be satisfied by taking $f(u, u^*) = \langle u, -u^* \rangle$, from which we draw similar conclusions. To record this fact, we define an element of $B_0(G)$ for each isomorphism $\gamma: G^* \rightarrow G$ by

$$d'_0(\gamma) = \left(\left(\begin{array}{cc} 0 & -\gamma^{*-1} \\ \gamma & 0 \end{array} \right), \langle u, -u^* \rangle \right).$$

As a final special case, we also consider the nontrivial Jordan block form where $\alpha = 1$, $\delta = 1$, and $\gamma = 0$. The equation $\sigma\sigma^I = 1$ then reduces to $\beta = \beta^*$, and the formulas from Section 1.3 show that any f satisfying Equation (6) must be of the form

$$f(u, u^*) = g(u)h(u^*),$$

where h is a character of G^* and g is a character of G of second degree associated to β . Using these observations to solve Equation (6) leads to the following formula, where f is a character of G of second degree and $\rho: G \rightarrow G^*$ is the symmetric morphism associated to f :

$$t_0(f) = \left(\left(\begin{array}{cc} 1 & \rho \\ 0 & 1 \end{array} \right), f \right).$$

The map $t_0: X_2(G) \rightarrow B_0(G)$ is a monomorphism. Of course, we can also define lower-triangular operators of the same flavor: for f' a character of G^* of second degree with associated symmetric morphism $\rho': G^* \rightarrow G$,

$$t'_0(f') = \left(\left(\begin{array}{cc} 1 & 0 \\ \rho' & 1 \end{array} \right), f' \right)$$

defines a monomorphism $t'_0: X_2(G^*) \rightarrow B_0(G)$.

If f is a character of G of second degree, and α is an automorphism of G , we will write¹⁰

$$f^\alpha(x) = f(\alpha^{-1}(x)).$$

With this notation, we have

$$d_0(\alpha)^{-1} t_0(f) d_0(\alpha) = t_0(f^\alpha), \quad d_0(\alpha) t'_0(f') d_0(\alpha)^{-1} = t'_0(f'^{\alpha^*}).$$

If α is as above and $\gamma: G^* \rightarrow G$ is an isomorphism, then one has

$$d'_0(\alpha \circ \gamma) = d'_0(\gamma) d_0(\alpha), \quad d'_0(\gamma \circ \alpha^{*-1}) = d_0(\alpha) d'_0(\gamma).$$

The first of these relations shows that the image of d'_0 in $B_0(G)$, if it is not empty, is a right-coset for the image of d_0 in $B_0(G)$. More generally, using Equation (6) one observes that if an element $s \in B_0(G)$ is of the form $(\sigma, 1)$, the bicharacter \mathbf{F} is invariant under σ . As $1 \times G^* \leq G \times G^*$ is the set of those z_1 such that the function $\mathbf{F}(z_1, -)$ is constant at 1, and $G \times 1 \leq G \times G^*$ is the set of those z_2 such the function $\mathbf{F}(-, z_2)$ is constant at 1, it follows that σ is diagonal and $s = d_0(\alpha)$. Equation (7) then shows that for two elements $s = (\sigma, f)$ and $s'' = (\sigma'', f'')$ of $B_0(G)$ which belong to the same right-coset for the image of d_0 , it is necessary and sufficient that $f = f''$.

1.5. Matrix decomposition when γ is invertible. Let $s \in B_0(G)$ be an automorphism, (σ, f) its components, and γ the lower-left matrix component of σ . We denote by $\Omega_0(G)$ the set of those $s \in B_0(G)$ such that $\gamma: G^* \rightarrow G$ is an isomorphism.¹¹

Proposition 1.5.1. *The set $\Omega_0(G)$ is the set of those elements $s \in B_0(G)$ of the form*

$$(8) \quad s = t_0(f_1) d'_0(\gamma) t_0(f_2),$$

where $\gamma: G^* \rightarrow G$ is an isomorphism and where f_1 and f_2 are characters of G of second degree. Every element of $\Omega_0(G)$ can be uniquely expressed in this way.

Proof. In one direction, if s is given as in Equation (8), then the γ matrix component of s is indeed γ . Since γ was chosen to be invertible, s is in $\Omega_0(G)$. Conversely, for an arbitrary $s = (\sigma, f) \in \Omega_0(G)$, if it is possible to satisfy Equation (8) then we will be forced to take γ in the equation to be γ from the matrix decomposition. Coupling Equation (8) to the matrix decomposition for σ , we can then solve for f_1 and f_2 in terms of the other matrix components:

$$f_1(u) = f(u, -\gamma^{-1}(\alpha(u))), \quad f_2(u) = f(0, \gamma^{-1}(u)). \quad \square$$

¹⁰This notation could lead to a confusing collision if $\alpha = -1$. We will instead write $f^-(x) = f(-x)$.

¹¹With assuming that G and G^* are abstractly isomorphic, this set may be empty.

We may also express Equation (8) directly in terms of the matrix presentation of σ . Starting with $s = (\sigma, f)$, the characters f_1 and f_2 of second degree extracted from s via Proposition 1.5.1 are associated to the symmetric homomorphisms

$$\gamma^{-1} \circ \alpha, \delta \circ \gamma^{-1}: G \rightarrow G^*.$$

This gives the following matrix decomposition:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & \gamma^{-1} \circ \alpha \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & -\gamma^{*-1} \\ \gamma & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & \delta \circ \gamma^{-1} \\ 0 & 1 \end{pmatrix}.$$

There is an additional important relation in the case where the component f is *nondegenerate* (i.e., when the associated morphism $\rho: G \rightarrow G^*$ is an isomorphism). Associated to the symmetric morphism $\rho^{-1}: G^* \rightarrow G$ yields a character f' of G^* of second degree, given explicitly by the formula

$$f'(x^*) = f(-\rho^{-1}(x^*)).$$

Proposition 1.5.1 applied to $t'_0(f')$ gives

$$t'_0(f') = t_0(f) d'_0(\rho^{-1}) t_0(f^-).$$

On the other hand, an easy calculation also shows

$$t'_0(f') = d'_0(\rho^{-1}) t_0(f^-) d'_0(-\rho^{-1}).$$

Noting $d'_0(\rho^{-1})^2 = d_0(-1)$, we deduce a coset relation:

$$(9) \quad d'_0(-\rho^{-1}) t_0(f) d'_0(\rho^{-1}) t_0(f^-) = t_0(f^-) d'_0(-\rho^{-1}).$$

We may also put Equation (9) into a simpler form by using the relations of Section 1.4:

$$(t_0(f) d'_0(-\rho^{-1}))^3 = 1.$$

In this form, we recognize our coset relation as a classical relation from the theory of modular groups. However, it is Equation (9) that we will make literal use of later on.

1.6. $B_0(G) \leq \text{Inn}(U(L^2(G)))$. In Section 1.3 we introduced a group $\mathbf{A}(G)$, isomorphic to $A(G)$, which carries an interpretation as a collection of certain unitary operators on a function space. In this section, we will study the automorphisms of $A(G)$ via $\mathbf{A}(G)$, where we will find that each $s \in B_0(G)$, reinterpreted as an automorphism of $\mathbf{A}(G)$ rather than of $A(G)$, is induced by an interior automorphism of the group of *all* unitary operators. This theorem is due to I. Segal [Seg63] in the case where multiplication by 2 is an automorphism of G , and we borrow his method of proof.

It will be convenient to introduce an “averaged” variant of the operators $U(w)$ from Section 1.3. For a function φ on $G \times G^*$, we set

$$U(\varphi) = \int U(w) \varphi(w) d w,$$

where $w = (u, u^*)$ and $d w = d u \cdot d u^*$ does not depend on the precise choice of Haar measure $d u$ on G . More explicitly, if Φ is a function on G , then $U(\varphi)\Phi$ is the function on G defined by

$$(10) \quad U(\varphi)\Phi(x) = \int U(w)\Phi(x) \cdot \varphi(w) d w = \int \Phi(x+u) \cdot \langle x, u^* \rangle \cdot \varphi(u, u^*) d u d u^*$$

where we suppose for the moment that φ and Φ are smooth and compactly supported, to avoid technical fuss.

This last equation can also be written as

$$(11) \quad U(\varphi)\Phi(x) = \int K(x, y)\Phi(y)dy,$$

where the integral kernel K is given by

$$K(x, y) = \int \varphi(y - x, u^*) \cdot \langle x, u^* \rangle du^*$$

or, equivalently, by

$$K(x, x + u) = \int \varphi(u, u^*) \langle x, u^* \rangle du^*.$$

Therefore, taking the Fourier transform of $\varphi(u, u^*)$, fixing u and considering the remainder as a function of u^* , we obtain $K(-x, -x + u)$. Under the conditions for which the Fourier transform is invertible, it follows that

$$\varphi(u, u^*) = \int K(x, x + u) \cdot \langle x, -u^* \rangle \cdot dx.$$

Applying Plancherel's theorem yields

$$\int |K(x, y)|^2 dx dy = \int |\varphi(u, u^*)|^2 du du^*,$$

which shows that the correspondence between functions φ on $G \times G^*$ and functions K on $G \times G$, defined by the above formulas, extends by continuity to an isomorphism

$$W: L^2(G \times G^*) \rightarrow L^2(G \times G),$$

without assumptions on compact support or smoothness.

Just as the group law on the operators $U(w)$ was of interest to us in Section 1.3, the group law on the operators $U(\varphi)$ is of interest to us now. Take φ_1, φ_2 to be two functions on $G \times G^*$, provisionally assumed to have compact support. Using Equation (10), we deduce

$$(12) \quad U(\varphi_1)U(\varphi_2) = U(\varphi_3),$$

$$\varphi_3(w) = \int \varphi_1(w - w_1)\varphi_2(w_1)\mathbf{F}(w - w_1, w_1)dw_1,$$

where \mathbf{F} denotes the function defined in Equation (3). We can also express the group law at the level of integral kernels: writing $K_i = W(\varphi_i)$, Equation (11) shows

$$(13) \quad K_3(x, y) = \int K_1(x, z)K_2(z, y)dz,$$

and we denote this operation by $K_3 = K_1 \times K_2$. These formulas given for the group laws above extend by continuity to the spaces $L^2(G \times G^*)$ and $L^2(G \times G)$, and we drop the compact support hypothesis.

9.

The next three numbered equations are out of order from the original text.

A particularly simple case of all this is when K separates as $K(x, y) = P(x)Q(y)$. In this case, we write $K = P \otimes Q$. If P and Q belong to $L^2(G)$, we denote their inner product by $(P, Q) = \int P(x)\overline{Q(x)}dx$, and in this notation we describe the associated function φ as

$$(14) \quad W^{-1}(P \otimes \overline{Q})(w) = (P, U(w)Q).$$

In order to make use of separability in our context, we will need the following results:

Lemme 2

Lemma 1.6.1. *Take $K \in L^2(G \times G)$. The following conditions are equivalent:*

- (1) *The kernel K is of the form $P \otimes Q$ with $P, Q \in L^2(G)$.*
- (2) *For all $K' \in L^2(G \times G)$, $K \times K' \times K$ differs from K by a scalar factor.*

Now take $K = P \otimes Q$ and $K' = P' \otimes Q'$ for $P, Q, P', Q' \in L^2(G)$. Again, the following conditions are equivalent:

- (1) *The factors P and P' differ by a scalar multiple, as do the factors Q and Q' .*
- (2) *For all $K'' = P'' \otimes Q''$ with $P'', Q'' \in L^2(G)$, $K \times K''$ and $K' \times K''$ differ by a scalar multiple, as do $K'' \times K$ and $K'' \times K'$.*

Proof. The second part is completely evident. In the first part, necessity is evident, and sufficiency follows from setting $K' = P' \otimes Q'$. \square

We will actually need only the following consequence of this Lemma:

Lemme 3

Lemma 1.6.2. *Let $K \mapsto K^s$ be an automorphism of the Hilbert space $L^2(G \times G)$ which preserves the composition law $(K_1, K_2) \mapsto K_1 \times K_2$. There is then an automorphism t of the Hilbert space $L^2(G)$ such that for $P, Q \in L^2(G)$,*

$$(P \otimes Q)^s = P^t \otimes \overline{Q^{\bar{t}}},$$

where \bar{t} is the “imaginary conjugate” of t , defined by $\overline{Q^{\bar{t}}} = \overline{Q^t}$.

Proof. Using Lemma 1.6.1, all elements $(P \otimes Q)^s$ of $L^2(G \times G)$ are of the form $P' \otimes Q'$. Let us choose P_0 such that $\|P_0\| = 1$; since s preserves the norm, we can put $(P_0 \otimes \overline{P_0})^s$ in the form $P'_0 \otimes Q'_0$ with $\|P'_0\| = \|Q'_0\| = 1$. The second part of Lemma 1.6.1 shows then that for $P, Q \in L^2(G)$, $(P \otimes \overline{P_0})^s$ and $(P_0 \otimes Q)^s$ can be uniquely written as $P' \otimes Q'_0$ and $P'_0 \otimes Q'$. We define assignments $t, u: L^2(G) \rightarrow L^2(G)$ by $P^t = P'$ and $Q^u = Q'$.

We turn to the properties of t and u . It is clear that these assignments are linear and that $P_0^t = P'_0, \overline{P_0^u} = Q'_0$. Since s preserves the norm in $L^2(G \times G)$, the same is true of t and u in $L^2(G)$. We can also deduce information from how these assignments interact with the composition law: note that $P \otimes Q = (P \otimes \overline{P_0}) \times (P_0 \otimes Q)$, from which we deduce

$$(P \otimes Q)^s = c \cdot P^t \otimes Q^u,$$

with $c = (P'_0, \overline{Q'_0})$. Setting $P = P_0$ and $Q = Q_0$, we find $c = 1$. Because in complete generality we have

$$(P \otimes Q) \times (P \otimes Q) = (P, \overline{Q}) \cdot P \otimes Q,$$

we see that for $P^t = P^t$ and $Q^u = Q^u$ we also have $(P', \overline{Q'}) = (P, \overline{Q})$, and it follows that $u = \bar{t}$. Finally, since s^{-1} has the same properties as s , it follows that t and u are invertible, and hence they are automorphisms of $L^2(G)$. \square

10.

We now return in earnest to our goal of describing $B_0(G)$ in terms of inner automorphisms. Let $s = (\sigma, f)$ be an automorphism of $A(G)$ belonging to $B_0(G)$, and we transfer s to an automorphism of $\mathbf{A}(G)$ using the isomorphism in Section 1.3:

$$U(w)^s = f(w) \cdot U(\sigma(w)).$$

We deduce from this an automorphism of the algebra of averaged operators $U(\varphi)$ introduced above:

$$U(\varphi)^s = \int U(\sigma(w))f(w)\varphi(w)d\mathcal{W},$$

which we might write as $U(\varphi)^s = U(\varphi^s)$, where φ^s is given by

$$\varphi^s(w) = f(\sigma^{-1}(w))\varphi(\sigma^{-1}(w)).$$

It follows that the assignment $\varphi \mapsto \varphi^s$ is a unitary operator on $L^2(G \times G^*)$ and that it preserves the composition law from Equation (12).¹² We may also transport this construction to the associated integral kernels: for a kernel $K \in L^2(G \times G)$, we set $K^s = W(W^{-1}(K)^s)$. This assignment $K \mapsto K^s$ satisfies the hypotheses of Lemma 1.6.2, and hence there is an automorphism t of $L^2(G)$ such that for $P, Q \in L^2(G)$,

$$(P \otimes Q)^s = P^t \otimes Q^{\bar{t}}.$$

Because we will soon make use of the inverse to the assignment t , we define $s^{-1}P = P^t$ to ease our notational burden. Replacing Q by \bar{Q} and applying Equation (14), we find

$$(P, U(w)Q)^s = (s^{-1}P, U(w)s^{-1}Q).$$

Using the definition of φ^s , the left-hand side has value

$$\begin{aligned} \text{(definition)} \quad & (P, U(w)Q)^s = f(\sigma^{-1}(w)) \cdot (P, U(\sigma^{-1}(w))Q) \\ \text{((}P, Q) \text{ is antilinear in } Q) \quad & = (P, \overline{f(\sigma^{-1}(w))}U(\sigma^{-1}(w))Q) \\ \text{(}f \text{ is valued in } T) \quad & = (P, f(\sigma^{-1}(w))^{-1}U(\sigma^{-1}(w))Q). \end{aligned}$$

Because s is unitary, the right-hand side is equal to $(P, sU(w)s^{-1}Q)$. Letting this relation range over P and Q , we conclude

$$f(\sigma^{-1}(w))^{-1}U(\sigma^{-1}(w)) = sU(w)s^{-1}.$$

Replacing w by $\sigma(w)$, we then have

$$(15) \quad s^{-1}U(w)s = f(w) \cdot U(w\sigma) = U(w)^s.$$

This shows that the interior automorphism determined by s of the unitary group induces the automorphism s on $\mathbf{A}(G)$.

Conversely, taking s as given, this same relation determines s as an element up to the centralizer of $\mathbf{A}(G)$. However, if a unitary operator commutes with each $U(w)$, it commutes also with each $U(\varphi)$, hence with operators formed from kernels K using Equation (11). Specializing to $K = P \otimes \bar{Q}$, the operator defined by Equation (11) is given by

$$\Phi \mapsto (\Phi, Q) \cdot P.$$

¹²It is also easy to verify this directly.

If the assignment $\Phi \mapsto \Phi^t$ commutes with this operator, then we additionally have

$$(\Phi, Q) \cdot P^t = (\Phi^t, Q) \cdot P.$$

It follows that $\Phi \mapsto \Phi^t$ is of the form $\Phi \mapsto t \cdot \Phi$, where t is a scalar—and if this operator is unitary, then $t \in T$. We denote by \mathbf{T} the operators of this form; they form the center of $\mathbf{A}(G)$ as well as the center of all automorphisms of $L^2(G)$. In summary, we have thus proved the following:

Théorème 1

Theorem 1.6.3. *The centralizer of $\mathbf{A}(G)$ in the group of all automorphisms of $L^2(G)$ is \mathbf{T} , which in turn is the center of both of these groups. Moreover, if $\mathbf{B}_0(G)$ is the normalizer of $\mathbf{A}(G)$ in $L^2(G)$, then every automorphism of $\mathbf{A}(G)$ inducing the identity on \mathbf{T} is the restriction of an interior automorphism determined by an element of $\mathbf{B}_0(G)$. Additionally, $\mathbf{B}_0(G)/\mathbf{T}$ is isomorphic to $B_0(G)$, i.e., the group of automorphisms of $A(G)$ inducing the identity on T . \square*

11.

1.7. The operators s are automorphisms of Schwartz space. The Fourier transform induces an automorphism on Schwartz space $\mathcal{S}(G)$, which consists (somewhat imprecisely) of functions which are “indefinitely differentiable and rapidly decreasing”.¹³ We will now see that the operators in $\mathbf{B}_0(G)$ have the same property.

Let us recall the definition of $\mathcal{S}(G)$ for a locally compact abelian group G , considering first an “elementary” group.

Definition 1.7.1. A group G is said to be elementary when it has the form

$$G = \mathbb{R}^n \times \mathbb{Z}^p \times T^q \times F,$$

for F a finite group. A *polynomial function* on G is a function which can be written as a polynomial in the \mathbb{R} - and \mathbb{Z} -coordinates of G . The Schwartz space $\mathcal{S}(G)$ is the set of those indefinitely differentiable functions Φ on G such that, for any translation-invariant differential operator D and polynomial P , $P \cdot D\Phi$ remains bounded on G . The set $\mathcal{S}(G)$ acquires a topology by considering the seminorms $\sup |P \cdot D\Phi|$.

To tackle the general case, we consider pairs (H, H') of subgroups of G with the following properties:

- (1) H is contained in a compact neighborhood of 0 (and hence is both open and closed in G).
- (2) H' is a compact subgroup of H and H/H' is isomorphic to an elementary group.

For such a pair, we form the family $\mathcal{S}(H, H')$ of continuous functions on G which are further subject to the following three properties:

- (1) Their support contained is in H .
- (2) They are constant on the cosets of H' .
- (3) The induced function on H/H' lies in $\mathcal{S}(H/H')$.

The Schwartz space $\mathcal{S}(G)$ is then the union of these $\mathcal{S}(H, H')$, and we give $\mathcal{S}(G)$ the “inductive limit” topology, i.e., a convex set X is a neighborhood of 0 in $\mathcal{S}(G)$ if for any pair (H, H') the image of $X \cap \mathcal{S}(H, H')$ in $\mathcal{S}(H/H')$ is a neighborhood of 0 there.

¹³This space was originally introduced by L. Schwartz [Sch51, Chap. VII] in the case of \mathbb{R}^n and by F. Bruhat [Bru61] in general.

With the domain of study $\mathcal{S}(G)$ established, we would now like to show that every $\mathfrak{s} \in \mathbf{B}_0(G)$ induces an automorphism of $\mathcal{S}(G)$. It suffices to show that \mathfrak{s} induces a continuous function from $\mathcal{S}(G)$ to itself, which we will show by retracting the steps of Theorem 1.6.3. We will return to writing t in favor of \mathfrak{s}^{-1} , and we will pursue a proof for the operator $P \mapsto P^t$. Taking $Q \neq 0$ in $\mathcal{S}(G)$, the map $P \mapsto P^t$ is the composite of the following stages:

- (a) $P \mapsto K = P \otimes Q$
- (b) $K \mapsto \varphi = W^{-1}K$
- (c) $\varphi \mapsto \varphi^{\mathfrak{s}}$
- (d) $\varphi^{\mathfrak{s}} \mapsto K^{\mathfrak{s}} = W(\varphi^{\mathfrak{s}})$
- (e) $K^{\mathfrak{s}} = P^t \otimes Q^{\bar{t}} \mapsto P^t$.

It suffices to show that each step in this chain is continuous. For (a), this is immediate. For (e), notice first that if a function $K \in \mathcal{S}(G \times G)$ is of the form $P \otimes Q$ with $P, Q \in L^2(G \times G)$, then it is also of the form $P \otimes Q$ for $P, Q \in L^2(G)$. Moreover, for $Q \neq 0$ the map $P \mapsto P \otimes Q$ is an isomorphism of $\mathcal{S}(G)$ onto a closed subspace of $\mathcal{S}(G \times G)$, from which (e) follows.

For (b) and (d), it suffices to show that W determines an isomorphism from $\mathcal{S}(G \times G^*)$ to $\mathcal{S}(G \times G)$. The map W is the composition of the operator $\mathbf{F}(x, y) \mapsto \mathbf{F}(y - x, -x)$, which evidently determines an automorphism of $\mathcal{S}(G \times G)$, and the partial Fourier transform relative to the second factor of the product $G \times G^*$. This remnant is an easy generalization of an analogous theorem about the ordinary Fourier transform.¹⁴

It remains to consider (c). As an automorphism σ of $G \times G^*$ determines an automorphism of $\mathcal{S}(G \times G^*)$, we are left (upon abbreviating $G \times G^*$ to G) with proving the following:

12.

Proposition 1.7.2. *Let f be a character of G of second degree. Then $\Phi \mapsto f \circ \Phi$ is an automorphism of $\mathcal{S}(G)$.*

Proposition 2

Proof. First, we consider the elementary case: take

$$G = \mathbb{R}^n \times \mathbb{Z}^p \times T^q \times F$$

for F finite. We need only show that for all differential operators D which are invariant under translation on G , there is a polynomial function P on G such that $|Df| \leq |P|$. Note first that restricted to $\mathbb{R}^n \times T^q$, the function f is necessarily of the form $e^{iF(x)}\chi(x, y)$, where $x \in \mathbb{R}^n$, $y \in T^q$, F is a quadratic form on \mathbb{R}^n and χ is a character of $\mathbb{R}^n \times T^q$. The rest follows by expressing f on the cosets of $\mathbb{R}^n \times T^q$ in G and considering Equation (1).

Passing to the general case, let $\rho: G \rightarrow G^*$ be the symmetric morphism associated to f , and select a subgroup $H \leq G$ contained in a compact neighborhood of 0. For any subgroup H' satisfying condition (2) of the definition of $\mathcal{S}(H, H')$, it is necessary

¹⁴It is possible to prove in generality that if A and B are locally compact abelian groups and if B^* is dual to B , then the partial Fourier transform

$$\mathcal{S}(A \times B) \rightarrow \mathcal{S}(A \times B^*), \quad f(a, b) \mapsto f'(a, b^*) = \int f(a, b) \cdot \langle b, b^* \rangle db$$

is an isomorphism.

and sufficient¹⁵ to show that the subgroup $H'_* \leq G^*$ orthogonal to H' lies in a compact neighborhood of 0. The group $H'_* + \rho(H)$ would then have the same property, and we could then replace H and H' by smaller groups so that $\rho(H) \subseteq H'_*$. Using Equation (1) of 1. , this gives $f(b + b') = f(b)f(b')$ for each $b \in H$, $b' \in H'$, from which it follows that f gives a character of H' , and by writing $f(b') = \langle b', a^* \rangle$ for some $a^* \in G^*$ this formula extends to a character of G . Replacing H'_* by the group generated by H'_* and a^* , we can guarantee that a^* is a member of H'_* , from which it follows that f sends H' to 1 and is constant on the cosets of H' in G .

Since we have finished the case where G is an elementary group, it follows by passing to the quotient H/H' that $\Phi \mapsto \Phi f$ determines an automorphism of $\mathcal{S}(H, H')$. Using the inductive limit topology on $\mathcal{S}(G)$, this concludes the proof for a generic G . \square

13.

1.8. **Lifts from $B_0(G)$ to $\mathbf{B}_0(G)$.** Equation (15) gives rise to a *canonical projection*:

$$\begin{aligned} \pi_0: \mathbf{B}_0(G) &\rightarrow B_0(G), \\ \mathbf{s} &\mapsto s = (\sigma, f) \end{aligned}$$

As we will explicitly later show in the case of groups of local type, this projection does not generally admit a section. However, one may at least define sections on the images of d , d' , and t as introduced in Section 1.4.

Definition 1.8.1. Let $\Phi \in L^2(G)$. For an automorphism α of G , we define

$$\mathbf{d}_0(\alpha)\Phi(x) = |\alpha|^{-\frac{1}{2}}\Phi(x\alpha).$$

For a character of G of second degree, we define

$$\mathbf{t}_0(f)\Phi(x) = \Phi(x)f(x).$$

For an isomorphism $\gamma: G^* \rightarrow G$, we define

$$\mathbf{d}'_0(\gamma)\Phi(x) = |\gamma|^{-\frac{1}{2}}\Phi^*(-x\gamma^{*-1}),$$

where, as before, Φ^* denotes the Fourier transform of Φ .

One checks without difficulty that \mathbf{d}_0 , \mathbf{t}_0 , and \mathbf{d}'_0 are “lifts” of those functions d_0 , t_0 , and d'_0 defined in Section 1.4, i.e., $d_0 = \pi_0 \circ \mathbf{d}_0$, $t_0 = \pi_0 \circ \mathbf{t}_0$, and $d'_0 = \pi_0 \circ \mathbf{d}'_0$. Moreover, \mathbf{d}_0 and \mathbf{t}_0 are monomorphisms into $\mathbf{B}_0(G)$ from the group of automorphisms of G and from the group $X_2(G)$ respectively. For α , f , and γ as above, we also have the relations

$$\mathbf{d}_0(\alpha)^{-1}\mathbf{t}_0(f)\mathbf{d}_0(\alpha) = \mathbf{t}_0(f^\alpha), \quad \mathbf{d}'_0(\gamma\alpha) = \mathbf{d}'_0(\gamma)\mathbf{d}_0(\alpha), \quad \mathbf{d}'_0(\alpha^{*-1}\gamma) = \mathbf{d}_0(\alpha)\mathbf{d}'_0(\gamma).$$

Given these, we may therefore lift all the elements of $\Omega_0(G)$ as defined in Proposition 1.5.1 into $\mathbf{B}_0(G)$. Each $s \in \Omega_0(G)$ can be written uniquely in the form of Equation (8), i.e.,

$$s = t_0(f_1)d'_0(\gamma)t_0(f_2),$$

by which we set

$$\mathbf{r}_0(s) = \mathbf{t}_0(f_1)\mathbf{d}'_0(\gamma)\mathbf{t}_0(f_2).$$

¹⁵cf. [?, 9., p. 60].

We may also make this explicit: by writing $s = (\sigma, f)$ and $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, we obtain

$$(16) \quad \mathbf{r}_0(s)\Phi(x) = |\gamma|^{\frac{1}{2}} \int \Phi(x\alpha + x^*\gamma)f(x, x^*)dx^*.$$

The conditions under which this formula is valid are the same as those in the formula “defining” the Fourier transform, which in turn was used to define \mathbf{d}'_0 : it holds almost everywhere when $\Phi \in L^2(G) \cap L^1(G)$, and it holds for all x if $\Phi \in \mathcal{S}(G)$, in which case the two members define the same function on $\mathcal{S}(G)$.

Along with the elements themselves, we may also lift the relations among elements—for instance, Equation (9). As in Section 1.5, we will consider a nondegenerate character f of G of second degree, associated to a symmetric isomorphism $\rho: G \rightarrow G^*$. For the moment, we will use \mathbf{s} and \mathbf{s}' to denote the operators stemming from the left- and right-hand sides of Equation (9) by replacing d'_0 and t_0 with \mathbf{d}'_0 and \mathbf{t}_0 respectively. Select a function Φ (which for now we take to be continuous and of compact support), and set

$$\Phi_1(x) = \Phi * f := \int \Phi(u)f(x-u)du.$$

An easy calculation shows that $\mathbf{s}\Phi$ and $\mathbf{s}'\Phi$ are given by

$$\mathbf{s}\Phi(x) = |\rho|\Phi_1^*(\rho(x)), \quad \mathbf{s}'\Phi(x) = |\rho|^{\frac{1}{2}}\Phi^*(\rho(x)) \cdot f(x)^{-1}.$$

These operators are both unitary, and it follows that $\Phi \mapsto \Phi * f$ is continuous on $L^2(G)$. Moreover, Equation (9) itself, which we can write now as $\pi_0\mathbf{s} = \pi_0\mathbf{s}'$, shows that \mathbf{s} and \mathbf{s}' differ only by a scalar of absolute value 1. We record this relation as $\mathbf{s} = \gamma(f)\mathbf{s}'$, and by substituting $\rho^{-1}(x^*)$ for x this gives

$$(17) \quad \mathcal{F}(\Phi * f) = \gamma(f)|\rho|^{-\frac{1}{2}}\mathcal{F}(\Phi) \cdot g,$$

where g is the character of G^* of second degree associated to $-\rho^{-1}$, defined by

$$g(x^*) = f(\rho^{-1}(x^*))^{-1}.$$

Following the usual conventions in Fourier theory, Equation (17) shows that $\gamma(f)|\rho|^{-\frac{1}{2}}$ is the Fourier transform of f . We have therefore proven the following:

Theorem 1.8.2. *Let f be a nondegenerate character of G of second degree, associated to a symmetric isomorphism $\rho: G \rightarrow G^*$. Then f possess a Fourier transform $\mathcal{F}(f)$, given by the formula*

$$\mathcal{F}(f)(x^*) = \gamma(f)|\rho|^{-\frac{1}{2}}f(\rho^{-1}(x^*))^{-1},$$

where $\gamma(f)$ is a scalar factor of absolute value 1.

We emphasize that this result should be understood in the following sense: the function $\Phi \mapsto \Phi * f$ extends by continuity to $L^2(G)$, and in the context of $\Phi \in L^2(G)$ we have $\mathcal{F}(\Phi * f) = \Phi^* \cdot \mathcal{F}(f)$. By transport of structure through the isomorphism $\rho: G \rightarrow G^*$, we conclude also that $\mathcal{F}(\Phi f) = \Phi^* * \mathcal{F}(f)$. In view of Proposition 1.7.2, we see moreover that for $\Phi \in \mathcal{S}(G)$, both sides of the last equation are continuous functions, and hence the equality holds not just as members of $L^2(G^*)$ but rather on-the-nose. This is also true for the relation preceding that one. We record this as follows:

14.

Théorème 2

Corollary 1.8.3. *In addition to the notations and hypotheses of Theorem 1.8.2, let $\Phi \in \mathcal{S}(G)$ and set $\Phi^* = \mathcal{F}(\Phi)$. For any $x^* \in G^*$, we then have*

$$\int (\Phi * f)(x) \cdot \langle x, x^* \rangle dx = \gamma(f) |\rho|^{-\frac{1}{2}} \Phi^*(x^*) f(\rho^{-1}(x^*))^{-1}.$$

In particular, setting $x^* = 0$, we obtain a formula which actually implies that of Corollary 1.8.3:

Corollary 1.8.4. *Retaining the notations and hypotheses of Theorem 1.8.2, for every function $\Phi \in \mathcal{S}(G)$ we have*

$$\int \left(\int \Phi(x-y) f(y) dy \right) dx = \gamma(f) |\rho|^{-\frac{1}{2}} \int \Phi(x) dx.$$

Remark 1.8.5. The number $\gamma(f)$, attached to characters of G of second degree via Theorem 1.8.2, have enormous importance in number theory. In the case where G is of local type, these are, as we will see later, the eighth roots of unity which appear in Gauss sums.

The lifting function $\mathbf{r}_0: \Omega_0(G) \rightarrow \mathbf{B}_0(G)$ is not quite a homomorphism, though it is nearly so. Take s, s', s'' to be three elements of $\Omega_0(G)$ with $s'' = s s'$; then because $\mathbf{r}_0(s) \mathbf{r}_0(s')$ has the same image as $\mathbf{r}_0(s'')$ in $B_0(G)$, they therefore differ by a scalar $\lambda(s, s') \in T$. This value $\lambda(s, s')$ records the failure of \mathbf{r}_0 to be a homomorphism, and we now focus on its determination.

Expanding out the definition of \mathbf{r}_0 , this factor is defined by

$$\mathbf{t}_0(f_1) \mathbf{d}'_0(\gamma) \mathbf{t}_0(f_2) \cdot \mathbf{t}_0(f_1') \mathbf{d}'_0(\gamma') \mathbf{t}_0(f_2') = \lambda(s, s') \mathbf{t}_0(f_1'') \mathbf{d}'_0(\gamma'') \mathbf{t}_0(f_2'').$$

Setting $f_0 = f_2 f_1'$, $f_3 = f_1^{-1} f_1''$, $f_4 = f_2'' f_2'^{-1}$, we then have

$$(18) \quad \mathbf{d}'_0(\gamma) \mathbf{t}_0(f_0) \mathbf{d}'_0(\gamma) = \lambda(s, s') \mathbf{t}_0(f_3) \mathbf{d}'_0(\gamma'') \mathbf{t}_0(f_4).$$

We begin by considering the operator given by the left-hand side of Equation (18). Applying it to a function Φ yields

$$|\gamma|^{-\frac{1}{2}} |\gamma'|^{-\frac{1}{2}} \Psi_1^*(\gamma^{*-1}(-x)),$$

where

$$\Psi(x) = \Phi^*(\gamma'^{-1}(-x)), \quad \Psi_1(x) = \Psi(x) f_0(x).$$

However, since f_0 is a character of G of second degree with associated symmetric morphism

$$\rho = \rho_2 + \rho_1' = \delta \circ \gamma^{-1} + \gamma'^{-1} \circ \alpha' = \gamma'^{-1} \circ \gamma'' \circ \gamma^{-1},$$

it follows that f_0 is nondegenerate with Fourier transform is given by Theorem 1.8.2. The Fourier transform Ψ_1^* is thus $\Psi^* * \mathcal{F}(f_0)$. As Ψ^* is given by the formula $\Omega^*(u^*) = |\gamma'| \cdot \Psi(u^* \gamma')$, we thus compute that the operator appearing as the left-hand side of Equation (18) applied to Φ is thus¹⁶

$$\gamma(f_0) \cdot |\gamma''|^{-\frac{1}{2}} \int \Phi(\gamma'(u^*)) f_0(\gamma'^*(\gamma''^{*-1}(-x)) - \gamma(\gamma''^{-1}(\gamma'(u^*))))^{-1} d(\gamma'(u^*)).$$

¹⁶In producing this formula, we have used $\rho = \rho^*$.

Turning to the operator determined by the right-hand side of Equation (18), the image of Φ is seen to be¹⁷

$$\lambda(s, s') f_3(x) \cdot |\gamma''|^{-\frac{1}{2}} \int \Phi(u) f_4(u) \cdot \langle u, \gamma''^{*-1}(-x) \rangle \cdot du.$$

Observe that we have made a change of variables $u = \gamma'(u^*)$ in the final integral expression for the left-hand operator. To determine $\lambda(s, s')$, it suffices to observe that now both integrals take the form $c_1 \int \Phi(u) g_1(x, u) du$ and $c_2 \int \Phi(u) g_2(x, u) du$ respectively, where c_1 and c_2 are constants and g_1 and g_2 are characters of $G \times G$ of second degree. In order for these to determine the same element for $\Phi \in L^2(G)$, we must have $c_1 = c_2$ and $g_1 = g_2$. It follows that $\lambda(s, s') = \gamma(f_0)$. We record this result as follows:

Theorem 1.8.6. *Let $s = (\sigma, f)$, $s' = (\sigma', f')$, and $s'' = (\sigma'', f'')$ be three elements of $B_0(G)$ such that $s'' = s s'$. Let us write*

$$\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \sigma' = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}, \quad \sigma'' = \begin{pmatrix} \alpha'' & \beta'' \\ \gamma'' & \delta'' \end{pmatrix},$$

where $\gamma, \gamma', \gamma'' : G^* \rightarrow G$ are each isomorphisms. Then, the formula

$$f_0(u) = f(0, \gamma^{-1}(u)) f'(u, \gamma'^{-1}(\alpha'(-u)))$$

defines a nondegenerate character of G of second degree, associated to the symmetric isomorphism $\gamma'^{-1} \circ \gamma'' \circ \gamma^{-1} : G \rightarrow G^*$. Moreover, the operators $\mathbf{r}_0(s)$, $\mathbf{r}_0(s')$, and $\mathbf{r}_0(s'')$ associated respectively to s , s' , and s'' via Equation (16) participate in the relation

$$\mathbf{r}_0(s) \mathbf{r}_0(s') = \gamma(f_0) \mathbf{r}_0(s''),$$

where $\gamma(f_0)$ is defined by Theorem 1.8.2.

1.9. Θ -functions. Taking Γ to be a closed subgroup of G , we now begin to turn our attention toward the following cases of number-theoretic interest:

Local type: G is a vector space X of finite dimension over a locally compact field k of discrete valuation, and Γ (for a convenient choice of basis for X) is the group of points whose coordinates are integers in k .

Real local type: G is a vector space X of finite dimension over \mathbb{R} or \mathbb{C} and Γ is the subgroup generated by a basis of G over \mathbb{R} .

Adelic type: G is of the form $X_A = X_k \otimes A_k$, and $\Gamma = X_k$.

Let Γ_* be the closed subgroup of G^* associated by duality, i.e.,

$$\Gamma_* = \{x^* \in G^* \mid \langle \xi, x^* \rangle = 1 \text{ for all } \xi \in \Gamma\}.$$

In these three cases, there exist isomorphisms between G and G^* which transform Γ into Γ_* . In the local case, Γ and Γ_* are compact with discrete quotient, and in the other two cases they are discrete with compact quotients. One can also make the identifications

$$\Gamma_* = (G/\Gamma)^*, \quad G^*/\Gamma_* = \Gamma^*.$$

¹⁷With enough labor, one can verify directly that these two formulas differ only by a constant factor. However, this result was already calculated above, and it is useless for our present objective, which is to expressly determine $\lambda(s, s')$.

Given elements $x \in G$ and $x^* \in G^*$, we denote by \dot{x} and \dot{x}^* their images in G/Γ and in G^*/Γ_* respectively. We may choose Haar measures $d\xi$ on Γ and $d\dot{x}$ on G/Γ so that

$$dx = d\xi d\dot{x},$$

from which it follows that for any function $\Phi \in L^1(G)$ we have the factorization formula¹⁸

$$\int_G \Phi(x) dx = \int_{G/\Gamma} \left(\int_\Gamma \Phi(x + \xi) d\xi \right) d\dot{x}.$$

For every function Φ on G and every $x \in G$, it will be useful to write Φ_x for the function on Γ defined by $\Phi_x(\xi) = \Phi(x + \xi)$, so that we can abbreviate the above equality to

$$\int \Phi dx = \int \left(\int \Phi_x d\xi \right) d\dot{x}.$$

Replacing Φ by $|\Phi|^2$, we obtain in particular

$$(19) \quad \|\Phi\|_G^2 = \int_{G/\Gamma} \|\Phi_x\|_\Gamma^2 d\dot{x},$$

where $\|\Phi\|_G$ and $\|\Phi_x\|_\Gamma$ denote respectively the norms in $L^2(G)$ and in $L^2(\Gamma)$.

Proceeding along the lines of Section 1.1 for the definition of the Fourier transform, we are also led to introduce the function Θ on $G \times G^*$, defined by the formula

$$(20) \quad \Theta(x, x^*) = \int_\Gamma \Phi(x + \xi) \cdot \langle \xi, x^* \rangle \cdot d\xi.$$

In terms of Θ , the Fourier transform Φ^* of Φ can thus be written

$$(21) \quad \Phi^*(x^*) = \int_{G/\Gamma} \Theta(x, x^*) \cdot \langle x, x^* \rangle \cdot d\dot{x}.$$

Note that if Γ is discrete, the integral defining Θ reduces to a sum. In the real local case and adelic case, functions of this form are very important: for a particular choice of Φ , this construction yields the classical θ -series.

To conclude generic properties of Θ , it will be convenient to assume for a moment that Φ is continuous with compact support, so that Θ is also continuous and given explicitly by the above integral formula. The integral formula for Θ shows that it satisfies the relation

$$\Theta(x + \xi, x^* + \xi^*) = \Theta(x, x^*) \cdot \langle \xi, -x^* \rangle.$$

By setting $z = (x, x^*)$ and $\zeta = (\xi, \xi^*)$, we can use the function \mathbf{F} from Equation (3) to rewrite this as

$$(22) \quad \Theta(z + \zeta) = \Theta(z) \mathbf{F}(\zeta, z)^{-1} \quad (z \in G \times G^*, \zeta \in \Gamma \times \Gamma_*).$$

In particular, we find Θ to be invariant under translation by elements $\zeta^* \in \Gamma_*$. For any solution Θ to Equation (22) and $x \in G$, we will denote by Θ_x the function defined on G^*/Γ_* by fixing its left-coordinate at x : $\Theta_x(\dot{x}^*) = \Theta(x, x^*)$. We can then summarize Equation (20) by saying that Θ_x is the Fourier transform of Φ_x .

¹⁸The notation is justified by the fact that the function to integrate over G/Γ in the second part, written as a function of $x \in G$, is invariant under $x \mapsto x + \xi$ for all $\xi \in \Gamma$, and hence can be considered in the evident sense as a function of \dot{x} on G/Γ .

Identifying G^*/Γ_* and Γ_* with the duals of Γ and G/Γ , we endow these groups with the Haar measures $d\dot{x}^*$ and $d\xi^*$ dual to $d\xi$ and $d\dot{x}$. For $z = (x, x^*)$, we denote by \dot{z} the image (\dot{x}, \dot{x}^*) of z in the group

$$Q = (G \times G^*) / (\Gamma \times \Gamma_*) = (G/\Gamma) \times (G^*/\Gamma_*),$$

and we set $d\dot{z} = d\dot{x}d\dot{x}^*$. Following Equation (22), $|\Theta|$ is invariant under $z \mapsto z + \zeta$ for $\zeta \in \Gamma \times \Gamma_*$, and can thus be considered as a function on Q . Applying Plancherel's formula to the Fourier dual functions Φ_x and Θ_x shows $\|\Phi_x\|_\Gamma = \|\Theta_x\|$, using the measure $d\dot{x}^*$ on $L^2(G^*/\Gamma_*)$.

Equation (19) would then give $\|\Phi\|_G^2 = \|\Theta\|_Q^2$ if for all solutions Θ to Equation (22) we had

$$\|\Theta\|_Q^2 = \int_Q |\Theta(z)|^2 d\dot{z}.$$

Replacing x^* by $x^* + \xi^*$ in Equation (21), we see that for all $x^* \in G^*$, the function

$$\xi^* \mapsto \Phi^*(x^* + \xi^*)$$

is the Fourier transform of

$$\dot{x} \mapsto \Theta(x, x^*) \cdot \langle x, x^* \rangle.$$

Applying Plancherel's formula to these functions and integrating over G^*/Γ_* , we obtain

$$\|\Theta\|_Q^2 = \int_{G^*/\Gamma_*} \left(\int_{\Gamma_*} |\Phi^*(x^* + \xi^*)|^2 d\xi^* \right) d\dot{x}^*.$$

Applying Plancherel's formula to Φ and Φ^* , we see that the second half of this equation is nothing other than $\|\Phi^*\|_{G^*}^2$, using the measure $d\dot{x}^*$ dual to $d\dot{x}$ —and, by consequence, $d\dot{x}^* = d\xi^*d\dot{x}$, so that we may conclude $\|\Theta\|_Q^2 = \int_Q |\Theta(z)|^2 d\dot{z}$ and hence $\|\Phi\|_G^2 = \|\Theta\|_Q^2$.

With this established, we now work to remove the hypothesis that Φ be continuous of compact support. Denote by $H(G, \Gamma)$ the Hilbert space of solutions Θ to Equation (22) which are everywhere locally integrable on $G \times G^*$ and which have $\|\Theta\|_Q < +\infty$, so that this space inherits the norm $\|\Theta\|_Q$. The assignment $\Phi \mapsto \Theta$, as defined by the integral equation in Equation (20) for Φ continuous and compactly supported, extends by continuity to a norm-preserving linear function

$$Z: L^2(G) \rightarrow H(G, \Gamma).$$

For a general $\Phi \in L^2(G)$, the integral equation laid out in Equation (20) does not make sense. However, if we further suppose that $\Phi \in L^2(G) \cap L^1(G)$, then there is a negligible part N of G/Γ such that Φ_x belongs to $L^2(\Gamma) \cap L^1(\Gamma)$ for each $\dot{x} \notin N$. In this case, we define Θ by

$$\Theta(x, x^*) = \begin{cases} \int_\Gamma \Phi(x + \xi) \cdot \langle \xi, x^* \rangle \cdot d\xi & \text{when } \dot{x} \notin N, \\ 0 & \text{when } \dot{x} \in N. \end{cases}$$

This is a solution to Equation (22), and Θ_x is the Fourier transform of Φ_x whenever $\dot{x} \notin N$. Plancherel's theorem applies to these functions, and it combined with Equation (19) recovers the formula $\|\Phi\|_G^2 = \|\Theta\|_Q^2$ in this less restrictive context. Moreover, if we write Φ as a limit (in the sense of $L^2(G)$) of a sequence Φ_n of continuous functions with compact

18.

support, and we set $\Theta_n = Z(\Phi_n)$, we see exactly that $\Theta - \Theta_n$ has the same norm as $\Phi - \Phi_n$. This norm tends to 0 for $n \rightarrow +\infty$, and hence $\Theta = Z(\Phi)$.

Conversely, let $\Theta \in H(G, \Gamma)$ be such that $|\Theta|$ is integrable over Q —such functions are everywhere dense in $H(G, \Gamma)$. There will be a negligible subset $N \subset G/\Gamma$ such that Θ_x belongs both to $L^2(G^*/\Gamma_*) \cap L^1(G^*/\Gamma_*)$ for x in the complement of N . Since N is negligible, we replace Θ by 0 whenever $x \in N$ and preserve it otherwise, and using this new Θ we set Φ to be the function on G defined by

$$(23) \quad \Phi(x) = \int_{G^*/\Gamma_*} \Theta(x, x^*) dx^*.$$

Since $|\Theta|$ is assumed to be integrable, Φ is everywhere locally integrable. Moreover, by substituting $x + \xi$ for x in Equation (23), we see that for all x , Φ_x is the Fourier transform of Θ_x . Based on this, we may repeat the arguments above: Plancherel's formula again applies to these functions, which we again combine with Equation (19) to see that $\|\Phi\|_G^2 = \|\Theta\|_Q^2$, and hence again $\Phi \in L^2(G)$. If we write Φ as a limit (in the sense of $L^2(G)$) of a sequence Φ_n of functions with compact support, we again see that Θ is the limit of $\Theta_n = Z(\Phi_n)$ in $H(G, \Gamma)$. We have thus shown that the function $\Theta \mapsto \Phi$ defined on a dense set by Equation (23) again extends continuously to a linear function on $H(G, \Gamma)$ which is inverse to Z . We therefore conclude that $Z: L^2(G) \rightarrow H(G, \Gamma)$ is an isomorphism.

Remark 1.9.1. Let us consider further the case of $\Phi \in \mathcal{S}(G)$. It is easy to check (first for an elementary group, then passing to the general case using the definitions recounted in Section 1.7) that $x \mapsto \Phi_x$ is a *continuous* function of G in $\mathcal{S}(\Gamma)$. It follows directly that the integral equation in Equation (20) is “uniformly convergent” and hence defines a continuous solution $\Theta = Z(\Phi)$ to Equation (22). As moreover the Fourier transform relative to Γ determines a continuous isomorphism $\mathcal{S}(\Gamma) \rightarrow \mathcal{S}(G^*/\Gamma_*)$, it follows that $x \mapsto \Theta_x$ is also a continuous function from $G \rightarrow \mathcal{S}(G^*/\Gamma_*)$, hence that the integral in Equation (23) defines a continuous function in G . Therefore, for $\Phi \in \mathcal{S}(G)$, it follows that $\Theta = Z(\Phi)$ is a continuous function, and thus Equation (20) and Equation (23) are valid everywhere (and not merely almost everywhere).

19.

1.10. **The standard unitary representation on $H(G, \Gamma)$.** Using the isomorphism Z , we can transport the unitary operators previously defined on $L^2(G)$ to this new setting of $H(G, \Gamma)$. Though it is an abuse of notation, we will reuse the notation $U(w)$ for the unitary operator acting on $H(G, \Gamma)$, even though this operator would be more literally expressed as $Z \cdot U(w) \cdot Z^{-1}$. The benefit to this abuse of notation is the appearance of such familiar formulas as

$$(24) \quad U(w)\Theta(z) = \Theta(z + w)\mathbf{F}(z, w).$$

In the same vein, we will reuse $\mathbf{A}(G)$ to denote the group formed by the operators $t \cdot U(w)$ on $H(G, \Gamma)$ for $t \in T$, and we will reuse $\mathbf{B}_0(G)$ to denote the normalizer of $\mathbf{A}(G)$ in the automorphism group of $H(G, \Gamma)$.¹⁹

Let $B_0(G, \Gamma)$ denote the subgroup of $B_0(G)$ consisting of those elements $s = (\sigma, f)$ such that f is the constant function 1 on $\Gamma \times \Gamma_*$. For every such s , we define an operator

¹⁹Of course, while it follows from the preceding formula that the operator $U(w)$ given by Equation (24) transforms every solution of Equation (22) into another solution of Equation (22), this fact is also self-evident.

$\mathbf{r}_\Gamma(s) \in H(G, \Gamma)$ by the formula

$$(25) \quad \mathbf{r}_\Gamma(s)\Theta(z) = \Theta(\sigma(z))f(z).$$

It can be verified immediately that this operator permutes the space of solutions of Equation (22) among itself, from which it follows that \mathbf{r}_Γ determines a unitary representation on $B_0(G, \Gamma)$. Moreover, we may combine Equation (24), Equation (25), and Equation (6) of Section 1.3 to deduce

$$U(w)\mathbf{r}_\Gamma(s) = f(w) \cdot \mathbf{r}_\Gamma(s)U(w\sigma),$$

from which it follows that $\mathbf{r}_\Gamma(s)$ belongs to $\mathbf{B}_0(G)$ and that its canonical projection into $B_0(G)$ is s . In other terms, \mathbf{r}_Γ lifts $B_0(G, \Gamma)$ to $\mathbf{B}_0(G)$. We will denote by $\mathbf{B}_0(G, \Gamma)$ the image of $B_0(G, \Gamma)$ through \mathbf{r}_Γ , and also (by abuse of notation) the group induced from that by transport to $L^2(G)$ through Z . Again, we will write \mathbf{r}_Γ for this function transported to the old context, rather than the clumsier $Z^{-1}\mathbf{r}_\Gamma Z$.

Remark 1.10.1. Let us examine the first trivial case $\Gamma = \{0\}$ and $\Gamma_* = G^*$. Then $H(G, \Gamma)$ can be identified with $L^2(G)$, and Z can be identified with the identity function. Using the notation of Section 1.3, the group $B_0(G, \Gamma)$ becomes the group of automorphisms of the form $d_0(\alpha)t_0(f)$. The function \mathbf{r}_Γ lifts $d_0(\alpha)t_0(f)$ to $\mathbf{d}_0(\alpha)t_0(f)$.

The second trivial case is to take $\Gamma = G$ and $\Gamma_* = \{0\}$. In this case, $H(G, \Gamma)$ can be identified with $L^2(G^*)$, and Z with the Fourier transform. The group $B_0(G, \Gamma)$ is then the group of automorphisms of the form $d_0(\alpha)t'_0(f')$. Finally, \mathbf{r}_Γ is whatever function one deduces from this by duality.

In the setting of Schwartz space, we can produce the following formula:

Theorem 1.10.2. *For every function $\Phi \in \mathcal{S}(G)$ and every $\mathbf{s} \in \mathbf{B}_0(G, \Gamma)$, we have*

$$(26) \quad \int_\Gamma \Phi(\xi)d\xi = \int_\Gamma \mathbf{s}\Phi(\xi)d\xi.$$

Proof. It follows from Section 1.7 that if Φ is a function with $\Phi \in \mathcal{S}(G)$, then the deduced function $\Phi' = \mathbf{s}\Phi$ also belongs to $\mathcal{S}(G)$. Then, the associated Θ -functions $\Theta = Z(\Phi)$ and $\Theta' = \mathbf{s}\Theta = Z(\Phi')$ are continuous and can be expressed everywhere by Equation (20):

$$\Theta(x, x^*) = \int_\Gamma \Phi(x + \xi) \cdot \langle \xi, x^* \rangle \cdot d\xi, \quad \Theta'(x, x^*) = \int_\Gamma \Phi'(x + \xi) \cdot \langle \xi, x^* \rangle \cdot d\xi.$$

But, if one writes $\mathbf{s} = \mathbf{r}_\Gamma(s)$ with $s \in B_0(G, \Gamma)$, Θ' and Θ must also satisfy Equation (25):

$$\mathbf{r}_\Gamma(s)\Theta(z) = \Theta(\sigma(z))f(z), \quad \mathbf{r}_\Gamma(s)\Theta'(z) = \Theta'(\sigma(z))f(z).$$

Equality here is initially taken in the sense of $H(G, \Gamma)$, but it is furthermore true that pointwise equality also holds on the continuous locus of these functions. Taking $z = 0$, this second pair of equalities shows $\Theta'(0) = \Theta(0)$. By expressing both sides in terms of Φ and Φ' using the previous pair, we conclude the announced result. \square

Corollary 1.10.3. *For every function $\Phi \in \mathcal{S}(G)$, the formula*

$$F : \mathbf{B}_0(G) \rightarrow \mathbb{C}, \\ \mathbf{s} \mapsto \int_\Gamma \mathbf{s}\Phi(\xi)d\xi$$

Théorème 4

Corollaire
(unnumbered)

defines a function invariant under left-translations.

Proof. Using Section 1.7, $\mathfrak{s}\Phi$ belongs to $\mathcal{S}(G)$ whenever $\mathfrak{s} \in \mathbf{B}_0(G)$. Using Section 1.9, this implies in particular that $\mathfrak{s}\Phi$ induces a function on Γ that belongs to $\mathcal{S}(\Gamma)$, hence to $L^1(\Gamma)$. The corollary thus follows immediately from Theorem 1.10.2. \square

As we will soon see, this Corollary provides one of the most powerful means available for the construction of automorphic functions.

20.

In Section 1.8, we defined a certain lifting function $\mathbf{r}_0: \Omega_0(G) \rightarrow \mathbf{B}_0(G)$. We now compare this definition with the version “relative to Γ ” given above. Let $\Omega_0(G, \Gamma)$ denote the set of those elements $s = (\sigma, f)$ in $B_0(G, \Gamma)$ such that $\gamma(s): G^* \rightarrow G$ is an isomorphism which furthermore induces an isomorphism of $\Gamma_* \rightarrow \Gamma$. This set is automatically contained within both $B_0(G, \Gamma)$ and $\Omega_0(G)$. We will now show that on this subset, \mathbf{r}_Γ and \mathbf{r}_0 coincide.

Recall the formula Equation (8) of Proposition 1.5.1:

$$s = t_0(f_1)d'_0(\gamma)t_0(f_2).$$

If s belongs to $\Omega_0(G, \Gamma)$, the three factors on the right-hand side then belong to $B_0(G, \Gamma)$. It therefore suffices to check that for $t_0(f) \in B_0(G, \Gamma)$ and $d'_0(\gamma) \in B_0(G, \Gamma)$, we have the equalities

$$\mathbf{r}_\Gamma(t_0(f)) = \mathbf{t}_0(f), \quad \mathbf{r}_\Gamma(d'_0(\gamma)) = \mathbf{d}'_0(\gamma).$$

The first equality is immediate from the first assumption. For the second equality, one sees via Equation (21) that the two operators in question differ only by a positive real scalar factor—and, since they are unitary, this real number must be 1.

Remark 1.10.4. By writing this all out more carefully, one discovers the following: taking $\gamma: (G^*, \Gamma_*) \rightarrow (G, \Gamma)$ to be an isomorphism,²⁰ then γ and the isomorphism $G^*/\Gamma_* \rightarrow G/\Gamma$ which it induces both have the same modulus, $|\gamma|^{\frac{1}{2}}$. In particular, this gives $|\gamma| = 1$ when Γ and Γ_* are either both discrete or both compact.

Remark 1.10.5. Using Equation (20) for Z , Equation (23) for Z^{-1} , and Equation (25) for \mathbf{r}_Γ in the case where $s \in \Omega_0(G, \Gamma)$, we obtain a formula for \mathbf{r}_Γ on $L^2(G)$ which coincides with that given by Equation (16) for \mathbf{r}_0 , up to a positive factor—which is again necessarily equal to 1.

We remark also that we may recover a classical formula of Poisson from these techniques. Take Γ and Γ_* to be discrete and take γ to be an isomorphism $\gamma: (G^*, \Gamma_*) \rightarrow (G, \Gamma)$. If we set $s = d'_0(\gamma)$ and thus $\mathbf{r}_\Gamma(s) = \mathbf{d}'_0(\gamma)$, then Theorem 1.10.2 entails the following:

Theorem 1.10.6. *Let f be a character of G of second degree, taking the value 1 on all elements of a closed subgroup Γ of G ; let G^* be the dual of G , and Γ_* the subgroup of G^* corresponding to Γ ; and suppose that the symmetric morphism $\rho: G \rightarrow G^*$ associated to f is an isomorphism $\rho: (G, \Gamma) \rightarrow (G^*, \Gamma_*)$. Then $\gamma(f) = 1$.*

Proof. Consider the proof of Theorem 1.8.2. There, we obtained $\gamma(f)$ as a difference factor between the two sides of Equation (9) after lifting them to $\mathbf{B}_0(G)$ using \mathbf{d}'_0 and \mathbf{t}_0 . With the hypotheses of Theorem 1.10.6, all the factors of Equation (9) are in $B_0(G, \Gamma)$,

²⁰By this we mean that $\gamma: G^* \rightarrow G$ is an isomorphism carrying Γ_* to Γ .

and hence their lifts by \mathbf{d}'_0 and \mathbf{t}_0 coincide with their lifts by \mathbf{r}_Γ . As \mathbf{r}_Γ constitutes a representation, the result follows.²¹ \square

As we will see in Section 2, Theorem 1.10.6 applied to a group of adelic type recovers the law of quadratic reciprocity. A more banal but still useful case is the following:

Corollary 1.10.7. *Let f be the character of $G \times G^*$ of second degree given by*

$$f(x, x^*) = \langle x, x^* \rangle.$$

Then f is nondegenerate and $\gamma(f) = 1$.

Proof. The dual of $G \times G^*$ can be identified in the evident manner with $G^* \times G$, and the morphism associated to f is that which exchanges the two factors of $G \times G^*$. Hence, f is nondegenerate. By substituting $G \times G^*$, $G^* \times G$, $G \times \{0\}$, and $\{0\} \times G$ for G , G^* , Γ , and Γ_* respectively in Theorem 1.10.6, we conclude the result. \square

1.11. Special cases. We close this section with some remarks relevant to our cases of interest: either Γ and Γ_* compact, or Γ compact and open in G . We choose dx and $d\xi$ so that Γ is of unit total volume under both measures. From this alone, it also follows that Γ_* has unit measure under dx^* and $d\xi^*$.

Remark 1.11.1. First, let Φ be the characteristic function φ_Γ of Γ . Its image $Z(\varphi_\Gamma)$ is then the characteristic function of $\Gamma \times \Gamma_*$. Using Equation (25), $Z(\varphi_\Gamma)$ is invariant under $\mathbf{B}_0(G, \Gamma)$, and this is thus also true of $\Phi = \varphi_\Gamma$.

Remark 1.11.2. Next, we take Φ to be the characteristic function of a coset for Γ , i.e., $\Phi(x) = \varphi_\Gamma(x - a)$ for some fixed choice of $a \in G$. We see again that $\Theta = Z(\Phi)$ is then the function $\varphi_\Gamma(x - a)\varphi_{\Gamma_*}(x^*)$, where φ_{Γ_*} is the characteristic function of Γ_* . For $s \in B_0(G, \Gamma)$, we would like to characterize what it means for Θ (and by consequence Φ) to be invariant under $\mathbf{r}_\Gamma(s)$. Expand s as $s = (\sigma, f)$, with $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. The invariance of Θ is then equivalent to $(\alpha - 1)(a) \in \Gamma$, $\beta \circ \alpha \in \Gamma_*$, and $f(a, 0) = 1$. These three conditions follow from Equation (25).

Remark 1.11.3. Let Γ' then be a compact open subgroup of G containing Γ . Retaining the notation of Section 1.7, the set $\mathcal{S}(\Gamma', \Gamma)$ is composed of those linear combinations of functions $\varphi_\Gamma(x - a)$ with constant coefficients and with $a \in \Gamma'$. This is a complex vector space whose dimension is equal to the finite index of Γ in Γ' . Let $s = (\sigma, f)$ be, as above, an element of $B_0(G, \Gamma)$. Then, for every function of $\mathcal{S}(\Gamma', \Gamma)$ to be invariant under $\mathbf{r}_\Gamma(s)$, it is necessary and sufficient that f takes the value 1 on $\Gamma' \times \Gamma_*$, that $\Gamma' \cdot (\alpha - 1) \subseteq \Gamma$, and that $\Gamma' \cdot \beta \subseteq \Gamma_*$. This is the same as saying that f must take the value 1 on $\Gamma' \times \Gamma_*$, that σ must induce an automorphism on $\Gamma' \times \Gamma_*$, and that σ determines the identity automorphism on the quotient $(\Gamma' \times \Gamma_*)/(\Gamma \times \Gamma_*)$.

Finally, we make note of some “functorial” properties. In the case of a product group $G = G_1 \times G_2$, we can identify its dual as $G^* = G_1^* \times G_2^*$ and moreover

$$G \times G^* = (G_1 \times G_1^*) \times (G_2 \times G_2^*).$$

²¹Though it is essentially the same proof, one could also make use of Theorem 1.8.6, in which one would investigate $s, s' \in B_0(G, \Gamma)$ such that $f_0 = f$.

Corollaire
(unnumbered)

21.

22.

Then, if σ_1 and σ_2 are respectively automorphisms of $G_1 \times G_1^*$ and $G_2 \times G_2^*$, we may extract an automorphism of $G \times G^*$ which we will denote (σ_1, σ_2) . If f_1 and f_2 are characters of $G_1 \times G_1^*$ and $G_2 \times G_2^*$ respectively, both of second degree, we then extract from this a character $f = f_1 \otimes f_2$ on $G \times G^*$ of second degree. If $s_1 = (\sigma_1, f_1)$ and $s_2 = (\sigma_2, f_2)$ are elements of $B_0(G_1)$ and $B_0(G_2)$ respectively, then $((\sigma_1, \sigma_2), f_1 \otimes f_2)$ will be an element of $B_0(G)$ which we denote by $s_1 \otimes s_2$. Whenever it is convenient, we will identify the product $B_0(G_1) \times B_0(G_2)$ with the group of those elements $s_1 \otimes s_2$ of $B_0(G)$, and we identify $B_0(G_1)$ and $B_0(G_2)$ with the factors of this group.

On the other hand, we can consider $L^2(G)$ as a ‘‘Hilbertian tensor product’’ of $L^2(G_1)$ and $L^2(G_2)$. Thus, if s_1 and s_2 are automorphisms of $L^2(G_1)$ and $L^2(G_2)$ respectively, then we extract from this an automorphism $s_1 \otimes s_2$ of $L^2(G)$. For each of the groups G_1 , G_2 , and G , just as in Section 1.3 we can define operators $U(w_1)$, $U(w_2)$, and $U(w)$ which are automorphisms respectively of $L^2(G_1)$, $L^2(G_2)$, and $L^2(G)$. For $w = (w_1, w_2)$, we then have

$$U(w) = U(w_1) \otimes U(w_2).$$

It follows that $(s_1, s_2) \mapsto s_1 \otimes s_2$ determines a homomorphism

$$\otimes: \mathbf{B}_0(G_1) \times \mathbf{B}_0(G_2) \rightarrow \mathbf{B}_0(G),$$

compatible with the canonical projections of $\mathbf{B}_0(G_1)$ to $B_0(G_1)$, $\mathbf{B}_0(G_2)$ to $B_0(G_2)$, and $\mathbf{B}_0(G)$ to $B_0(G)$. The kernel of this homomorphism is formed by those elements $(\mathbf{t}, \mathbf{t}^{-1})$ of the center, and it induces monomorphisms from $\mathbf{B}_0(G_1)$ and $\mathbf{B}_0(G_2)$ to $\mathbf{B}_0(G)$. Using this we can, whenever convenient, identify these subgroups with their images inside $\mathbf{B}_0(G)$.

2. APPLICATION TO THE LAW OF QUADRATIC RECIPROCITY

2.1. Basic definitions, part II. We will now begin to investigate situations of interest to algebraic number theorists in earnest. The notational conventions of that field are in very mild conflict with the conventions established in Section 1, so we begin by reconciling any such differences.

For every vector space X over a field k , we will designate its linear dual by X^* , and for every $x \in X$ and $x^* \in X^*$ we will denote by $[x, x^*]$ the value of x^* on x . For X of finite dimension, we may identify it with its bidual $(X^*)^*$ using the formula

$$[x, x^*] = [x^*, x].$$

For a linear function $\alpha: X \rightarrow Y$, we will denote its transpose as α^* , which is the linear function $\alpha^*: Y^* \rightarrow X^*$ defined by the formula

$$[\alpha(x), y^*] = [x, \alpha^*(y^*)].$$

Every bilinear form on $X \times Y$ can be written as $[x, \alpha(y)]$, where $\alpha: Y \rightarrow X^*$ is a linear morphism. For $X = Y$, we say that α is symmetric if $[x, \alpha(y)]$ is symmetric in x and y , i.e., if $\alpha = \alpha^*$.

We will denote by $Q(X)$ the vector space of quadratic forms on X . For f such a quadratic form, we may produce a symmetric morphism ρ for which we have

$$f(x + y) - f(x) - f(y) = [x, \rho(y)].$$

We then say that f and ρ are *associated* to one another. The quadratic form f is said to be *nondegenerate* when $\rho: X \rightarrow X^*$ is an isomorphism; at the other extreme, we say

that f is additive if $\rho = 0$. We will denote the subspace of $Q(X)$ consisting of the additive forms by $Q_a(X)$. If k is not of characteristic 2, then one can show that there are no additive quadratic forms other than 0. Moreover, in this case every symmetric morphism $\rho: X \rightarrow X^*$ is associated to a unique quadratic form f :²²

$$f(x) = [x, 2^{-1}\rho(x)].$$

Suppose that k is a local field, i.e., a locally compact non-discrete commutative field. Such fields are either isomorphic to \mathbb{R} , isomorphic to \mathbb{C} , or of discrete valuation. If this last condition holds, then if the characteristic of k is zero then k must be isomorphic to a finite extension either of \mathbb{Q}_p (the p -adic completion of the rationals), or if it is positive characteristic p then k must be isomorphic to the field of Laurent series in a single variable over \mathbb{F}_p . In the case of discrete valuation, we denote by \mathfrak{o} the ring of integers of k , by π a prime element of \mathfrak{o} (i.e., a generator of the prime ideal \mathfrak{p} of \mathfrak{o}), and by q the number of elements of the finite field $\mathfrak{o}/\mathfrak{p}$.

Once and for all, we choose a nontrivial²³ character χ of the additive group of k .²⁴ The function $(x, y) \mapsto \chi(xy)$ is a bicharacter of $k \times k$ which identifies k with its own Pontryagin dual (i.e., the dual in the sense of locally compact abelian groups). More generally, let X be a vector space of finite dimension over k , let X^* be its dual, and endow X and X^* with the evident topologies. We can then identify X^* with the Pontryagin dual of X so that

$$\langle x, x^* \rangle = \chi([x, x^*]).$$

This identification, which clearly depends upon the choice of χ , we make once and for all, so that we need not distinguish between the algebraic dual and the Pontryagin dual.

If f is a quadratic form on a vector space X over k , $\chi \circ f$ is a character of X of second degree in the sense of Section 1.1. The morphism $X \rightarrow X^*$ associated to $\chi \circ f$ is the same as that associated to f . In particular, $\chi \circ f$ is nondegenerate if and only if f is nondegenerate.

Remark 2.1.1. When this is the case, we can apply Theorem 1.8.2 to extract a number $\gamma(\chi \circ f)$ of absolute value 1. We abbreviate this value to $\gamma(f)$, though one must remember that this value depends upon the choice of χ . However, because $|\gamma(f)| = 1$, it does not depend on the choices of Haar measures which appear in its definition: when one changes the measures, this modifies the formulas of Theorem 1.8.2 and its Corollaries only by positive real factors, which collectively must have no effect. This remark allows us to abandon the convention established in Section 1 of always taking on the dual G^* of a group G the measure dual to that taken on G .

The remark above shows in particular that these γ -values are stable under isomorphism: if $f' = f \circ \alpha$ for an isomorphism $\alpha: X' \rightarrow X$ of vector spaces, then $\gamma(f') = \gamma(f)$. We note also that $\gamma(-f) = \gamma(f)^{-1}$, because $\chi \circ (-f)$ is the imaginary conjugate of $\chi \circ f$. If -1 is a square in k (and k is of characteristic 2), then $-f$ is equivalent to f . In this case, it then follows that for any f we have $\gamma(f) = \pm 1$.

²²Irrespective of the characteristic of k , one always has $2f(x) = [x, \rho(x)]$.

²³“Nontrivial” here means that it is not the constant function valued at 1.

²⁴It is possible to choose χ in a “canonical” manner, but this will not be useful for us.

24.

25.

Proposition 3

Proposition 2.1.2. *The function $f \mapsto \gamma(f)$ determines a character of the Witt group of the local field k .*

Proof. Let f_1, f_2 be nondegenerate forms on the vector spaces X_1, X_2 , and let f be the form given by $f(x_1, x_2) = f_1(x_1) + f_2(x_2)$ on $X_1 \oplus X_2$. Then f is nondegenerate, and by using the definition of γ in Theorem 1.8.2 it is clear that $\gamma(f) = \gamma(f_1)\gamma(f_2)$.

Second, a nondegenerate form f corresponds to the trivial element of the Witt group exactly when for some n it is equivalent to the form $\sum_{i=1}^n x_i x_{n+1}$ on k^{2n} . This coincides with the claim that f is equivalent to the form $[x, x^*]$ on $X \times X^*$ for some choice of X . In this case, Corollary 1.10.7 shows that $\gamma(f) = 1$, so that γ factors through the Witt group. \square

26.

2.2. Some calculations of γ over local fields. Our goal in this Section is to make calculations of the value $\gamma(f)$ where we can. We begin this project by considering the case where k is a local field. We will denote by q_m the form $q_m(x) = \sum_{i=1}^m x_i^2$ on the space k^m , which is automatically nondegenerate whenever k is not of characteristic 2.

Let us first consider $k = \mathbb{R}$. Over \mathbb{R} , all nondegenerate quadratic forms are equivalent to one of the form $q_a(x) - q_b(y)$ on the space $\mathbb{R}^a \times \mathbb{R}^b$. We call the pair (a, b) the *inertia type* of the form. Proposition 2.1.2 shows that if f has inertia type (a, b) , then $\gamma(f) = \gamma(q_1)^{a-b}$, so that we may consider only the calculation of $\gamma(q_1) = \gamma(\chi \circ q_1)$. Whatever character χ on \mathbb{R} we have chosen must be of the form $\chi(x) = e^{2\pi i \lambda x}$ for some real nonzero λ . To calculate $\gamma(q_1)$, we recall the following formula from Corollary 1.8.4:

$$\int \left(\int \Phi(x-y)f(y)dy \right) dx = \gamma(f)|\rho|^{-\frac{1}{2}} \int \Phi(x)dx.$$

By selecting a particularly simple choice of Φ , such as $\Phi = e^{-\pi x^2}$, we then compute

$$\gamma(q_1) = \begin{cases} e^{\pi i/4} & \text{if } \lambda > 0, \\ e^{-\pi i/4} & \text{if } \lambda < 0. \end{cases}$$

Remark 2.2.1. In both cases, $\gamma(f)$ is an eighth root of unity. Independent of λ , we thus have that $\gamma(q_4) = -1$ for the form $q_4(x)$, which is the norm for real quaternions.

We next consider $k = \mathbb{C}$. Every complex quadratic form is equivalent to q_m for some m , and for the same reason as in the real case it suffices to determine $\gamma(q_1)$. Let χ_0 denote the character $e^{2\pi i x}$ on \mathbb{R} ; our chosen character χ on \mathbb{C} is then necessarily of the form $\chi(z) = \chi_0(\lambda z + \bar{\lambda} \bar{z})$ for some $\lambda \neq 0$. For $f = \lambda^{-1} q_1$, we have $\chi \circ f = \chi_0 \circ f_0$, where f_0 is given by $f_0(z) = z^2 + \bar{z}^2$. On the 2-dimensional real vector space underlying \mathbb{C} , f_0 is a quadratic form of inertia type $(1, 1)$, from which it follows from the real case that $\gamma(\chi_0 \circ f_0) = 1$, hence $\gamma(\chi \circ f) = 1$, hence $\gamma(f) = 1$, hence $\gamma(q_1) = 1$. The assignment γ is thus constant for all nondegenerate quadratic forms on \mathbb{C} .

27.

Finally, we turn to the case where k is of discrete valuation. Before we begin, we establish some relevant tools. For X a vector space over k , an open compact subgroup of X which is a module over \mathfrak{o} will be called a *lattice*. If L is a lattice in X , its dual L_* also yields a lattice in X^* . If $L \supset L'$ are two lattices in X , we may define the analogue $\mathcal{S}(L, L')$ of the function space considered in Section 1.7: $\mathcal{S}(L, L')$ consists of functions which are supported on L and which are constant on cosets of L' . The union of the families $\mathcal{S}(L, L')$ for all possible choices of L and L' recovers $\mathcal{S}(X)$.

Let f be a nondegenerate quadratic form on X with associated symmetric isomorphism $\rho: X \rightarrow X^*$. As the kernel of χ is an open subgroup of k , $\chi \circ f$ has the constant value 1 on a neighborhood of 0 in X , hence on all sufficiently small lattices. Choosing such a small lattice L , we find that for $x, y \in L$ we have $\chi([y, \rho(x)]) = 1$,²⁵ and hence $\rho(L) \subset L_*$. We set $L' = \rho^{-1}(L_*)$, we satisfies $L' \supset L$. If φ_L is the characteristic function of L , we then have

$$(\varphi_L * (\chi \circ f))(x) = \int_L \chi(f(x-y)) dy = \chi(f(x)) \int_L \chi([y, -\rho(x)]) dy.$$

Depending on whether $\rho(x)$ is or is not in L_* , the right-hand integral attains either the value $m(L) = \int_L dy$ or zero, i.e.,

$$(27) \quad \varphi_L * (\chi \circ f) = m(L) \varphi_{L'} \cdot (\chi \circ f).$$

We now apply Corollary 1.8.4 to $\chi \circ f$ and $\Phi = \varphi_L$. It follows that $\gamma(f)$ differs only by a positive real factor from $\int_{L'} \chi \circ f dx$. We investigate this as a function of a lattice M :

$$g(f, M) = \int_M \chi(f(x)) dx.$$

If $M \supset L$, this can be written

$$g(f, M) = \sum_{x \in M/L} \chi(f(x)) \int_L \chi([y, -\rho(x)]) dy,$$

or, by setting $M' = M \cap L'$ and using the preceding discussion,

$$g(f, M) = m(L) \sum_{x \in M'/L} \chi(f(x)).$$

This right-hand summation is a Gauss sum, which is moreover independent of M once M is sufficiently large, as we required only $M \supset L'$. We therefore drop it from the notation and write $g(f) = g(f, L')$ for M sufficiently large, and we have thus learned

$$\gamma(f) = g(f) / |g(f)|.$$

This permits, if one likes, to apply the calculation of $\gamma(f)$ to the theory of Gauss sums.

Leaving that aside for now, we turn instead to the following result, which is more directly connected to our goal of quadratic reciprocity.

Proposition 2.2.2. *Let \mathfrak{k} be the algebra of quaternions over the local field k , and let n be the norm on \mathfrak{k} : $n(z) = z \cdot \bar{z}$. Then $\gamma(n) = -1$.*

Proof. Pursuant to the preceding discussion, all that remains to show is that $g(n, M)$ is real and negative when M is a sufficiently large lattice in \mathfrak{k} .

It is well-known that $n: \mathfrak{k}^\times \rightarrow k^\times$, considered as a homomorphism of multiplicative groups, is surjective with compact kernel. Moreover, for any integer v , the set M_v of those $z \in \mathfrak{k}$ such that $n(z) \in \pi^{-v} \mathfrak{o}$ is an ideal in \mathfrak{k} (and, indeed, a lattice), which can be made as large as one likes by taking v sufficiently large. Let dx and dz denote the Haar measures on the additive groups of k and \mathfrak{k} , and let the first be normalized so that \mathfrak{o} has unit measure. As usual, for $a \in k$ we define $|a|$ to be the modulus of the homothety

²⁵The converse is true whenever 2 is invertible in \mathfrak{o} .

28.

Proposition 4

$x \mapsto ax$. It follows that $|x|^{-1}dx$ is a Haar measure on the multiplicative group of k . For $c \in \mathfrak{k}$, the modulus of the analogous homothety $z \mapsto cz$ is $|n(c)|^2$, and hence $|n(z)|^{-2}dz$ is a Haar measure on the multiplicative group of \mathfrak{k} .

Writing φ for the characteristic function of \mathfrak{o} on k , the lattice $M_v \subseteq \mathfrak{k}$ can then be written as $M_v = \varphi(\pi^v n(z))$ and hence the value $g(n, M_v)$ can be written as

$$g(n, M_v) = \int \chi(n(z))\varphi(\pi^v n(z))dz.$$

By setting $\psi_v(x) = \chi(x)\varphi(\pi^v x)|x|^2$, we can rewrite this in terms of the Haar measure on the multiplicative group:

$$g(n, M_v) = \int_{\mathfrak{k}^\times} \psi_v(n(z)) \cdot |n(z)|^{-2} dz.$$

Let us first integrate over the cosets of kernel of n , since the integrand is constant on these sets, giving

$$g(n, M_v) = \lambda \int_{\mathfrak{k}^\times / \ker n} \psi_v(x) \cdot |x|^{-1} dx = \lambda \int_{k^\times / \ker n} \chi(x)\varphi(\pi^v x)|x| dx,$$

where the positive real value λ is the measure of the kernel of n for the quotient measure of $|n(z)|^{-2}dz$ by $|x|^{-1}dx$. We may rewrite part of the integrand as

$$\varphi(\pi^v x) \cdot |x| = \sum_{\mu=-v}^{+\infty} |\pi^\mu| \cdot (\varphi(\pi^{-\mu} x) - \varphi(\pi^{-\mu-1} x)).$$

Then, since $|\pi| = q^{-1}$, we can then further rewrite the whole expression as

$$g(n, M_v) = \lambda \sum_{\mu=-v}^{+\infty} q^{-\mu} \left(\int_{\pi^\mu \mathfrak{o}} \chi(x) dx - \int_{\pi^{\mu+1} \mathfrak{o}} \chi(x) dx \right).$$

Take j to be the least of those integers μ such that the ideal $\mathfrak{p}^\mu = \pi^\mu \mathfrak{o}$ is contained in the kernel of the character χ of k . When χ induces the constant function 1 on \mathfrak{p}^μ (i.e., for $\mu \geq j$), its integral merely computes its measure $m(\mathfrak{p}^\mu) = q^{-\mu}$. In the alternative, χ is a nontrivial character and hence this integral vanishes. For $v \geq 1 - j$ we thus obtain:

$$g(n, M_v) = -\lambda \cdot \frac{q^{1-2j}}{1+q^{-1}}. \quad \square$$

Remark 2.2.3. We note that our previous computations in $k = \mathbb{R}$ and $k = \mathbb{C}$ show that Proposition 2.2.2 is valid also in those settings.²⁶ It is thus valid for all local fields.

Over every local field k other than \mathbb{C} , it can be shown that there are only two classes of nondegenerate quadratic forms in 4 variables whose discriminant is a square in k : the form n studied above and the “trivial” form $xy + zt$. Proposition 2.1.2 and Proposition 2.2.2 show that the symbol $\gamma(f)$ can be used to distinguish between these classes of forms, as it takes the value -1 for the first and the value 1 for the second—at least, if k is not of characteristic 2. In this case, all quadratic forms determined by $x^2 - ay^2 - bz^2 + abt^2$ for

²⁶Indeed, trivially so for $k = \mathbb{C}$, as there is no complex quaternion algebra over.

nonzero $a, b \in k$ belong to one or the other of these two classes, depending on whether a is the norm or not of an element of $k(b^{\frac{1}{2}})$. We record this in the formula

$$\gamma(x^2 - ay^2 - bz^2 + abt^2) = \left(\frac{a}{b}\right),$$

where $\left(\frac{a}{b}\right)$ denotes the norm residue symbol. Setting $b = -1$ and applying Proposition 2.1.2,

$$\gamma(q_1)^2 \gamma(-aq_1)^2 = \left(\frac{a}{-1}\right),$$

which can also be written as

$$\gamma(aq_1)^2 = \left(\frac{a}{-1}\right) \gamma(q_1)^2.$$

As all nondegenerate forms can be written as $\sum_i a_i x_i^2$ under a convenient choice of basis, we may therefore apply Proposition 2.1.2 again to conclude

$$(28) \quad \gamma(f)^2 = \left(\frac{D}{-1}\right) \gamma(q_1)^{2m},$$

where m is the dimension of the vector space where f is defined and D is the discriminant of f .²⁷ We thus conclude $\gamma(f)^4 = \gamma(q_4)^m$, where $\gamma(f)^8 = 1$ as q_4 is nondegenerate of discriminant 1. This echoes results already achieved in other settings: if -1 is a square in k , then $\gamma(f) = \pm 1$ for any such f , and hence $\left(\frac{D}{-1}\right) = 1$ no matter what D is. In fact, $\gamma(f)^8 = 1$ is valid even in characteristic 2, as one again has $\gamma(f) = \pm 1$.

2.3. Calculation of γ over adelic rings and the law of quadratic reciprocity. We are now in a position to pursue the law of quadratic reciprocity in earnest: we will assemble it out of the local results from the preceding section, bundling them together by using Theorem 1.10.6 in the context of a group of adelic type.

Let k be a global field, by which we mean a number field (i.e., a finite extension of \mathbb{Q}) or a function field (i.e., a transcendental extension of dimension 1 over a finite field). We will use k_v to denote the completion of k at the place v , \mathfrak{o}_v to denote the ring of integers of k_v each time that k_v is of discrete valuation, and A_k to denote the ring of adeles of k . For X_k a vector space (of finite dimension) over k , we set $X_A = X_k \otimes A_k$ and, for every v , $X_v = X_k \otimes k_v$. If X° is a basis of X_k over k , for k_v of discrete valuation we denote by X_v° the set of those points of X_v whose coordinates in the basis X° lie in \mathfrak{o}_v ; this gives a lattice in X_v . We denote by S all finite sets of completions of k , including those which are isomorphic to \mathbb{R} or to \mathbb{C} . Under these conditions and with these notations, X_A is the union (and, in the sense of topological spaces, the direct limit) of the products

$$(29) \quad X_S^\circ = \prod_{v \in S} X_v \times \prod_{v \notin S} X_v^\circ.$$

All compact subsets of X_A are contained in a set of the form $\prod_v C_v$, where C_v is a compact subset of X_v for each v and where $C_v = X_v^\circ$ for almost all²⁸ v . We conclude from this that any subgroup of X_A contained in a compact neighborhood of 0 is contained in

²⁷This value is independent of choice of basis, as a change of basis does not modify D except by a square.

²⁸That is, for all v save for a finite number of them.

a subgroup of the form $H = \prod_v H_v$, where H_v is equal to X_v each time that k_v is isomorphic to \mathbb{R} or to \mathbb{C} , H_v is equal to a lattice in X_v each time k_v is of discrete valuation, and $H_v = X_v^\circ$ for almost all v . Supposing that we have chosen such an H , all compact subgroups H' of H such that H/H' is an elementary group contain an analogous subgroup of the form $H' = \prod_v H'_v$, where H'_v is equal to $\{0\}$ each time k_v is isomorphic to \mathbb{R} or to \mathbb{C} , H'_v is equal to a lattice of X_v contained in H_v each time k_v has discrete valuation, and $H'_v = X_v^\circ$ for almost all v . From this we may make deductions about the structure of $\mathcal{S}(X_A)$. For one, it contains all functions of the form $(x_v) \mapsto \prod \Phi_v(x_v)$, where $\Phi_v \in \mathcal{S}(X_v)$ for all v , and where for almost all v the function Φ_v is the characteristic function of X_v° . For $k = \mathbb{Q}$, or for k of characteristic $p \neq 0$, $\mathcal{S}(X_A)$ is in fact the set of finite linear combinations of these functions using constant coefficients.²⁹

As in the local case, we choose once and for all a nontrivial character χ of A_k which takes the value 1 on k . Such a character is necessarily of the form

$$\chi(t) = \prod_v \chi_v(t_v), \quad (t = (t_v) \in A_k),$$

where for each v the character χ_v is a nontrivial character of k_v , and where for almost all v the lattice \mathfrak{o}_v corresponds to itself by duality.³⁰ This implies that, for almost all v , χ_v reduces to the constant function 1 on \mathfrak{o}_v , so that in the infinite product expression of $\chi(t)$ almost all the factors are 1. We will use χ_v to put X_v in duality with X_v^* . Hence, if X° is a basis of X_k over k and $(X^*)^\circ$ is a basis of X_k^* over k (for example, the basis dual to X°), the lattice $(X_v^\circ)_*$ in X_v^* which corresponds by duality to X_v° is $(X^*)^\circ_v$ for almost all v .

If χ is chosen as above, the bicharacter $(x, y) \mapsto \chi(xy)$ of $A_k \times A_k$ puts A_k in duality with itself, in such a way that the discrete subgroup k of A_k is self-dual. Hence, if X_k is as above, one can identify the linear dual X_A^* with the Pontryagin dual of X_A using the formula

$$\langle x, x^* \rangle = \chi([x, x^*]), \quad (x \in X_A, x^* \in X_A^*)$$

where $[x, x^*]$ denotes the extension of the bilinear form on $X_k \times X_k^*$ defined in Section 2.1 to a bilinear function $X_A \times X_A \rightarrow A_k$. Under this duality, the discrete subgroups X_k and X_k^* of X_A and X_A^* correspond one to the other. Otherwise said, we can take $G = X_A$, $G^* = X_A^*$, $\Gamma = X_k$, and $\Gamma_* = X_k^*$ in Section 1.9 and Section 1.10.

30.

Fix a nondegenerate quadratic form f on X_k . We may then extract the following data:

- A family of quadratic forms on the spaces X_v .
- A function $x \mapsto f(x)$ from X_A to A_k .
- Characters of second degree, $\chi_v \circ f$ and $\chi \circ f$, on X_v and on X_A .

For brevity, we will write $\gamma_v(f)$ and $\gamma(f)$ in lieu of $\gamma(\chi_v \circ f)$ and $\gamma(\chi \circ f)$.

 Proposition 5

Proposition 2.3.1. *Let f be a nondegenerate quadratic form on a vector space X_k over k . One then has*

$$\gamma(f) = \prod_v \gamma_v(f) = 1.$$

²⁹For the corresponding assertion where k is an algebraic number field other than \mathbb{Q} , see 39.

³⁰Here we identify k_v with its dual through χ_v , in the manner explained in Section 2.1.

Proof. Theorem 1.10.6 applied to $\chi \circ f$, X_A , and X_k yields $\gamma(f) = 1$. It remains to show locality:

$$\gamma(f) = \prod_v \gamma_v(f).$$

Toward that, select a fleet of functions $\Phi_v \in \mathcal{S}(X_v)$ for all v and with Φ_v the characteristic function of X_v° for almost all v , and set Φ be the function on X_A defined by

$$\Phi(x) = \prod_v \Phi_v(x_v).$$

We intend to apply Corollary 1.8.4 to Φ as well as to the function Φ_v . As noted previously, the choice of Haar measure will not affect the outcome, and so we may freely pick the normalization of the measure m_v on X_v . It will be most convenient to assume $m_v(X_v^\circ) = 1$ for almost all v , from which one extracts a product measure on each of the sets X_S° defined by Equation (29), and hence a measure on X_A . Following Equation (27), we find that for almost all v , $\Phi_v * (\chi_v \circ f) = \Phi_v$, from which we see $\gamma_v(f) = 1$ using Corollary 1.8.4. Applying this to X_A , we write the integrals over X_A as the limit of the corresponding integrals on the sets X_S° , which yields the result. \square

To obtain the law of quadratic reciprocities, we simply apply Proposition 2.3.1 to a nondegenerate form of 4 variables whose discriminant is a square in k . For the form obtained as the norm of an algebra \mathfrak{k} of quaternions over k , this gives the reciprocity law in the form due to Hasse:

$$\prod_v h_v(\mathfrak{k}) = 1,$$

where $h_v(\mathfrak{k})$ has the value 1 or -1 depending on whether $\mathfrak{k} \otimes k_v$ is a matrix algebra or a “true” algebra of quaternions over k_v . For the form $x^2 - ay^2 - bz^2 + abt^2$ over a field k of characteristic other than 2, this gives the reciprocity law in the form of Hilbert:

$$\prod_v \left(\frac{a}{b} \right)_v = 1.$$

Remark 2.3.2. Despite appearances, the proof of the law of quadratic reciprocity given above does not differ substantially from the classical proof by Gauss, which uses theta functions and their sums. In view of this, we note the essential role played by generalized theta functions in the proof of Theorem 1.10.6.

Remark 2.3.3. Let us write $P(k, m)$ for the property $\prod_v \gamma_v = 1$ for nondegenerate quadratic forms on spaces X_k of dimension m over a field k . Following Proposition 2.1.2, $P(k, m)$ follows formally from $P(k, 1)$ if k is not of characteristic 2, and from $P(k, 2)$ if k is of characteristic 2.

Taking k' to be a separable extension of k of degree d , τ to be the trace taken in k'/k , one then has $A_{k'} = k' \otimes A_k$. If χ is the character introduced above for A_k , then $\chi' = \chi \circ \tau$ is an analogous character for $A_{k'}$. If f' is a nondegenerate quadratic form on a vector space $X_{k'}$ of dimension m over k' , then $\tau \circ f'$ is a nondegenerate quadratic form on the vector space over k underlying $X_{k'}$. We conclude thus that $P(k, md)$ implies $P(k', m)$.

Taking k to have characteristic 0 for example, it follows from these remarks that $P(k, m)$ is a formal consequence of $P(\mathbb{Q}, 1)$ for all m and for all algebraic number fields k . We note also that the law of quadratic reciprocity, whether in the form of Hasse or of Hilbert, is

contained in $P(k, 4)$ so that, at least formally, this law is weaker than $P(k, 1)$ if k is not of characteristic 2 or weaker than $P(k, 2)$ if k is of characteristic 2.

Question 2.3.4. Can one give the general reciprocity law a proof analogous to that which we have given for the law of quadratic reciprocity?

III. Le groupe métaplectique (cas local et cas adélique)

3. THE METAPLECTIC GROUP (LOCAL CASE AND ADELIC CASE)

3.1. Basic definitions, part III. We return to the case that X is a finite-dimensional vector space over an arbitrary field k . Continuing to use the notation of Section 2.1, we will consider automorphisms $z \mapsto \sigma(z)$ of $X \times X^*$, which as in Section 1.2 we will denote in matrix form

$$\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

As before, we set

$$\sigma^I = \begin{pmatrix} \delta^* & -\beta^* \\ -\gamma^* & \alpha^* \end{pmatrix}.$$

On $(X \times X^*) \times (X \times X^*)$, we will consider the bilinear form

$$B(z_1, z_2) = [x_1, x_2^*] \quad (z_1 = (x_1, x_1^*), z_2 = (x_2, x_2^*))$$

and we say that an automorphism σ of $X \times X^*$ is *symplectic* if it preserves the bilinear form $B(z_1, z_2) - B(z_2, z_1)$, i.e., if $\sigma\sigma^I = 1$. These automorphisms form a group $Sp(X)$, named the *symplectic group* of X .

We construct a group $\mathfrak{A}(X)$ by imbuing $X \times X^* \times k$ with a composition law analogous to Equation (4):

$$(30) \quad (z_1, t_1) \cdot (z_2, t_2) = (z_1 + z_2, B(z_1, z_2) + t_1 + t_2).$$

For σ an automorphism of $X \times X^*$ and f a quadratic form on $X \times X^*$, consider the function

$$(31) \quad (z, t) \mapsto (\sigma(z), f(z) + t). \quad (z \in X \times X^*, t \in k)$$

This function is an automorphism of $\mathfrak{A}(X)$ if and only if σ and f satisfy the relation

$$(32) \quad f(z_1 + z_2) - f(z_1) - f(z_2) = B(\sigma(z_1), \sigma(z_2)) - B(z_1, z_2),$$

analogous to Equation (6). In this situation, we will denote by (σ, f) the automorphism of $\mathfrak{A}(X)$ defined by Equation (31). The group $Ps(X)$ formed by these automorphisms is named the *pseudosymplectic group* of X , and the group law is given in terms of coordinates by

$$(\sigma, f) \cdot (\sigma', f') = (\sigma' \circ \sigma, f''),$$

where f'' is the quadratic form defined by

$$f''(z) = f(z) + f'(\sigma(z)).$$

If Equation (32) is satisfied, the right-hand side will be symmetric in z_1 and z_2 , which holds if and only if σ is symmetric. Thus, the projection $(\sigma, f) \mapsto \sigma$ defines a homomorphism $Ps(X) \rightarrow Sp(X)$. If k is not of characteristic 2, Equation (32) produces for all $\sigma \in Sp(X)$ a unique quadratic form f on $X \times X^*$. In this case the projection is thus an isomorphism, and hence we may, as it suits us, identify these groups with each other. On

the contrary, when k is of characteristic 2, we see by taking $z_1 = z_2$ in Equation (32) that σ leaves invariant the nondegenerate quadratic form $B(z, z)$ on $X \times X^*$. Hence, $(\sigma, f) \mapsto \sigma$ is a homomorphism from $Ps(X)$ to the orthogonal group $O(B)$ for this quadratic form. One may verify that it is surjective and that its kernel is formed by the elements $(1, f)$ of $Ps(X)$ where f is additive.

Remark 3.1.1. We may depart from a physical vector space X_k over k and consider the extension X of X_k to an affine space over k , in which context these same definitions may be given. The resulting groups $Sp(X)$, $\mathfrak{A}(X)$, and $Ps(X)$ are then algebraic groups defined over k , and $Sp(X_k)$, $\mathfrak{A}(X_k)$, and $Ps(X_k)$ are the groups of rational points $Sp(X)_k$, $\mathfrak{A}(X)_k$, and $Ps(X)_k$ formed by those elements of $Sp(X)$, $\mathfrak{A}(X)$, and $Ps(X)$. We note that $Ps(X)$ is of dimension $m(2m + 1)$ if $m = \dim(X)$, independent of the characteristic of k . One can say (in a sense that schemes make precise) that in characteristic 2 the group $Ps(X)$ is a degeneration from characteristic 0 of the symplectic group. It is well-known that the symplectic group in any characteristic is connected, simply connected, and semisimple. This is therefore also true for $Ps(X)$ when k is not of characteristic 2, but it fails when k is of characteristic 2.

3.2. The standard pseudosymplectic representation and its automorphisms: local case. We have results for $Ps(X)$ entirely analogous to those of Section 1.3 for $B_0(G)$, which we rapidly lay out. We first define analogues of the four maps d , d' , t , and t' .

32.

- There is a monomorphism $d: \text{Aut}(X) \rightarrow Ps(X)$ given by

$$d(\alpha) = \left(\left(\begin{array}{cc} \alpha & 0 \\ 0 & \alpha^{*-1} \end{array} \right), 0 \right).$$

- There is a function $d': \text{Is}(X^*, X) \rightarrow Ps(X)$, where the set $\text{Is}(X^*, X)$ consists of isomorphisms from X^* to X , given by

$$d'(\gamma) = \left(\left(\begin{array}{cc} 0 & -\gamma^{*-1} \\ \gamma & 0 \end{array} \right), [x, -x^*] \right).$$

- There is a monomorphism $t: Q(X) \rightarrow Ps(X)$ from the additive group of quadratic forms on X given by

$$t(f) = \left(\left(\begin{array}{cc} 1 & \rho \\ 0 & 1 \end{array} \right), f \right),$$

where $\rho: X \rightarrow X^*$ is the symmetric morphism associated to f .

- Similarly, there is a function $t': Q(X) \rightarrow Ps(X)$ given by

$$t'(f') = \left(\left(\begin{array}{cc} 1 & 0 \\ \rho' & 0 \end{array} \right), f' \right),$$

where $\rho': X^* \rightarrow X$ is the symmetric morphism associated to f' .

Between elements, we have relations analogous to those of Section 1.3:

$$\begin{aligned} d(\alpha)^{-1} t(f) d(\alpha) &= t(f^\alpha), & d'(\alpha \circ \gamma) &= d'(\gamma) d(\alpha), \\ d(\alpha) t'(f') d(\alpha)^{-1} &= t'(f'^{\alpha*}), & d'(\gamma \circ \alpha^{*-1}) &= d(\alpha) d'(\gamma), \end{aligned}$$

for $\alpha \in \text{Aut}(X)$, $\gamma \in \text{Is}(X^*, X)$, and where f^α is defined by $f^\alpha(x) = f(\alpha^{-1}(x))$.

Consider the matrix expansion of an element $s = (\sigma, f)$ into $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. We will use $\Omega(X)$ to denote the set of those $s \in Ps(X)$ for which the component $\gamma: X^* \rightarrow X$ is an isomorphism. Appropriately modifying the proof of Proposition 1.5.1, one sees that all $s \in \Omega(X)$ can be placed uniquely in the form

$$(33) \quad s = t(f_1)d'(\gamma)t(f_2),$$

for $f_1, f_2 \in Q(X)$ and $\gamma \in \text{Is}(X^*, X)$ is that associated to the matrix expansion of s .

We may draw from this some basic results about the schematic structures of $Ps(X)$ and $\Omega(X)$. By definition, $\Omega(X)$ is the complement of the Zariski closed subset of $Ps(X)$ determined by $\det \gamma(s) = 0$, and hence is itself a Zariski open. Equation (33) furthermore determines a k -isomorphism of algebraic varieties

$$Q(X) \times \text{Is}(X, X^*) \times Q(X) \rightarrow \Omega(X).$$

Since $\text{Is}(X^*, X)$ is nonempty, $\Omega(X)$ is nonempty.

It follows that $\Omega(X)$ is the union of varieties of codimension 0 or 1 in $Ps(X)$. Whenever the characteristic of k is not 2, it is in fact the union of varieties of codimension 1, hence $Ps(X)$ is isomorphic to the connected group $Sp(X)$ and $\Omega(X)$ is nonempty. If on the other hand the characteristic of k is 2, one then knows that the orthogonal group $O(B)$ has two connected components, $O^+(B)$ and $O^-(B)$, formed respectively from the elements $\sigma \in O(B)$ for which $\text{tr}(\gamma^* \circ \beta)$ has the value 0 or 1. Denoting by $Ps^+(X)$ and $Ps^-(X)$ their respective inverse images in $Ps(X)$, one may verify that $\Omega(X)$ is contained in $Ps^+(X)$ or $Ps^-(X)$ depending on whether the dimension of m is even or odd.³¹

33.

We now specialize further to the case where k is a local field. As in Section 2.1, we will suppose that we have fixed a nontrivial character χ of k , which we use to identify the linear dual of a vector space X with its Pontryagin dual. The topology on k permits us to consider the groups $\mathfrak{A}(X)$ and $Ps(X)$ as locally compact topological groups, and hence we may tie the functions defined above directly to those developed in Section 1.3. The functions

$$\begin{aligned} \mathfrak{A}(X) &\rightarrow A(X), & \mu: Ps(X) &\rightarrow B_0(X) \\ (w, t) &\mapsto (w, \chi(t)) & (\sigma, f) &\mapsto (\sigma, \chi \circ f) \end{aligned}$$

are both homomorphisms. The kernel of μ is formed of those elements of the form $(1, f) \in Ps(X)$, where $f \in Q(X)$ satisfies $\chi \circ f = 1$. As the symmetric morphism from $X \rightarrow X^*$ associated to $\chi \circ f$ is the same as that associated to f , we find that f is additive.

When k is not of characteristic 2, there are no nontrivial additive forms, from which we conclude that μ is injective. In this case, this permits us to identify $Ps(X)$ with its image in $B_0(X)$, and then the functions d, d', t , and t' defined in this section become the restrictions of d_0, d'_0, t_0 , and t'_0 of Section 1.1. When k is of characteristic 2, this is no longer the case, and one can say only that the functions from this section are “compatible”, in the evident sense, with those of Section 1.1.

34.

In Theorem 1.6.3 we gave a description of $B_0(G)$ in terms of inner automorphisms, and to do so we defined a group $\mathbf{B}_0(G)$ of automorphisms of $L^2(G)$, as well as the canonical

³¹More precisely, take $s = (\sigma, f) \in Ps(X)$, let σ be as above, and let r be the rank of γ . We then see that $r \equiv \text{tr}(\beta\gamma^*) \pmod{2}$, which gives $\text{tr}(\beta\gamma^*) \equiv m \pmod{2}$ for $r = m$, which implies that γ is invertible.

projection $\pi_0: \mathbf{B}_0(G) \rightarrow B_0(G)$. Taking $G = X$ to be our finite dimensional vector space over our local field k , we define $Mp(X)$, the *metaplectic group* of X , to be the subgroup of $Ps(X) \times \mathbf{B}_0(G)$ formed from those elements $\mathbf{S} = (s, \mathbf{s})$ such that $\mu(s) = \pi_0(\mathbf{s})$. This group is equipped with a projection $\pi: Mp(X) \rightarrow Ps(X)$, $\pi(s, \mathbf{s}) = s$. Because π_0 is surjective, so is π , and its kernel is given by $\{e\} \times \mathbf{T}$, where \mathbf{T} is the group of operators $\Phi \mapsto t\Phi$ for $t \in T$. To simplify bookkeeping, we freely identify this kernel with \mathbf{T} or with T . It is contained in the center of $Mp(X)$. The other projection, $(s, \mathbf{s}) \mapsto \mathbf{s}$, determines a representation of $Mp(X)$ in the group of automorphisms of $L^2(X)$. For $\mathbf{S} = (s, \mathbf{s}) \in Mp(X)$ and $\Phi \in L^2(X)$, we will often write $\mathbf{S}\Phi$ to mean the action through this projection: $\mathbf{s}\Phi$.

Coupling the lifting functions of Section 1.8 to the functions defined above yields more general lifting functions from $\text{Aut}(X)$, $\text{Is}(X^*, X)$, and $Q(X)$ to $Mp(X)$:³²

$$\begin{aligned} \mathbf{d}(\alpha) &= (d(\alpha), \mathbf{d}_0(\alpha)) & (\alpha \in \text{Aut}(X)) \\ \mathbf{d}'(\gamma) &= (d'(\gamma), \mathbf{d}'_0(\gamma)) & (\gamma \in \text{Is}(X^*, X)) \\ \mathbf{t}(f) &= (t(f), \mathbf{t}_0(\chi \circ f)) & (f \in Q(X)). \end{aligned}$$

In the case where f' is an additive quadratic form³³ on X^* , we will define also a lift $\mathbf{t}'(f')$ of $t'(f')$ to $Mp(X)$. The additivity of f' entails that $\chi \circ f'$ is a character of X^* , and there is thus an element $a \in X$ such that

$$\chi(f'(x^*)) = \chi([a, x^*]).$$

We then define $\mathbf{t}'_0(f')$ to be the operator whose action on $\Phi \in L^2(X)$ is

$$\mathbf{t}'_0(f')\Phi(x) = \Phi(x - a).$$

One checks immediately that $\mathbf{t}'_0(f') = \pi_0(\mathbf{t}'(f'))$, and hence we set

$$\mathbf{t}'(f') = (t'(f'), \mathbf{t}'_0(f')).$$

The relations among \mathbf{d}_0 , \mathbf{t}_0 , and \mathbf{d}'_0 entail entirely analogous relations among \mathbf{d} , \mathbf{t} , and \mathbf{d}' , which we decline to write out.

As in Section 1.8, we define also a section $\mathbf{r}: Mp(X) \rightarrow \Omega(X)$ by expanding $s \in Mp(X)$ using Equation (33) and then setting

$$\mathbf{r}(s) = \mathbf{t}(r_1)\mathbf{d}'(\gamma)\mathbf{t}(f_2),$$

or, equivalently, by writing

$$(34) \quad \mathbf{r}(s) = (s, \mathbf{r}_0(\mu(s))),$$

for \mathbf{r}_0 as in Section 1.8 and μ as above. It should be noted that because k is a local field, $\Omega(X)$ is an open subset of $Ps(X)$, complementary to $\det \gamma = 0$, and is thus locally compact, and again Equation (33) determines an isomorphism of k -analytic varieties

$$Q(X) \times \text{Is}(X^*, X) \times Q(X) \rightarrow \Omega(X).$$

If k is not of characteristic 2, this is thus a closed k -analytic subset of codimension 1 of $Ps(X)$, and hence $\Omega(X)$ generates $Ps(X)$. If k is of characteristic 2, $\Omega(X)$ is contained

³²In the definition of \mathbf{d}_0 and of \mathbf{d}'_0 , we operated under particular conventions about Haar measures on X and X^* and about the definitions of $|\alpha|$ and $|\gamma|$. We are free to abandon these conventions, which would only modify \mathbf{d}_0 and \mathbf{d}'_0 by positive real factors, which would leave them unitary.

³³In this case and *only* in this case.

in $Ps^+(X)$ or $Ps^-(X)$ depending on the parity of m , and in either case it generates the relevant parent group.

Remark 3.2.1. We have already remarked that μ is injective when k is not of characteristic 2, which thus permits us to identify $Ps(X)$ with its image in $B_0(X)$. For the same reason, we may (at least from the view of points) identify $Mp(X)$ with its projection to $\mathbf{B}_0(X)$.

35.

We may also consider $Mp(X)$ as a topological group by using the subgroup relation $Mp(X) \leq Ps(X) \times \mathbf{B}_0(X)$ and inheriting from the topology on $Ps(X)$ introduced above and the topology on $\mathbf{B}_0(X)$ in turn inherited from the “strong” topology of the automorphism group of $L^2(X)$.³⁴ This topology makes the projection $\pi: Mp(X) \rightarrow Ps(X)$ into a continuous map and the representation $(s, \mathbf{s}) \mapsto \mathbf{s}$ of $Mp(X)$ in the automorphism group of $L^2(X)$ a continuous representation.³⁵

The functions \mathbf{d}_0 , \mathbf{d}'_0 , and \mathbf{t}_0 are continuous functions from $\text{Aut}(X)$, $\text{Is}(X^*, X)$, and $Q(X)$ respectively to the group of automorphisms of $L^2(X)$ with the strong topology. It then follows that \mathbf{d} , \mathbf{d}' , and \mathbf{t} are homeomorphisms from these same sets to their images in $Mp(X)$. We can also conclude that \mathbf{r} is a homeomorphism from $\Omega(X)$ to its image in $Mp(X)$, and hence that the function

$$(35) \quad (s, \tau) \mapsto \tau \mathbf{r}(s) = (s, \tau \mathbf{r}_0(\mu(s)))$$

determines a homeomorphism $\Omega(X) \times T \rightarrow \pi^{-1}(\Omega(X))$. As this last set is open in $Mp(X)$, we learn that $Mp(X)$ is locally compact and that $\pi: Mp(X) \rightarrow Ps(X)$ is an open map.

For $\mathbf{S} \in Mp(X)$, it follows from Section 1.7 that the automorphism $\Phi \mapsto \mathbf{S}\Phi$ of $L^2(X)$ induces an automorphism on $\mathcal{S}(X)$, which we now seek to show to be continuous. For this, it suffices to show that

$$\begin{aligned} \Omega(X) \times \mathcal{S}(X) &\rightarrow \mathcal{S}(X) \\ (s, \Phi) &\mapsto \mathbf{r}(s)\Phi \end{aligned}$$

is continuous. Note that Equation (33) entails a homeomorphism between $\Omega(X)$ and $Q(X) \times \text{Is}(X^*, X) \times Q(X)$, and that for any $\gamma_0 \in \text{Is}(X^*, X)$ the map $\alpha \mapsto \gamma_0 \alpha$ is a homeomorphism from $\text{Aut}(X) \rightarrow \text{Is}(X^*, X)$. It then remains to demonstrate the continuity of the function

$$\begin{aligned} Q(X) \times \text{Aut}(X) \times Q(X) \times \mathcal{S}(X) &\rightarrow \mathcal{S}(X) \\ (f_1, \alpha, f_2, \Phi) &\mapsto \mathbf{t}_0(\chi \circ f_1) \mathbf{d}'_0(\gamma_0) \mathbf{d}_0(\alpha) \mathbf{t}_0(\chi \circ f_2) \Phi \end{aligned}$$

which reduces to the continuity of the individual factors

$$\begin{aligned} Q(X) \times \mathcal{S}(X) &\rightarrow \mathcal{S}(X) & \text{Aut}(X) \times \mathcal{S}(X) &\rightarrow \mathcal{S}(X) \\ (f, \Phi) &\mapsto \mathbf{t}_0(\chi \circ f) \Phi, & (\alpha, \Phi) &\mapsto \mathbf{d}_0(\alpha) \Phi. \end{aligned}$$

These are immediate.

³⁴This is defined by the fundamental system of neighborhoods of 1 formed by the sets

$$\{s \mid \|s\Phi_i - \Phi_i\| \leq 1, 1 \leq i \leq n\},$$

where Φ_1, \dots, Φ_n are any finite collection of elements from $L^2(X)$.

³⁵This can be stated as $(\mathbf{S}, \Phi) \mapsto \mathbf{S}\Phi$ being a continuous function of $Mp(X) \times L^2(X)$ to $L^2(X)$.

Remark 3.2.2. When X decomposes as a direct sum $X_1 \oplus X_2$, the map $Mp(X_1) \times Mp(X_2) \rightarrow Mp(X)$ observed in Section 1.11 is continuous. This same map also induces isomorphisms between $Mp(X_1)$ and its image in $Mp(X)$, and similarly for $Mp(X_2)$. We will freely make use of these identifications.

Taking k to be a field of discrete valuation and choosing for Γ a lattice $L \leq X$, we can also produce an analogue of the lift r_Γ defined in Section 1.10. In the notation of Section 1, we substitute X , X^* , L , and L_* in place of G , G^* , Γ , and Γ_* respectively.³⁶ Just as we have substituted $Ps(X)$ in place of $B_0(X)$, we replace $B_0(X, L)$ (as laid out in Section 1.10) by the subgroup $Ps(X, L) \leq Ps(X)$ formed by those elements $s = (\sigma, f)$ such that $\chi \circ f|_{L \times L_*} = 1$ and $\sigma|_{L \times L_*}$ restricts to an automorphism. This is an open subgroup of $Ps(X)$, compact whenever k is not of characteristic 2, and the homomorphism $\mu: Ps(X) \rightarrow B_0(X)$ restricts to a map $\mu: Ps(X, L) \rightarrow B_0(X, L)$.

Considering $H(X, L)$ as a Hilbert space, Equation (25) defines a representation

$$r_L: B_0(X, L) \rightarrow \text{Aut } H(X, L),$$

hence a representation $r_L \circ \mu$ of $Ps(X, L)$ in this same automorphism group. By transport of structure along the isomorphism $Z^{-1}: H(X, L) \rightarrow L^2(X)$, we produce representations of $B_0(X, L)$ and of $Ps(X, L)$ in $\text{Aut } L^2(X)$, also written r_L and $r_L \circ \mu$. Equation (25) shows immediately that the representation $r_L \circ \mu$ of $Ps(X, L)$ is continuous when the target is endowed with the strong topology. Let r'_L be the representation

$$\begin{aligned} r'_L: Ps(X, L) &\rightarrow Mp(X) \\ s &\mapsto (s, r_L(\mu(s))). \end{aligned}$$

This map r'_L determines an isomorphism of $Ps(X, L)$ with its image in $Mp(X)$, such that $(s, \tau) \mapsto \tau r'_L(s)$ is an isomorphism of $Ps(X, L) \times T$ onto an open subgroup of $Mp(X)$. Moreover, it follows from Section 1.11 that for any $\Phi \in \mathcal{S}(X)$ the function

$$\begin{aligned} Ps(X, L) &\rightarrow \mathcal{S}(X) \\ s &\mapsto r'_L(s)\Phi \end{aligned}$$

is locally constant.

3.3. The standard pseudosymplectic representation and its automorphisms: adelic case. We now extend our analysis of the pseudosymplectic and metaplectic groups into the adelic case. We first recall the hypotheses and notation of Section 2.3, extending them as needed to other algebraic groups defined over the base field k .

- We will write $Ps(X)_v$ and $Ps(X)_v$ for the groups formed by the elements of the algebraic group $Ps(X)$ which are rational over k and over k_v respectively, and we will also write $Ps(X)_A$ for the adelic group attached to $Ps(X)$.
- We will write X° for a chosen basis of X and $(X^*)^\circ$ for a chosen basis of X^* . There is no harm in supposing that the latter is the dual basis of the former.

³⁶As a reminder, L_* is the lattice in X^* which corresponds to L by duality, i.e., the set of those $x^* \in X^*$ such that $\chi([x, x^*]) = 1$ for any $x \in L$.

- For k_v a completion of k with discrete valuation, we will write $Ps(X)_v^\circ \leq Ps(X)_v$ for the subgroup of those elements (σ, f) such that σ induces an automorphism of the lattice $X_v^\circ \times (X^*)_v^\circ$ and f induces on that same lattice an integer-valued function (i.e., valued in \mathfrak{o}_v).
- We will write $Ps(X)_A$ for the union (i.e., the inductive limit) of the groups

$$Ps(X)_S^\circ = \prod_{v \in S} Ps(X)_v \times \prod_{x \notin S} Ps(X)_v^\circ,$$

where S is the collection of all finite sets of completions of k which are required to contain the set S_∞ of those completions which are isomorphic to \mathbb{R} or to \mathbb{C} .

Just as in the local case of Section 3.2, there are homomorphisms

$$\begin{aligned} \mathfrak{A}(X)_A &\rightarrow A(X_A) & \mu_A: Ps(X)_A &\rightarrow B_0(X_A) \\ (\omega, t) &\mapsto (\omega, \chi(t)), & (\sigma, f) &\mapsto (\sigma, \chi \circ f). \end{aligned}$$

The map μ_A is injective when k is not of characteristic 2. Using this, we then define the metaplectic group to be the subgroup specified by

$$Mp(X)_A = \{(s, \mathbf{s}) \in Ps(X)_A \times \mathbf{B}_0(X_A) \mid \mu_A(s) = \pi_0(\mathbf{s})\}.$$

Giving $\mathbf{B}_0(X_A)$ the strong topology and on $Ps(X)_A$ the usual adelic topology, we endow the metaplectic group with the subspace topology. We will again denote the projection by $\pi: Mp(X)_A \rightarrow Ps(X)_A$, which is again surjective, and its kernel is again the group $\{e\} \times \mathbf{T}$, which will denote more simply as \mathbf{T} .

38.

We will now describe a continuous lift of an open set of $Ps(X)_A$ to $Mp(X)_A$. As before, this will also permit us to conclude that $Mp(X)_A$ is locally compact, that it is locally homeomorphic to $Ps(X)_A \times T$, and that $\pi: Mp(X)_A \rightarrow Ps(X)_A$ is an open map.

To do this, for each v we set $\Omega_v = \Omega(X)_v$, which is a nonempty open set in $Ps(X)_v$. For each finite set S of completions of k which contain S_∞ , we consider the relevant factor of the adelic product:

$$\Omega_S = \prod_{v \in S} \Omega_v \times \prod_{v \notin S} Ps(X)_v^\circ.$$

This yields an open subset of $Ps(X)_S^\circ$, hence also an open set of $Ps(X)_A$.

We use Section 3.2 to avail ourselves of the following lifts:

- On each Ω_v , there is a lift $\mathbf{r}_v: \Omega_v \rightarrow Mp(X)_v$.
- For each v such that k_v is of discrete valuation and for each lattice L in X_v , there is a lift $\mathbf{r}'_L: Ps(X_v, L) \rightarrow Mp(X_v)$.

We would like to assemble these local lifts into an adelic object. As was shown in Section 1.11, the image of $Ps(X_v, L)$ through \mathbf{r}'_L yields a subgroup of $Mp(X_v)$ which leaves invariant the characteristic function of the lattice L . For almost all v , setting $L = X_v^\circ$ shows $Ps(X)_v^\circ$ to be a subgroup of $Ps(X_v, L)$. We denote by S_0 the set of those completions of k for which this does not hold, and note that it is finite and contains S_∞ . For $v \notin S_0$, we denote by \mathbf{r}'_v the induced lift \mathbf{r}'_L on $Ps(X)_v^\circ$ for $L = X_v^\circ$.

On the other hand, (Φ_v) be a sequence of functions belonging to $L^2(X_v)$, and suppose that for almost all v , Φ_v is the characteristic function of X_v° , so that $\Phi = (\Phi_v)$ determines

a function on X_A by the formula

$$(36) \quad \Phi(x) = \prod_v \Phi_v(x_v).$$

Select a measure on X_A according to the theory laid out in Section 2.3, so that Φ then belongs to $L^2(X_A)$. The linear span of functions of this form forms an everywhere dense set in $L^2(X_A)$; we then set for all $S \supset S_0$ and all $s = (s_v) \in \Omega_S$,

$$\mathbf{r}_S(s)\Phi(x) = \prod_{v \in S} \mathbf{r}_v(s_v)\Phi_v(x_v) \times \prod_{v \notin S} \mathbf{r}'_v(s_v)\Phi_v(x_v).$$

As almost all the factors of the second product are equal to characteristic functions of the lattices X_v° , $\mathbf{r}_S(s)\Phi$ also lies in this everywhere dense set. It then follows from Section 1.11 that the function $\mathbf{r}_S(s)$ defined above for the functions of the form Equation (36) extends to an automorphism of $L^2(X_A)$, and hence we have defined a continuous lift \mathbf{r}_S of Ω_S to $Mp(X)_A$.

We are now going to show the action

$$\begin{aligned} Mp(X_A) \times \mathcal{S}(X_A) &\rightarrow \mathcal{S}(X_A) \\ (\mathbf{S}, \Phi) &\mapsto \mathbf{S}\Phi \end{aligned}$$

is continuous. As in Section 3.2, it suffices to show that the following assignment is continuous:

$$\begin{aligned} \Omega_S \times \mathcal{S}(X_A) &\rightarrow \mathcal{S}(X_A) \\ (s, \Phi) &\mapsto \mathbf{r}_S(s)\Phi \end{aligned}$$

Collecting definitions from Section 1.7 and results from Section 2.3, we see that $\mathcal{S}(X_A)$ is composed of finite linear combinations of functions of the form

$$(37) \quad \Phi_\infty(x_\infty) \prod_{v \notin S_\infty} \Phi_v(x_v),$$

where $\Phi_\infty \in \mathcal{S}(X_\infty)$ belongs to the Schwartz space of $X_\infty = \prod_{v \in S_\infty} X_v$,³⁷ where Φ_v belongs to $\mathcal{S}(X_v)$ for all finite places v , and where for almost all v the component Φ_v is equal to the characteristic function of X_v° .

Take Φ to be a function of the form of Equation (37), and take $s \in \Omega_S$. Because it preserves the characteristic function of the lattice, \mathbf{r}'_v leaves Φ_v invariant for almost all v . For all $v \in S \setminus S_\infty$, the function

$$\begin{aligned} Ps(X)_v^\circ &\rightarrow \mathcal{S}(X_v) \\ s_v &\mapsto \mathbf{r}_v(s_v)\Phi_v \end{aligned}$$

is continuous—indeed, even locally constant. We may combine the local results of Section 3.2 as applied to the product X_∞ via the results of Section 1.11 so show that for $v \in S_\infty$ the elements s_v (resp., the elements $\mathbf{r}_v(s_v)$) determine a “tensor product” element $s_\infty \in Ps(X_\infty)$ (resp., an element $\mathbf{r}_\infty(s_\infty) \in Mp(X_\infty)$) such that

$$s_\infty \mapsto \mathbf{r}_\infty(s_\infty)\Phi_\infty$$

³⁷At these real places $v \in S_\infty$, X_v is as usual a finite dimensional real vector space.

is a continuous assignment. It follows that

$$\begin{aligned} \mathcal{S}(X_\infty) &\rightarrow \prod_v \Omega_v \\ s_\infty &\mapsto \mathbf{r}_\infty(s_\infty)\Phi_\infty \end{aligned}$$

is a continuous function extended to $v \in S_\infty$. Taking all these results together, we conclude that

$$\begin{aligned} \Omega_S &\rightarrow \mathcal{S}(X_A) \\ s &\mapsto \mathbf{r}_S(s)\Phi, \end{aligned}$$

is a continuous assignment, hence that for any fixed $\Phi \in \mathcal{S}(X_A)$, the following assignment is also continuous:

$$\begin{aligned} Mp(X)_A &\rightarrow \mathcal{S}(X_A) \\ \mathbf{S} &\mapsto \mathbf{S}\Phi. \end{aligned}$$

We complete the proof of continuity of $(s, \Phi) \mapsto \mathbf{r}_S(s)\Phi$ as follows. Take K to be a compact subset of Ω_S , and take U to be a convex neighborhood of 0 in $\mathcal{S}(X_A)$. The set U' given by

$$U' = \{\Phi \in \mathcal{S}(X_A) \mid \text{for all } s \in K, \mathbf{r}_S(s)\Phi \in U\}$$

is then a neighborhood of 0 in $\mathcal{S}(X_A)$. As U' is convex, we may use the definition of the topology on $\mathcal{S}(X_A)$ as the inductive limit of those on $\mathcal{S}(H, H')$ to see that this is equivalent to showing that, for each choice of H and H' , $U' \cap \mathcal{S}(H, H')$ is a neighborhood of 0 in $\mathcal{S}(H, H')$. However, for a given K, H , and H' , there is a finite set S' of completions of k which have the following properties:

- (1) For all $s = (s_v) \in K$ and all $v \notin S'$, we have $s_v \in Ps(X)_v^\circ$.
- (2) Every function $\Phi \in \mathcal{S}(H, H')$ when expressed a linear combination of functions of the form of Equation (37) has, for all $v \notin S'$, Φ_v the characteristic function of X_v° .

From here, we may draw directly from Section 3.2 to conclude the proof.

3.4. Theta series and the metaplectic group. We now apply the results of Section 1.9 and Section 1.10 to our case of interest: $G = X_A$, $G^* = X_A^*$, $\Gamma = X_k$, and $\Gamma_* = X_k^*$. It follows from those results that the homomorphism $\mu_A: Ps(X)_A \rightarrow B_0(X_A)$ carries $Ps(X)_k$ to the subgroup of $B_0(X_A, X_k) \leq B_0(X_A)$, in the language of Section 1.10. Using the lifting \mathbf{r}_Γ from that same Section, we can thus define a lift of $Ps(X)_k \rightarrow \mathbf{B}_0(G)$ and hence a lift of $\mathbf{r}_k: Ps(X)_k \rightarrow Mp(X)_A$. We will now make this lift explicit.

It will suffice to make $\mathbf{r}_k(s)\Phi$ explicit for $\Phi \in \mathcal{S}(X_A)$ and $s \in Ps(X)_k$. We will need the following Θ -function:

$$\Theta(x, x^*) = \sum_{\xi \in X_k} \Phi(x + \xi) \chi([\xi, x^*]) \quad (x \in X_A, x^* \in X_A^*).$$

As noted in Section 1.9, because we have taken $\Phi \in \mathcal{S}(X_A)$, the series on the right-hand side is uniformly convergent on all compact subsets. As a particular case of Equation (23), we therefore obtain

$$\Phi(x) = \int_{X_A^*/X_k^*} \Theta(x, x^*) d\dot{x}^*,$$

where $d\dot{x}^*$ is the measure on the compact group X_A^*/X_k^* which takes the value 1 on the entire group.

Let $s = (\sigma, f)$ be an element of $Ps(X)_k$ with matrix expansion $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. The function $\Phi' = r_k(s)\Phi$ is then defined for $z = (x, x^*) \in X_A \times X_A'$ by the formulas

$$\begin{aligned} \Theta'(z) &= \Theta(\sigma(z))\chi(f(z)) \\ \Phi'(x) &= \int \Theta'(x, x^*)d\dot{x}^*, \end{aligned}$$

or alternatively,

$$\Phi'(x) = \int_{X_A^*/X_k^*} \Theta(\alpha(x) + \gamma(x^*), \beta(x) + \delta(x^*))\chi(f(x, x^*))d\dot{x}^*.$$

Let $N \leq X^*$ be the kernel of γ , and set $Y = X^*/N$. We may identify N_A/N_k with a closed subgroup of X_A^*/X_k^* and hence Y_A/Y_k with the quotient of X_A^*/X_k^* by N_A/N_k . Writing \bar{x}^* for the image of \dot{x}^* in this last quotient, we can thus write

$$\Theta'(x) = \int \Psi(x, \bar{x}^*)d\bar{x}^*,$$

where Ψ is given by the formula

$$\Psi(x, \bar{x}^*) = \int_{N_A/N_k} \sum_{\xi \in X_k} \Phi(\alpha(x) + \gamma(x^*) + \xi)\chi([\xi, \beta(x) + \delta(x^* + n)] + f(x, x^* + n))d\dot{n}.$$

Here, \dot{n} denotes the image of $n \in N_A$ in N_A/N_k and $d\dot{n}$ denotes the measure on N_A/N_k for which the total volume is 1. Similarly, $d\bar{x}^*$ is the measure on Y_A/Y_k for which Y_A/Y_k has total volume 1.

Using Equation (32), we then have

$$f(x, x_1^* + x_2^*) = f(0, x_1^*) + f(x, x_2^*) + [\gamma(x_1^*), \beta(x) + \delta(x_2^*)],$$

and for $n \in N$ we thus conclude

$$f(x, x^* + n) = f(0, n) + f(x, x^*).$$

In particular, it follows that $f(0, n)$ is an additive form on N . As f is rational on k , we can define a character φ on X_A^*/X_k^* simply by setting $\varphi(\dot{n}) = \chi(f(0, n))$ on the generating set $n \in N_A$.³⁸

With this established, we may rewrite the expression for Ψ given above as

$$\Psi(x, \bar{x}^*) = \int \sum_{\xi} \Phi(\alpha(x) + \gamma(x^*) + \xi)\chi([\xi, \beta(x) + \delta(x^*)] + f(x, x^*))\chi([\delta^*(\xi) + \xi_0, n])d\dot{n},$$

³⁸Equivalently, φ is a character of X_A^* which takes the value 1 on X_k^* and which coincides with $\chi(f(0, n))$ on N_A . This may be relaxed even further to the claim that there exists $\xi_0 \in X_k$ such that one has

$$\chi(f(0, n)) = \chi([\xi_0, n])$$

for all $n \in N_A$. Of course, when k is not of characteristic 2, the additive form $f(0, n)$ on N reduces to 0, so that one can take $\xi_0 = 0$.

or more simply

$$\Psi(x, \bar{x}^*) = \sum_{\xi \in L} \Phi(\alpha(x) + \gamma(x^*) + \xi) \chi([\xi, \beta(x) + \delta(x^*)] + f(x, x^*)),$$

where

$$L = \{\xi \in X_k \mid \delta^*(\xi) + \xi_0 \in N_*\}$$

Note $N_* = \gamma^*(X^*)$, so that L may also be expressed as the set of those $\xi \in X_k$ such that we may find a solution $\xi^* \in X_k^*$ to

$$\delta^*(\xi) + \xi_0 = \gamma^*(\xi^*).$$

As we have

$$\sigma^{-1} = \begin{pmatrix} \delta^* & -\beta^* \\ -\gamma^* & \alpha^* \end{pmatrix},$$

this is equivalent to saying that $\sigma^{-1}(\xi, \xi^*)$ is of the form $(-\xi_0, \xi_1^*)$ for $\xi_1^* \in X_k$, which shows L to be the image of X_k^* under the function

$$\begin{aligned} X_k^* &\rightarrow X_k \\ \xi_1^* &\mapsto -\alpha(\xi_0) + \gamma(\xi_1^*). \end{aligned}$$

Using Equation (32), we can thus write

$$\Psi(x, \bar{x}^*) = \sum_{\xi_1^* \in X_k^*/N_k} \Phi(\alpha(x - \xi_0) + \gamma(x^* + \xi_1^*)) \chi(f(x - \xi_0, x^* + \xi_1^*) - [\xi_0, x^*]).$$

We observe that the function

$$\chi(f(x, x^*) - [\xi_0, x^*])$$

does not change if we replace x^* by $x^* + n$ for $n \in N_A$. Taking $x \in X_A$ and y the image of x^* in $Y_A = X_A^*/N_A$, we can thus define a function Ω on $X_A \times Y_A$ by the formula

$$\Omega(x, y) = \Phi(\alpha(x) + \gamma(x^*)) \chi(f(x, x^*) - [\xi_0, x^*]).$$

Using the same notation, this gives

$$\Psi(x, \bar{x}^*) = \sum_{\eta \in Y_k} \Omega(x - \xi_0, y + \eta)$$

and hence

$$\Phi'(x) = \int_{Y_A} \Omega(x - \xi_0, y) dy,$$

where dy is the Tamagawa measure on Y_A (for which Y_A/Y_k has measure 1). As γ^* determines an isomorphism from $Y \rightarrow Z = \gamma(X^*)$, we ultimately obtain

$$(38) \quad \mathbf{r}_k(s) \Phi(x) = \int_{Z_A} \Phi(\alpha(x - \xi_0) + z) \psi(x - \xi_0, z) dz,$$

where dz is the Tamagawa measure on Z_A and ψ is the character of second degree of $X_A \times Z_A$ defined by

$$\psi(x, \gamma(x^*)) = \chi(f(x, x^*) - [\xi_0, x^*]) \quad (x \in X_A, x^* \in X_A^*).$$

Theorem 1.10.2 will now give us the main result of this memoir. For $s \in Ps(X)_k$ and $\Phi \in \mathcal{S}(X_A)$, this theorem gives³⁹

$$(39) \quad \sum_{\xi \in X_k} \Phi(\xi) = \sum_{\xi \in X_k} \mathbf{r}_k(s)\Phi(\xi),$$

From there—or, equivalently, using Corollary 1.10.3—we would like to deduce the following:

Theorem 3.4.1. *Let X_k be a vector space of finite dimension over k , and let $\Phi \in \mathcal{S}(X_A)$. For $\mathbf{S} \in Mp(X)_A$, let Θ be the function on $Mp(X)_A$ defined by*

Théorème 6

$$\Theta(\mathbf{S}) = \sum_{\xi \in X_k} \mathbf{S}\Phi(\xi).$$

The function Θ is continuous and invariant under left-translations by elements of the form $\mathbf{r}_k(s)$, $s \in Ps(X)_k$.

The invariance of Θ is evident from Equation (39), or from Corollary 1.10.3. It does not follow from those results that Θ is continuous, which we will instead deduce from the following lemmas.

Lemma 3.4.2. *Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of positive real numbers. Then there exists a function $\varphi \in \mathcal{S}(\mathbb{R})$ such that for every x ,*⁴⁰

Lemme 4

$$\varphi(x) \geq \inf_{n \in \mathbb{N}} (a_n |x|^{-n}).$$

Proof. Set $f(x) = \inf_{n \in \mathbb{N}} (a_n |x|^{-n})$, take g to be a nonnegative indefinitely differentiable function on \mathbb{R} with support contained in $[-1, +1]$ and with unit volume, and set $h = f * g$. We then have

$$\begin{aligned} f(x-1) &\geq h(x) \geq f(x+1), & \text{for } x \geq +1, \\ f(x-1) &\leq h(x) \leq f(x+1), & \text{for } x \leq -1. \end{aligned}$$

Moreover, for $D = d/dx$ and $p > 0$, we have $D^p h = f * D^p g$, from which we immediately conclude that $|x^n D^p h|$ is bounded for all $n \geq 0$ and all $p \geq 0$, hence h belongs to $\mathcal{S}(\mathbb{R})$.

Now take $h_0 \in \mathcal{S}(\mathbb{R})$ to be nonnegative and $h_0(x) \geq a_0$ for $-2 \leq x \leq +2$. For any x , we then conclude:

$$f(x) \leq h(x-1) + h(x+1) + h_0(x). \quad \square$$

Lemma 3.4.3. *Let G be a locally compact abelian group and let C be a compact subset of $\mathcal{S}(G)$. Then there exists $\Phi_0 \in \mathcal{S}(G)$ such that $|\Phi(x)| \leq \Phi_0(x)$ for all $\Phi \in C$ and $x \in G$.*

Lemme 5

Proof. Every compact set of $\mathcal{S}(G)$ is contained in some $\mathcal{S}(H, H')$, re-using the notation from Section 1.7. It therefore suffices to take G to be an elementary group:

$$G = \mathbb{R}^n \times \mathbb{Z}^p \times T^q \times F,$$

³⁹When $\gamma: X_k^* \rightarrow X_k$ an isomorphism, setting $s = d'(\gamma)$ reduces this to Poisson's formula.

⁴⁰This proof was communicated to me by J. Dieudonné.

for F finite. Let $x \in G$, and let $x_1, \dots, x_n, y_1, \dots, y_p$ be its coordinates in $\mathbb{R}^n \times \mathbb{Z}^p \leq G$. We set

$$r(x) = \sum_{i=1}^n x_i^2 + \sum_{j=1}^p y_j^2,$$

$$a_n = \sup_{\substack{x \in G \\ \Phi \in C}} |r(x)^n \Phi(x)|.$$

Using the definition of $\mathcal{S}(G)$, we have $a_n < +\infty$ for all n . For this sequence (a_n) , use Lemma 3.4.2 to choose $\varphi \in \mathcal{S}(\mathbb{R})$ with $\varphi(x) \geq \inf_{n \in \mathbb{N}} (a_n |x|^{-n})$. The function $\Phi_0 = \varphi \circ r$ then has the desired property. \square

Proof of Theorem 3.4.1. Fixing $x_0 \in X$ on the left and $\Phi \in \mathcal{S}(X_A)$ on the right, the functions

$$\begin{array}{ccc} \mathcal{S}(X_A) & \rightarrow & \mathbb{C} \\ \Phi & \mapsto & \Phi(x_0) \end{array} \qquad \begin{array}{ccc} Mp(X)_A & \rightarrow & \mathcal{S}(X_A) \\ S & \mapsto & S\Phi \end{array}$$

are each continuous. From this, it follows that the terms in the series which defines Θ are each continuous functions on $Mp(X)_A$.

Take C to be a compact set of $Mp(X)_A$. For $\Phi \in \mathcal{S}(X_A)$, the image $C\Phi$ is a compact subset of $\mathcal{S}(X_A)$, and Lemma 3.4.3 shows that there exists $\Phi_0 \in \mathcal{S}(X_A)$ with $|S\Phi| \leq \Phi_0$ for any $S \in C$. The series which defines $F(S)$ is thus dominated term-by-term by the series $\sum \Phi_0(\xi)$. It follows that this sum is convergent. \square

Remark 3.4.4. The definition of modular functions using theta-series is naturally a particular case of the method of definition of automorphic functions which is contained in Theorem 3.4.1.

IV. Réduction du groupe métaplectique

4. THE METAPLECTIC GROUP AS A DOUBLE COVER

42.

Just as in the local case, the adelic metaplectic group is a central extension of the pseudosymplectic group by the group T . In this section we show that (in general) this extension is not trivial, but it does always reduce to a nontrivial extension by the group with two elements $\{\pm 1\}$. Accordingly, this core of the metaplectic group can be viewed as a double cover of the pseudosymplectic group.

4.1. A group-theoretic lemma. To get off the ground, we will need a lemma from pure group theory, unrelated to our study of metaplectic groups.

Lemme 6

Lemma 4.1.1. *Let G be a group, and let U be a subset of G such that for all $a, b, c \in G$, we have*

$$U^{-1} \cap Ua \cap Ub \cap Uc \neq \emptyset.$$

Then G is generated by U . Moreover, let R be the set of those elements $(u, u', u'') \in U^{\times 3}$ such that $u'' = uu'$. The group G can then be identified with the group with generators U and relations $u'' = uu'$ for $(u, u', u'') \in R$.

Proof. Let $x \in G$ and let $v \in U^{-1} \cap Ux^{-1}$. One then has $x = v^{-1}(vx)$, $v^{-1} \in U$, and $vx \in U$, which justifies the first assertion.

Let \overline{G} be the group explicitly presented as

$$\overline{G} = \langle \bar{u} \mid u \in U \rangle / (\bar{u}'' = \bar{u}\bar{u}' \mid (u, u', u'') \in R).$$

The assignment $\bar{u} \mapsto u$ determines a homomorphism $h: \overline{G} \rightarrow G$, and we would like to show that its kernel N is trivial.

We will first show that if an element of N is of the form $\bar{u}_0^{-1}\bar{u}_1\bar{u}_2 \cdots \bar{u}_n$, then it is \bar{e} . For $n = 1$ this is trivial, and for $n = 2$ this follows from the definition of \overline{G} , therefore we may proceed by induction. Fixing such an element of N , let v be an element of the set

$$U^{-1} \cap Uu_0^{-1} \cap Uu_1^{-1} \cap Uu_2^{-1}u_1^{-1}.$$

The elements $u = v^{-1}$, $u' = vu_0$, $u'' = vu_1$, and $u''' = vu_1u_2$ thus belong to U , hence we have the relations

$$u_0 = uu', \quad u_1 = uu'', \quad u''u_2 = u''', \quad u' = u'''u_3 \cdots u_n,$$

which, using the definition of \overline{G} and the inductive hypothesis, entails

$$\bar{u}_0 = \bar{u}\bar{u}', \quad \bar{u}_1 = \bar{u}\bar{u}'', \quad \bar{u}''\bar{u}_2 = \bar{u}''', \quad \bar{u}' = \bar{u}'''\bar{u}_3 \cdots \bar{u}_n,$$

hence $\bar{u}_0 = \bar{u}_1 \cdots \bar{u}_n$.

We will now show that elements of N of the form $\bar{u}_1 \cdots \bar{u}_n$ (where $n \geq 1$) must equal \bar{e} . Our goal is to reduce to the previous case, by using the form $\bar{u}^{-1}\bar{u}\bar{u}_1 \cdots \bar{u}_n$. Set $V = U \cap U^{-1}$, and let S be the set of those elements $(v, v', v'') \in V \times V \times V$ such that $v'' = vv'$. We extend the function $U \rightarrow \overline{G}$ over V by setting $\bar{v} = \bar{u}^{-1}$ for $v = u^{-1}$. Using the first case, we see $\bar{v}'' = \bar{v}\bar{v}'$ for $(v, v', v'') \in S$ —but then the reasoning made above for U and R applies to V and S , from which we conclude that all elements of N of the form $\bar{v}_1 \cdots \bar{v}_n$ are equal to \bar{e} .

As all elements of \overline{G} can be put in this form, this finishes the proof. \square

Setting G to be an analytic group over a local field k and U the complement of a union of analytic subvarieties of G of codimension ≥ 1 , the hypotheses of Lemma 4.1.1 are satisfied. In particular, we may follow Section 3.2 and either take k to be a local field of characteristic other than 2, X a finite dimensional vector space over k , $G = Ps(X)$, $U = \Omega(X)$; or k a local field of characteristic 2, X a vector space of even dimension over k , $G = Ps^+(X)$, and $U = \Omega(X)$.

4.2. Reduction to the two-sheeted cover. Let k be a local field of characteristic not equal to 2, and let X be a finite dimensional vector over k . In this case, we may identify $Ps(X)$ with the symplectic group $Sp(X)$. Recall that $Mp(X)$ is said to be a trivial extension of $Sp(X)$ by \mathbf{T} if there is a decomposition

$$Mp(X) = Sp_1(X) \times \mathbf{T},$$

where $Sp_1(X)$ is the subgroup of $Mp(X)$ on which $\pi|_{Sp_1(X)}: Sp_1(X) \rightarrow Sp(X)$ is an isomorphism. Just as in classical group theory, producing such a decomposition is equivalent to defining a “section”, i.e., a homomorphism $Mp(X) \rightarrow \mathbf{T}$ which reduces to the identity on \mathbf{T} . Similarly, producing such a decomposition as topological groups is equivalent to producing a continuous such section. Denoting the isomorphism $(e, t) \mapsto t$ as $\theta: \mathbf{T} \rightarrow T$, the

production of a (continuous) section is equivalent to producing a (continuous) character of $Mp(X)$ which coincides with θ on \mathbf{T} .

More generally, a character⁴¹ φ of $Mp(X)$ which coincides with θ^n on \mathbf{T} for some $n > 0$ yields a decomposition

$$Mp(X) = N \cdot \mathbf{T},$$

where N is the kernel of φ . The intersection $N \cap \mathbf{T}$ is the subgroup \mathbf{T}_n of order n in \mathbf{T} , and π therefore induces a homomorphism from $N \rightarrow Sp(X)$ with kernel \mathbf{T}_n . Moreover, if φ is continuous then N is closed in $Mp(X)$ and π induces on N an open homomorphism from N to $Sp(X)$ of kernel \mathbf{T}_n . This shows π to be a local isomorphism, so that N becomes an n -sheeted cover of $Sp(X)$.

We will now apply our theory of the metaplectic group to construct such a character φ . Following Lemma 4.1.1, if $Sp(X)$ is generated by $\Omega(X)$, then the character φ is entirely determined by the values of $\psi = \varphi \circ \mathbf{r}$ on $\Omega(X)$. Moreover, we deduce from Theorem 1.8.6 that if $s'' = ss'$ for $s, s', s'' \in \Omega(X)$, we have

$$\mathbf{r}(s)\mathbf{r}(s') = \gamma(f_0)\mathbf{r}(s''),$$

where f_0 is the quadratic form on X associated to the morphism

$$\gamma(s)^{-1}\gamma(s'')\gamma(s')^{-1}: X \rightarrow X^*.$$

For ψ is as above, it follows that

$$(40) \quad \psi(s'') = \gamma(f_0)^{-n} \psi(s)\psi(s').$$

Conversely, one concludes from Lemma 4.1.1 that if $\psi: \Omega(X) \rightarrow T$ is a function satisfying Equation (40), there is a unique character φ of $Mp(X)$ which satisfies $\varphi \circ \mathbf{r} = \psi$ and which coincides with θ^n on \mathbf{T} . Moreover, φ is continuous when ψ is continuous.

Now we consider specific cases of k , first taking $k = \mathbb{C}$. Following Section 2.2, we have $\gamma(f) = 1$ for all nondegenerate quadratic forms f on X . We thus satisfy Equation (40) by taking $n = 1$ and $\psi = 1$, and hence there is a unique character of $Mp(X)$ which takes the value 1 on $\mathbf{r}(\Omega(X))$ and which coincides with θ on \mathbf{T} . We denote this character by φ_1 and its kernel by $Sp_1(X)$. Our discussion above gives $Mp(X) = Sp_1(X) \times \mathbf{T}$, where $Sp_1(X)$ is a closed subgroup of $Mp(X)$, isomorphic to $Sp(X)$, and generated by $\mathbf{r}(\Omega(X))$.

For $k \neq \mathbb{C}$, we will suppose first that -1 is a square in k . Applying the results of Section 2.1, for all nondegenerate quadratic forms f on X we have $\gamma(f) = \pm 1$. We thus satisfy Equation (40) by taking $n = 2$ and $\psi = 1$. We denote by φ_2 the corresponding character of $Mp(X)$ and by $Sp_2(X)$ its kernel, which is a closed subgroup of $Mp(X)$ and is a two-sheeted cover of $Sp(X)$.

Now suppose that -1 is not a square in k . Choosing a basis of X over k , for all $s \in \Omega(X)$ we denote by $D(s)$ the determinant of $\gamma(s)$ in terms of this basis and its dual. By definition of $\Omega(X)$, we have $D(s) \neq 0$ for all $s \in \Omega(X)$. Equation (28) thus shows that we satisfy Equation (40) by taking $n = 2$ and

$$\psi(s) = \left(\frac{D(s)}{-1} \right) \gamma(q_1)^{2m}$$

⁴¹Perhaps discontinuous.

for $s \in \Omega(X)$, where m denotes the dimension of X and q_1 denotes the form $q_1(x) = x^2$ on k .⁴² As above, we denote by φ_2 the corresponding character on $Mp(X)$ and by $Sp_2(X)$ its kernel, and it has the same properties as above.

Let us now suppose that k is a local field of characteristic 2. In this case, we will denote by $Mp^+(X)$ the inverse image of $Ps^+(X)$ in $Mp(X)$, which is an open subgroup of index 2. How we proceed from here depends on the dimension of X .

- Taking X to be of even dimension, Lemma 4.1.1 applies to $Ps^+(X)$ and $\Omega(X)$, and Section 2.1 shows $\gamma(f) = \pm 1$ for all nondegenerate forms f on X . In Equation (40), we can thus take $n = 2$ and $\psi = 1$, which defines a character φ_2 of $Mp^+(X)$ which coincides with θ^2 on \mathbf{T} . Its kernel $Ps_2^+(X)$ is a closed subgroup of $Mp^+(X)$ and a two-sheeted cover of $Ps^+(X)$.
- If X is of odd dimension, we can apply the preceding to the space $X' = X \oplus k$, which is of even dimension. The character φ_2' of $Mp^+(X')$ defined as above induces on $Mp^+(X) \leq Mp^+(X')$ (cf. Section 3.2) an analogous character φ_2 , and we will again denote its kernel by $Ps_2^+(X)$.
- Finally, if X is of dimension 1 on k , all elements of $Ps^+(X)$ are of the form $d(\alpha)t(f)$, and denoting the set of those elements of $Mp(X)$ of the form $\mathbf{d}(\alpha)\mathbf{t}(f)$ by $Ps_1^+(X)$, one checks that this gives a closed subgroup of $Mp^+(X)$, isomorphic to $Ps^+(X)$, and $Mp^+(X) = Ps_1^+(X) \times \mathbf{T}$.

4.3. Nontriviality of the extension. We will now show that the results of Section 4.2 are the best possible, in the following senses:

44.

- If k is not of characteristic 2 (and, in particular, if $k = \mathbb{C}$), then $Mp(X)$ is not a trivial extension of $Sp(X)$.
- If k is of characteristic 2 and if X is not of dimension 1, then $Mp^+(X)$ is not a trivial extension of $Ps^+(X)$.
- If k is of characteristic 2, if $X = X_1 \oplus X_2$, and if there exists a character φ of $Mp(X)$ (resp., of $Mp^+(X)$) coinciding with θ on \mathbf{T} , then these induce an analogous character on $Mp(X_1)$ (resp., on $Mp^+(X_1)$).

It suffices thus to give the proof for $X = k^2$ or $X = k$, depending on whether the characteristic of k is or is not equal to 2.

For the sake of contradiction, let φ be a character of $Mp(X)$ (resp., of $Mp^+(X)$), continuous or not, which coincides with θ on \mathbf{T} (i.e., $n = 1$). Note that φ restricts to the constant function 1 on the group of commutators of $Mp(X)$ (resp., of $Mp^+(X)$). We will consider the subgroup of $Mp(X)$ (resp., of $Mp^+(X)$) formed by those elements of the form $\mathbf{d}(\alpha)\mathbf{t}(f)$ with $\alpha \in \text{Aut}(X)$, $f \in Q(X)$. One checks easily that its group of commutators contains all the elements of the form $\mathbf{t}(f)$, and we thus have $\varphi(\mathbf{t}(f)) = 1$ for any $f \in Q(X)$.

For $\alpha \in \text{Aut}(X)$ we set $\lambda(\alpha) = \varphi(\mathbf{d}(\alpha))$, and for $\gamma: X^* \rightarrow X$ an isomorphism we set $\mu(\gamma) = \varphi(\mathbf{d}'(\gamma))$. We see that λ is a character of $\text{Aut}(X)$, and the relation $\mathbf{d}'(\gamma\alpha) = \mathbf{d}'(\gamma)\mathbf{d}(\alpha)$ shows that μ has the semilinearity property

$$\mu(\gamma\alpha) = \mu(\gamma)\lambda(\alpha).$$

⁴²The character ψ is continuous—even locally constant—on $\Omega(X)$.

Let f be a nondegenerate quadratic form over X with associated morphism $\rho: X \rightarrow X^*$. Equation (9) then gives

$$d'(-\rho^{-1})t(f)d'(\rho^{-1})t(f) = t(-f)d'(-\rho^{-1})$$

and the definition of $\gamma(f)$ in Section 1.8 gives

$$\mathbf{d}'(-\rho^{-1})\mathbf{t}(f)\mathbf{d}'(\rho^{-1})\mathbf{t}(f) = \gamma(f)\mathbf{t}(-f)\mathbf{d}'(-\rho^{-1}).$$

Applying φ to both sides gives

$$\mu(\rho^{-1}) = \gamma(f).$$

Let us apply this to the case where k is not of characteristic 2 and $X = k$. The form f associated to ρ is then given by $f(x) = \rho x^2/2$, hence we obtain

$$\gamma(\rho x^2/2) = \mu(\rho^{-1}) = \mu(1)\lambda(\rho)^{-1}.$$

By means of Proposition 2.1.2, we conclude that $\gamma(f)$, for a form f of m variables $f(x) = \sum_i a_i x_i^2$, depends only on m and on the discriminant $\prod_i a_i$ of f , hence $\gamma(f)$ takes the same value on all forms of 4 variables with discriminant 1. This contradicts Proposition 2.2.2.

Let us consider the case $X = k^2$ with k of characteristic 2. In this setting, all nondegenerate forms over X are equivalent to one of the form $f_1(x, y) = ax^2 + xy + by^2$ for a choice of $a, b \in k$. As all these forms are associated to the same morphism, independent of a and b , the formula $\gamma(f) = \mu(\rho^{-1})$ shows that γ takes the same value for all nondegenerate quadratic forms on X . Following Proposition 2.1.2, it follows that γ takes the same value for all nondegenerate quadratic forms of 4 variables, which is again in contradiction with Proposition 2.2.2 of the same chapter.

It follows that the covers $Sp_2(X)$ and $Ps_2^+(X)$ defined in Section 4.2 are not trivial.

Remark 4.3.1. For $k = \mathbb{R}$, the existence of a nontrivial double cover of the symplectic group is naturally a consequence of the fact that it is a connected Lie group whose fundamental group is \mathbb{Z} . The unitary representation of this cover is given by

$$(\mathbf{S}, \Phi) \mapsto \mathbf{S}\Phi \quad (\mathbf{S} \in Sp_2(X), \Phi \in L^2(X)),$$

which is the representation constructed and studied by D. Shale [Sha62]. For $k = \mathbb{C}$, the symplectic group is simply connected, and hence one may deduce the triviality of $Mp(X)$ over $Sp(X)$. It does not seem that the existence of these covers $Sp_2(X)$ and their unitary representations $(\mathbf{S}, \Phi) \mapsto \mathbf{S}\Phi$ were previously known for k of discrete valuation.

4.4. Nontriviality of the extension: adelic case. The results above extend to the adelic case, but we will not have use for these results, so we limit ourselves to just a summary.

Retaining the notation of Section 3.3, we will complete those results with some remarks on those sections. Given a place v , we will denote the canonical projection by

$$\pi_v: Mp(X_v) \rightarrow Ps(X_v) = Ps(X)_v,$$

the kernel of π_v as \mathbf{T}_v , and the character $(e_v, t) \mapsto t$ of \mathbf{T}_v by θ_v . For $v \notin S_0$, recall that we previously defined a lift $\mathbf{r}'_v: Ps(X)_v^\circ \rightarrow Mp(X_v)$. Let $Mp(X)_v^\circ$ be the image of $Ps(X)_v^\circ$ under \mathbf{r}'_v , which gives a closed subgroup of $Mp(X_v)$. For all $S \supset S_0$, set

$$M(S) = \prod_{v \in S} Mp(X_v) \times \prod_{v \notin S} Mp(X)_v^\circ.$$

For every element $(\mathbf{S}_v) \in M(S)$ there is a unique element $\mathbf{S} \in Mp(X)_A$ such that for all functions $\Phi \in L^2(X_A)$ of the form of Equation (36), one has

$$(41) \quad \mathbf{S}\Phi(x) = \prod_v \mathbf{S}_v \Phi_v(x_v).$$

This formula defines an open homomorphism $(\mathbf{S}_v) \mapsto \mathbf{S}$ from $M(S)$ to the open subgroup

$$Mp(X)_S^\circ = \pi^{-1}(Ps(X)_S^\circ) \subseteq Mp(X)_A,$$

with kernel consisting of elements $(t_v \in \mathbf{T}_v) \in M(S)$ such that for any $v \notin S$, $\theta_v(t_v) = 1$ holds⁴³ and $\prod_v \theta_v(t_v) = 1$.

Following Section 1.10, \mathbf{r}_v coincides with \mathbf{r}'_v on the set Ω_v° of those elements $s \in Ps(X)_v^\circ$ such that $\gamma(s)$ induces an isomorphism $\gamma(s): (X^*)_v^\circ \rightarrow X_v^\circ$. Moreover, if \mathfrak{p}_v is the maximal ideal of \mathfrak{o}_v , then reduction modulo \mathfrak{p}_v determines surjective homomorphisms

$$X_v^\circ \rightarrow \mathcal{X}_v, \quad (X^*)_v^\circ \rightarrow \mathcal{X}_v^*$$

over the finite field $\mathfrak{k}_v = \mathfrak{o}_v/\mathfrak{p}_v$. We conclude that, for almost all v , reduction modulo \mathfrak{p}_v determines a surjective homomorphism

$$Ps(X)_v^\circ \rightarrow Ps(\mathcal{X}_v)$$

which sends Ω_v° onto $\Omega(\mathcal{X}_v)$.

We will need to know that for all vector spaces \mathcal{X} of finite dimension over a field \mathfrak{k} , $\Omega(\mathcal{X})^{-1} \cdot \Omega(\mathcal{X})$ is equal to $Ps(\mathcal{X})$ or to $Ps^+(\mathcal{X})$ depending on whether \mathfrak{k} is of characteristic other than 2 or equal to 2. We will later prove this as Corollary 5.1.4. Granting this for now, let us suppose that k is not of characteristic 2; then for almost all v , we have

$$Ps(\mathcal{X}_v) = \Omega(\mathcal{X}_v)^{-1} \cdot \Omega(\mathcal{X}_v).$$

From this we conclude that for almost all v , $Ps(X)_v^\circ = (\Omega_v^\circ)^{-1} \cdot \Omega_v^\circ$. For each v , Section 4.2 grants a continuous character φ_v of $Mp(X)_v$ which coincides with θ_v^2 on \mathbf{T}_v . The character φ_v is uniquely determined by the condition that for all $s \in \Omega_v$, we have

$$\varphi_v(\mathbf{r}_v(s)) = \left(\frac{D(s)}{-1} \right)_v \gamma_v(q_1)^{2m},$$

where $m = \dim(X)$. When -1 is a square in k_v , the right-hand side is equal to 1 for all $s \in \Omega_v$. For almost all v , $k_v(\sqrt{-1})$ is equal either to k_v or to a nonramified quadratic extension of k_v , and in either case we have $\left(\frac{u}{-1} \right)_v = 1$ for all units $u \in \mathfrak{o}_v$. On the other hand, Section 2.3 shows for almost all v that $\gamma_n(q_1) = 1$ for almost all v , which entails that φ_v takes the constant value 1 on $\mathbf{r}_v(\Omega_v^\circ)$ for almost all v . The map \mathbf{r}_v coincides with \mathbf{r}'_v on Ω_v° for almost all v , and in this case one has $Ps(X)_v^\circ = (\Omega_v^\circ)^{-1} \cdot \Omega_v^\circ$. Hence, φ_v takes the constant value 1 on $Mp(X)_v^\circ$ for almost all v . From this we have that

$$(\mathbf{S}_v) \mapsto \prod_v \varphi_v(\mathbf{S}_v)$$

is a character of $M(S)$, equal to 1 on the kernel of the homomorphism $M(S) \rightarrow Mp(X)_S^\circ$ defined in Equation (41). By passing to the quotient by the kernel and then to the inductive limit determining S , this gives a continuous character φ_A of $Mp(X)_A$ which on \mathbf{T}

⁴³That is, for $v \notin S$, t_v is the neutral element of $Mp(X)_v$.

coincides with $(e, t) \mapsto t^2$. If we denote by $Sp_2(X)_A$ the kernel of φ_A , we conclude that $Mp(X)_A = Sp_2(X)_A \cdot \mathbf{T}$ and that $Sp_2(X)_A$ is a two-sheeted cover of $Sp(X)_A$.

We will now show that $Sp_2(X)_A \supseteq \mathbf{r}_k(Ps(X)_k)$. As $Ps(X)_k = \Omega(X)_k^{-1} \cdot \Omega(X)_k$, it will suffice to show $Sp_2(X)_A \supseteq \mathbf{r}_k(\Omega(X)_k)$. We therefore take $s \in \Omega(X)_k$, for which $s \in \Omega_v$ for all v and $s \in \Omega_v^\circ$ for almost all v . As \mathbf{r}_v coincides with \mathbf{r}'_v over Ω_v° for almost all v , it follows that $(\mathbf{r}_v(s))$ lies in $M(S)$ for S sufficiently large. On the other hand, Equation (38) gives

$$\mathbf{r}_k(s)\Phi(x) = \int_{X_A^*} \Phi(x\alpha + x^*\gamma)\chi(f(x, x^*))dx^*,$$

and comparing this formula with Equation (16) (i.e., the definition of \mathbf{r}_v) shows that $\mathbf{r}_k(s) \in Mp(X)_A$ is the image of $(\mathbf{r}_v(s)) \in M(S)$ through the homomorphism defined by Equation (41). Under these conditions, we have $\mathbf{r}_k(s) \in Sp_2(X)_A$ if and only if $\prod_v \varphi_v(\mathbf{r}_v(s)) = 1$ —but this is either true evidently if -1 is a square in k or true via Proposition 2.3.1 and the law of quadratic reciprocity.

In the case where k is of characteristic 2, we may proceed as elsewhere: we substitute $Ps^+(X)$ for $Ps(X)$; if $m \geq 3$ is even, we extend X to a vector space X' of dimension $m+1$; and case $m=1$ consists of trivial results.

V. Compléments

5. COMPLEMENTS

To close this memoir, we begin to consider modules over k -algebras \mathcal{A} rather than modules directly over k . From our perspective, the primary complication that this introduces into the theory is that our reliance on k -linear duality and biduals is no longer sufficient to select \mathcal{A} -linear complements to submodules, which is a basic tool we have rested on throughout our discussion. In this section, we give this wrinkle careful consideration.

46.

5.1. Decompositions of elements in the pseudosymplectic group. We will first consider a vector space X over a field k as well as its pseudosymplectic group $Ps(X)$. For $s = (\sigma, f) \in Ps(X)$, we consider the matrix expansion

$$\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

We will denote the *parabolic subgroup* of $Ps(X)$ by $P(X)$, which consists of those $s \in Ps(X)$ for which $\gamma = 0$. We propose to write every element of $Ps(X)$ in a “normal” form using the left-cosets of $P(X)$ in $Ps(X)$, which generalizes Equation (33).

Taking $s = (\sigma, f) \in P(X)$ and using Equation (32), we obtain

$$f(x_2, x_1^* + x_2^*) = f(0, x_1^*) + f(x_2, x_1^*).$$

Denoting by g and h the forms respectively induced by f on X and on X^* , we that h is additive and that $f(x, x^*) = g(x) + h(x^*)$. We see immediately that $t(g)^{-1}t'(h)^{-1}s$ is of the form $(\sigma', 0)$, and by using the remarks at the end of Section 1.2 it is therefore equal to $d(\lambda)$ for some $\lambda \in \text{Aut}(X)$. This yields

$$(42) \quad s = t'(h)t(g)d(\lambda)$$

for some $h \in Q_a(X^*)$, $g \in Q(X)$, and $\lambda \in \text{Aut}(X)$, and this factorization is unique.

Now let $s = (\sigma, f)$ be an arbitrary element of $Ps(X)$, let N be the kernel of the attached morphism γ , let W be a complement of N in X^* , and let $Z = \gamma(X^*)$. The morphism γ then induces on W an isomorphism $\gamma: W \rightarrow Z$, and hence there is a quadratic form f_1 on Z such that $f(0, w) = f_1(w\gamma)$ for all $w \in W$. Otherwise said, the formula

$$f_0(x^*) = f(0, x^*) - f_1(x^*\gamma)$$

defines a quadratic form f_0 on X^* which vanishes on W . On the other hand, for $n \in N$ and $x^* \in X^*$ we may conclude from Equation (32) that

$$f(0, n + x^*) = f(0, n) + f(0, x^*),$$

which implies in particular that $f(0, n)$ is additive on N . It follows that $f_0(n + w) = f(0, n)$ for $n \in N$ and $w \in W$, which shows that f_0 is an additive form on X^* .

We will now deduce the following result:

Proposition 5.1.1. *Let $s = (\sigma, f)$, $s' = (\sigma', f')$ be two elements of $Ps(X)$ with matrix components γ, γ' . Then s and s' belong to the same left-coset of $P(X)$ if and only if both $X^*\gamma = X^*\gamma'$ and there is an $f_1 \in Q(X^*\gamma)$ such that*

$$x^* \mapsto f(0, x^*) - f_1(x^*\gamma), \quad x^* \mapsto f'(0, x^*) - f_1(x^*\gamma')$$

are additive forms on X^* .

Proof. Begin by choosing f_1 so that the first of these forms is additive, and let us suppose that we have $s' = s''s$ for an $s'' = (\sigma'', f'')$ with $\gamma'' = 0$. The morphism δ'' is then an automorphism of X^* , and $\gamma' = \delta''\gamma$, hence $\gamma(X^*) = \gamma'(X^*)$. We conclude that $f''(0, x^*)$ is an additive form. Because

$$f'(0, x^*) = f''(0, x^*) + f(0, x^*\delta''),$$

we also have

$$f'(0, x^*) - f_1(x^*\gamma') = f''(0, x^*) + f(0, x^*\delta'') - f_1(x^*\delta''\gamma),$$

and the right-hand side is an additive form by virtue of the definition of f_1 . We therefore conclude necessity.

To show sufficiency, suppose that this condition is satisfied. We then set $s'' = s's^{-1}$, from which we have $\gamma'' = \delta^* \circ \gamma' - \gamma^* \circ \delta'$. Let us set $Z = \gamma(X^*) = \gamma'(X^*)$, and let $j: Z \rightarrow X$ be the induced injection. We can then write $\gamma = j \circ \bar{\gamma}$ and $\gamma' = j \circ \bar{\gamma}'$, where $\bar{\gamma}, \bar{\gamma}': X^* \rightarrow Z$ are morphisms of the indicated type. Let $\rho_1: Z \rightarrow Z^*$ be the morphism associated to f_1 ; using Equation (32), the morphism associated to $x^* \mapsto f(0, x^*)$ is $\delta^* \circ \gamma$, and the hypothesis placed on $f(0, x^*)$ gives

$$\delta^* \circ \gamma = \bar{\gamma}^* \circ \rho_1 \circ \bar{\gamma},$$

which can be written also as

$$(\delta^* \circ j - \bar{\gamma}^* \circ \rho_1) \circ \bar{\gamma} = 0.$$

When $\bar{\gamma}$ is surjective, we thus conclude $\delta^* \circ j = \bar{\gamma}^* \circ \rho_1$. This yields $j^* \circ \delta = \rho_1 \circ \bar{\gamma}$, and by replacing f with f' this gives $j^* \circ \delta' = \rho_1 \circ \bar{\gamma}'$, hence

$$\bar{\gamma}^* \circ \rho_1 \circ \bar{\gamma}' = \delta^* \circ j \circ \bar{\gamma}' = \bar{\gamma}^* \circ j^* \circ \delta'.$$

47.

Proposition 6

By consequence, $\delta^* \circ \gamma' = \gamma^* \circ \delta'$, i.e., $\gamma'' = 0$. \square

It follows that the set of left cosets of $P(X) \leq Ps(X)$ corresponding to those elements $s \in Ps(X)$ for which γ is of rank r can be identified with a vector bundle on the Grassmannian of subspaces $Z \leq X$ of dimension r , with fiber over Z the vector space $Q(Z)/Q_a(Z)$.

Corollaire 1

Corollary 5.1.2. *Let $s \in Ps(X)$, let X_1 be the image of X^* through γ , let X_2 be a complement to X_1 in X , let $\gamma_1: X_1^* \rightarrow X_1$ be an isomorphism, and let Q'_1 be a complement to $Q_a(X_1)$ in $Q(X_1)$. Then s can be written uniquely in the form*

$$(43) \quad s = t'(h)t(g)d(\lambda)(d'(\gamma_1)t(f_1) \otimes e_2)$$

with $h \in Q_a(X^*)$, $g \in Q(X)$, $\lambda \in \text{Aut}(X)$, and $f_1 \in Q'_1$, where e_2 denotes the neutral element of $Ps(X_2)$.

Proof. By setting $s' = s$, it follows from Proposition 5.1.1 that there exists an $f_1 \in Q(X_1)$ such that the form

$$x^* \mapsto f(0, x^*) - f_1(\gamma(x^*))$$

is additive. As this condition determines f_1 uniquely up to an additive form, this relation is satisfied if and only if $f_1 \in Q'_1$.

We thus set $s' = d'(\gamma_1)t(f_1) \otimes e_2$. Proposition 5.1.1 shows immediately that s and s' belong to the same left-coset of $P(X)$. We can thus put $s's^{-1}$ into the form Equation (42) in a unique way, which puts s into the form of Equation (43).

Conversely, suppose that s has the form of Equation (43) and define s' as above. Using Corollary 5.1.2, we also see that f_1 is as described above. The unicity of Equation (42) finishes the proof. \square

Corollaire 2

Corollary 5.1.3. *Retain the notation and the hypotheses of Corollary 5.1.2. The element s can be written uniquely in the form*

$$(44) \quad s = t(g)d(\lambda)(d'(\gamma_1)t(g_1) \otimes t'(h_2))$$

with $g \in Q(X)$, $\lambda \in \text{Aut}(X)$, $g_1 \in Q(X_1)$, and $h_2 \in Q_a(X_2^*)$.

Proof. Proceeding as in Corollary 5.1.2, but applying Equation (42) to $s's^{-1}$, we obtain an expression

$$s = d(\lambda)^{-1}t(g)^{-1}t'(h)^{-1}(d'(\gamma_1)t(f_1) \otimes e_2).$$

For $x^* = (x_1^*, x_2^*) \in X^*$, we may write

$$-h(x^*) = h_1(x_1^*) + h_2(x_2^*),$$

for $h_1 \in Q_a(X_1^*)$ and $h_2 \in Q_a(X_2^*)$. From this we conclude $t'(h)^{-1} = t'(h_1) \otimes t'(h_2)$. One checks that we may also write $t'(h_1)d'(\gamma_1) = d'(\gamma_1)t(b'_1)$ with $b'_1 \in Q_a(X_1)$. It also follows from Section 3.2 that $d(\lambda)^{-1}t(g)^{-1}$ can be written in the form $t(g')d(\lambda')$. By writing g , λ , and g_1 in place of g' , λ' , and $f_1 + b'_1$, we obtain Equation (44).

The unicity results from that of Equation (43) and from the calculation we just finished, taken in reverse. \square

If we suppose $\gamma = 0$ (or, equivalently, $X_1 = \{0\}$ and $X_2 = X$) in Corollary 5.1.2 and Corollary 5.1.3, we recover Equation (42) (or an equivalent formula). At the other extreme, the case $X_1 = X$ and $X_2 = \{0\}$ is equivalent to $s \in \Omega(X)$, and in terms of Section 3.2, Corollary 5.1.3 recovers Equation (33). We are also going to deduce from these a result which we left unproven in Section 4:

Corollary 5.1.4. *Let X be a vector space over a field k . Then $\Omega(X)^{-1} \cdot \Omega(X)$ is equal to $Ps^+(X)$ or to $Ps(X)$, depending on whether k is or is not of characteristic 2.*

Corollaire 3

Proof. Take $s \in Ps(X)$ and $s' \in \Omega(X)$. Put s' in the form of Equation (33)⁴⁴ by writing $s' = t(g)d'(\gamma)t(g_1)$. The composite $s's$ belongs to $\Omega(X)$ (i.e., the attached function $\gamma: X^* \rightarrow X$ to be an isomorphism) if and only if $\alpha(t(g_1)s)$ is an automorphism of X . Otherwise said, s belongs to $\Omega(X)^{-1}\Omega(X)$ if and only if there exists $f \in Q(X)$ with $\alpha(t(f)s) \in \text{Aut}(X)$. Moreover, Equation (33) shows that $\Omega(X)$ is a doubling of $P(X)$, and hence $\Omega(X)^{-1} \cdot \Omega(X)$ is a union of such doublings. By putting s^{-1} in the form of Equation (43) and setting $s_1 = d'(\gamma_1)t(f_1) \otimes e_2$, we thus see that s belongs to $\Omega(X)^{-1} \cdot \Omega(X)$ if and only if s_1^{-1} does as well, or if and only if there is an $f \in Q(X)$ for which $\alpha(t(f)s_1^{-1})$ is an automorphism of X .

It is immediate that this last condition is satisfied by taking for f a form which vanishes on X_2 and such that $f - f_1$ is nondegenerate on X_1 . Such a choice is always possible when k is not of characteristic 2 as well as when k is of characteristic 2 and X is of even dimension. As we have already observed in Section 3.2, this last condition is equivalent to $s \in Ps^+(X)$, and the conclusion follows. \square

Remark 5.1.5. Suppose that k is of characteristic 2. Using Proposition 5.1.1 and the Corollaries above, we could deduce the results recalled in Section 3.2 about $Ps^+(X)$ and $Ps^-(X)$.

5.2. Lifts of the pseudosymplectic group to the metaplectic group. In the preceding chapters, we made use of the Equation (33)⁴⁵ to lift elements of $\Omega(X)$ to $Mp(X)$. Corollary 5.1.2 and Corollary 5.1.3 permit us to do the same for elements of $Ps(X)$, where X is a vector space over a local field. We will apply this to the proof of the following result:

48.

Proposition 5.2.1. *Let X be a vector space over a local field k , let $S \in Mp(X)$, and let $s = \pi(S)$ with matrix component γ . For ξ and ξ' two automorphisms of X with $sd(\xi) = d(\xi')s$, we also have $Sd(\xi) = d(\xi')S$, provided one of the following holds:*

Proposition 7

- (1) *We can write s in the form of Equation (44) with $h_2 = 0$.*
- (2) *There is a complement X_2 of $X_1 = \gamma(X^*)$ in X which is stable under ξ .*

Remark 5.2.2. As h_2 in Equation (44) is an additive form, case (1) always holds if k is not of characteristic 2. Note also that the condition $h_2 = 0$, which appears to depend on the choice of X_2 and of γ_1 of ??, is in fact equivalent to the following statement, evidently independent of X_2 and γ_1 :

- (1') For $s = (\sigma, f)$, the kernel of f contains the kernel of γ .

⁴⁴Equivalently, we may use the form of Equation (44).

⁴⁵Equivalently, Proposition 1.5.1.

Proof of Proposition 5.2.1. Taking X_2 and γ_1 to have been chosen as in Corollary 5.1.2, we may place s in the form of Equation (44). There is then an element $a_2 \in X_2$ such that, for all $x_2^* \in X_2^*$,

$$\chi(b_2(x_2^*)) = \chi([a_2, x_2^*]).$$

On the other hand, the relation $sd(\xi) = d(\xi')s$ gives $\xi \circ \gamma = \gamma \circ \xi'^{-1}$, hence $\xi(X_1) = X_1$, i.e., X_1 is stable under ξ . Using the decomposition $X = X_1 \oplus X_2$ to put ξ in matrix form, we have

$$\xi = \begin{pmatrix} \xi_1 & 0 \\ \eta & \xi_2 \end{pmatrix},$$

and (2) becomes equivalent to saying that there is some choice of X_2 for which $\eta = 0$.

Taking this to be true, \mathbf{S} differs from

$$\mathbf{S}' = \mathbf{t}(g)\mathbf{d}(\lambda)(\mathbf{d}'(\gamma_1)\mathbf{t}(g_1) \otimes \mathbf{t}'(b_2)) \in Mp(X)$$

by a scalar factor. At the same time, the hypotheses placed on s imply that $\mathbf{S}\mathbf{d}(\xi)$ and $\mathbf{d}(\xi)\mathbf{S}$ differ only by a scalar factor $\theta \in T$. For all $\Phi \in \mathcal{S}(X)$, it follows that $\mathbf{S}'\mathbf{d}(\xi)\Phi(0)$ and $\mathbf{d}(\xi')\mathbf{S}'\Phi(0)$ differ by the same scalar factor θ . By writing out the definitions of the operators which appear in the definition of \mathbf{S}' , we see that this is equivalent to saying that there exists $c > 0$ such that

$$\int \Phi(\xi_1(x_1) - \eta(a_2), -\xi_2(a_2))\chi(g_1(x_1))dx_1 = c\theta \int \Phi(x_1, -a_2)\chi(g_1(x_1))dx_1$$

for all $\Phi \in \mathcal{S}(X)$. This holds if and only if $\xi_2(a_2) = a_2$ and for any x_1 we have

$$\chi(g_1(x_1)) = \theta \cdot \chi(g_1(\xi_1(x_1) - \eta(a_2))).$$

Specializing to $x_1 = 0$ and either to $a_2 = 0$ in case (1) or $\eta = 0$ in case (2), this gives $\theta = 1$. \square

Corollaire
(unnumbered)

Corollary 5.2.3. *Let X be a vector space over a local field k , take $G \leq \text{Aut}(X)$, and suppose either that k does not have characteristic 2 or that G is completely reducible. Let \mathbf{S} be an element of $Mp(X)$ such that $s = \pi(\mathbf{S})$ commutes with $d(\xi)$ for all $\xi \in G$. Then \mathbf{S} commutes with $\mathbf{d}(\xi)$ for any $\xi \in G$.*

Proof. If k is not of characteristic 2, we need only apply Proposition 5.2.1 with $\xi = \xi'$. Regardless, the proof of Proposition 5.2.1 shows that $X_1 = \gamma(X^*)$ is stable under G . If G is completely reducible, X_1 has a complement stable under G , and we are in case (2). \square

49.

5.3. Algebras with involutions. We will now define some subgroups of the pseudosymplectic group which will play a large role in the applications to the arithmetic theory of the classical groups.

Let \mathcal{A} be an algebra over a base field k , which we will always assume to be associative, of finite dimension over k , and endowed with a unit element 1. We will freely identify k with its image in \mathcal{A} via the morphism $t \mapsto t \cdot 1$. We will eventually write \mathcal{A}_k in lieu of \mathcal{A} , particularly when we would like to consider the ‘‘adelic case’’.

We will suppose moreover that \mathcal{A} is endowed with an involution ι , i.e., an involutive antiautomorphism of \mathcal{A} as an algebra over k . By definition, ι induces the identity automorphism on $k \leq \mathcal{A}$. The choice of ι permits us to consider every right \mathcal{A} -module Y as a left \mathcal{A} -module via the formula $ty = yt'$ ($t \in \mathcal{A}$, $y \in Y$). In particular, for a left \mathcal{A} -module X , the \mathcal{A} -linear dual $\text{Hom}_{\mathcal{A}}(X, \mathcal{A}_s)$ is naturally endowed with a right

\mathcal{A} -module structure, and we will denote by X^* this same dual with the left \mathcal{A} -module structure instead. For $x \in X$ and $x^* \in X^*$, we will write $\{x, x^*\}$ for the value on x of the \mathcal{A} -linear form on X which corresponds to x^* . We thus have

$$\{tx, ux^*\} = t\{x, x^*\}u',$$

i.e., $\{x, x^*\}$ is a sesquilinear form on $X \times X^*$.

Finally, we will also suppose that \mathcal{A} carries a *trace function* τ . By “trace function”, we mean a k -linear form on \mathcal{A} which is invariant under the involution ι and such that $(t, u) \mapsto \tau(tu)$ is a nondegenerate symmetric bilinear form on $\mathcal{A} \times \mathcal{A}$. For X a left \mathcal{A} -module and f a k -linear form on X , for all $x \in X$ the assignment $t \mapsto f(tx)$ yields a k -linear form on \mathcal{A} , and there is thus a unique element $F(x)$ of \mathcal{A} such that $f(tx) = \tau(tF(x))$ for any t . The assignment $x \mapsto F(x)$ to Riesz representatives is itself an \mathcal{A} -linear form on X for which $f = \tau \circ F$, and the formula $f = \tau \circ F$ puts the k -linear forms f on X in bijection with the \mathcal{A} -linear forms F on X . By consequence, we may identify X^* with the dual of the underlying k -vector space of X via the formula

$$[x, x^*] = \tau(\{x, x^*\}).$$

Since we are considering only \mathcal{A} -modules of finite dimension over k , it follows that we may identify all left \mathcal{A} -modules X with their biduals $(X^*)^*$ via the formula

$$\{x, x^*\} = \{x^*, x\}'.$$

For X and Y two left \mathcal{A} -modules and a morphism $\alpha: X \rightarrow Y$ an \mathcal{A} -module morphism, the transpose α^* considered as a morphism of k -vector spaces belongs to $\text{Hom}_{\mathcal{A}}(Y^*, X^*)$, and for $x \in X, y^* \in Y^*$ we have

$$\{\alpha(x), y^*\} = \{x, \alpha^*(y^*)\}.$$

If F is a sesquilinear form on $X \times Y$, there is a unique morphism $\alpha: Y \rightarrow X^*$ such that

$$F(x, y) = \{x, \alpha(y)\} = \{y, \alpha^*(x)\}'.$$

If $X = Y$, then F is Hermitian if and only if $\alpha = \alpha^*$.

We will say that a quadratic form f on the k -vector space underlying X is \mathcal{A} -quadratic if it can be written as $f(x) = \tau(F(x, x))$ for F a sesquilinear (but not necessarily Hermitian) form on $X \times X$, and we will denote by $Q_{\mathcal{A}}(X)$ the space of such forms. Equivalently, $Q_{\mathcal{A}}(X)$ is the set of those forms which can be written

$$f(x) = \tau(\{x, \lambda(x)\}),$$

where $\lambda: X \rightarrow X^*$ is a not-necessarily-symmetric morphism, and in this language, the morphism associated to this form f is given by the formula $\rho = \lambda + \lambda^*$. We denote by $Q_{\mathcal{A}, \mathcal{A}}(X)$ the space of additive \mathcal{A} -quadratic forms on X . If k is not of characteristic 2, and if ρ is the symmetric morphism associated to a form $f \in Q_{\mathcal{A}}(X)$, the form F defined on $X \times X$ by $F(x, y) = \{x, 2^{-1}\rho(y)\}$ will be Hermitian and will satisfy the formula $f(x) = \tau(F(x, x))$. This gives a bijection between $Q_{\mathcal{A}}(X)$ and the space of Hermitian forms on $X \times X$.

The group of automorphisms of an \mathcal{A} -module X will be denoted by $\text{Aut}_{\mathcal{A}}(X)$. Likewise, if X and Y are \mathcal{A} -modules, the set of isomorphisms from X to Y will be denoted as $\text{Is}_{\mathcal{A}}(X, Y)$. We will denote by $Ps_k(X)$ the pseudosymplectic group attached to the k -vector space underlying X and by $Ps_{\mathcal{A}}(X)$ the subgroup of $Ps_k(X)$ of those elements (σ, f)

for which $\sigma \in \text{Aut}_{\mathcal{A}}(X \oplus X^*)$ and $f \in Q_{\mathcal{A}}(X \oplus X^*)$. In the notation of Section 3.2, the morphisms $d(\alpha)$, $d'(\gamma)$, $t(f)$, and $t'(f')$ will belong to $Ps_{\mathcal{A}}(X)$ whenever α , γ , f , and f' belong respectively to $\text{Aut}_{\mathcal{A}}(X)$, $\text{Is}_{\mathcal{A}}(X^*, X)$, $Q_{\mathcal{A}}(X)$, and $Q_{\mathcal{A}}(X^*)$. The identities among d , d' , t , and t' of course remain valid in these subgroups.

50.

In our intended applications, we will almost always assume the algebra \mathcal{A} to be semisimple. When this is so, all submodules of an \mathcal{A} -module X possess a complement. If \mathcal{A} is moreover *simple*, then $\text{Is}_{\mathcal{A}}(X^*, X)$ is never empty, i.e., all left \mathcal{A} -modules are isomorphic to their duals. If \mathcal{A} is *absolutely semisimple*⁴⁶ and it is endowed with an involution ι , then there always exists a trace function on \mathcal{A} .⁴⁷

When \mathcal{A} is not simple, we can make use of the following result instead:

Lemme 7

Lemma 5.3.1. *Take $s \in Ps(X/\mathcal{A})$ with matrix component γ , and set $Z = \gamma(X^*)$. In order for Z to admit a complement in X , it is necessary and sufficient that the kernel N of γ has a complement in X^* . If moreover, \mathcal{A} is semisimple, les \mathcal{A} -modules Z and Z^* are isomorphic.*

Proof. Let Z_* be the orthogonal to Z in X^* , which is the kernel of γ^* . By the same token, if N_* is the orthogonal to N in X , we have $N_* = X^*\gamma^*$. Let us set $s = (\sigma, f)$ with matrix expansion $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. We then set U to be the image of $\{0\} \times X^*$ (considered as a submodule of $X \times X^*$) under σ , or, equivalently, the image of X^* through the morphism $x^* \mapsto (\gamma^*(x^*), \delta(x^*))$. As σ is an automorphism of $X \times X^*$, this determines an isomorphism $\theta: X^* \rightarrow U$. The set U also consists of those elements $u \in X \times X^*$ such that $\sigma^{-1}(u) \in \{0\} \times X^*$, or, by setting $u = (y, y^*)$, such that $\delta^*(y) - \gamma^*(y^*) = 0$.

Set $V = U \cap (\{0\} \times X^*)$, which also has a few equivalent characterizations. As Z_* is the kernel of γ^* , we have $V = \{0\} \times Z_*$. At the same time, we have $V = \theta(N)$ and $N = \theta^{-1}(V)$.

Let us suppose now that Z has a complement Z' in X . The orthogonal Z'_* of Z' in X^* is then a complement to Z_* in X^* , and $X \times Z'_*$ is a complement of V in $X \times X^*$. It follows that $U_1 = U \cap (X \times Z'_*)$ is a complement to V in U , hence that $\theta^{-1}(U_1)$ is a complement of N in X^* . Let us now apply what which we have just shown for s to s^{-1} instead. As $\gamma(s^{-1}) = -\gamma^*$, it follows that if N_* has a complement in X , then Z_* has one in X^* . Finally, by passing to the quotient, γ determines an isomorphism $X^*/N \rightarrow Z$.

On the other hand, we may identify X^*/Z_* with Z^* , and the above shows that Z_* is isomorphic to V and hence to N . If \mathcal{A} is semisimple, the fact that N and Z_* are isomorphic entails that X^*/N and X^*/Z_* are also isomorphic, from which we have the final assertion of the Lemma. \square

51.

5.4. Parabolic subgroups and \mathcal{A} . As in Section 5.1, we will denote by $P_k(X) \leq Ps_k(X)$ and $P_{\mathcal{A}}(X) \leq Ps_{\mathcal{A}}(X)$ the subgroups whose elements s have matrix expansions with $\gamma = 0$. Recall that all elements of $P_k(X)$ can be uniquely written in the form of Equation (42); for $s \in P_{\mathcal{A}}(X)$, the same calculation shows that s can be placed in the form of Equation (42) with $h \in Q_{\mathcal{A}, \mathcal{A}}(X^*)$, $g \in Q_{\mathcal{A}}(X)$, and $\lambda \in \text{Aut}_{\mathcal{A}}(X)$. Moreover, if $s \in Ps_{\mathcal{A}}(X)$ is arbitrary, the calculation made in the second part of Section 5.1 remains

⁴⁶That is: its extension to the algebraic closure of k is semisimple.

⁴⁷For example, one may take for τ the k -linear form which, on each simple composite of \mathcal{A} , induces the “reduced trace”.

valid, showing that f_1 belongs to $Q_{\mathcal{A}}(Z)$ whenever W is an \mathcal{A} -module complement to the kernel N of γ . If we then suppose in Proposition 5.1.1 that s and s' belong to $Ps_{\mathcal{A}}(X)$, we may take $f_1 \in Q_{\mathcal{A}}(\gamma(X^*))$ whenever the kernel of γ admits a complement in X^* —or equivalently, following Lemma 5.3.1, each time that $\gamma(X^*)$ admits a complement in X . Analogous remarks apply to Corollary 5.1.2 and Corollary 5.1.3. With our ultimate applications in mind, we will make use of the following result:

Proposition 5.4.1 (cf. Corollary 5.1.3). *Let \mathcal{A} be a k -algebra with an involution and a trace function, let X be a left-module over \mathcal{A} , let $s \in Ps_{\mathcal{A}}(X)$, and take $X_1 = \gamma(X^*)$. Suppose that X_2 is a complement to X_1 in X and $\gamma_1: X_1^* \rightarrow X_1$ is an isomorphism. We may then write s uniquely in the form*

$$s = t(g)d(\lambda)(d'(\gamma_1)t(g_1) \otimes t'(h_2)),$$

with $g \in Q_{\mathcal{A}}(X)$, $\lambda \in \text{Aut}_{\mathcal{A}}(X)$, $g_1 \in Q_{\mathcal{A}}(X_1)$, and $h_2 \in Q_{\mathcal{A},a}(X_2^*)$. \square

Remark 5.4.2. The existence of a complement X_2 to X_1 and, by using Lemma 5.3.1, the existence of the isomorphism γ are assured whenever \mathcal{A} is semisimple.

If k is a local field, we will denote by $Mp_k(X)$ the metaplectic group attached to the k -vector space underlying X , and by $Mp_{\mathcal{A}}(X)$ the inverse image of $Ps_{\mathcal{A}}(X)$ in $Mp_k(X)$ through the canonical projection $Mp_k(X) \rightarrow Ps_k(X)$. In the adelic case, the objects $Mp_k(X)_A$ and $Mp_{\mathcal{A}}(X)_A$ are defined in the same way. Applying Proposition 5.2.1 then gives the following results:

Proposition 5.4.3. *Let \mathcal{A} be an algebra over a local field k , endowed with an involution ι and a trace function. Suppose that k is not of characteristic 2 or that \mathcal{A} is semisimple. For X a left \mathcal{A} -module and $Mp_{\mathcal{A}}(X)$ its metaplectic group, every element of $Mp_{\mathcal{A}}(X)$ commutes with all elements of $Mp_k(X)$ of the form $\mathbf{d}(\xi_a)$, where ξ_a is the homothety $x \mapsto a \cdot x$ determined by an element $a \in \mathcal{A}$ satisfying $a \cdot a' = 1$.*

Proof. If a is an invertible element in \mathcal{A} , it follows from the left \mathcal{A} -module structure on X^* that $d(\xi_a) = (\sigma, 0)$, where σ is the automorphism $(x, x^*) \mapsto (ax, (a')^{-1}x^*)$ of the underlying k -vector space of $X \oplus X^*$. For $a \cdot a' = 1$, σ is thus the homothety $z \mapsto az$ of $X \oplus X^*$. By definition of $Ps_{\mathcal{A}}(X)$, it follows that every element of $Ps_{\mathcal{A}}(X)$ thus commutes with $d(\xi_a)$. We conclude the Lemma by applying case (1) of Proposition 5.2.1 if k is not of characteristic 2 and case (2) if \mathcal{A} is semisimple. \square

Corollary 5.4.4. *Let A_k be the ring of adèles attached to a number field or a function field k , and let \mathcal{A}_k be a k -algebra over k with an involution ι and a trace function. Suppose either that k is not of characteristic 2 or that \mathcal{A} is semisimple.*

Let G_k be the group of those elements $a \in \mathcal{A}_k$ such that $a \cdot a' = 1$, and let G_A be the corresponding adelic group. Let X_k be a left \mathcal{A}_k -module, and let $X_A = X_k \otimes A_k$. Then, for all $\mathbf{S} \in Mp(X/\mathcal{A})_A$ and all $a \in G_A$, the operators $\Phi \mapsto \mathbf{S}\Phi$ and $\Phi(x) \mapsto \Phi(ax)$ in $L^2(X_A)$ commute.

Proof. It suffices to check that they commute when applied to functions Φ of the form of Equation (36). We are thus brought back to the local case, i.e., to Proposition 5.4.3. \square

Proposition 8

Proposition 9

Corollaire
(unnumbered)

5.5. **Teaser.** To conclude, we will announce the principal result to be proven in the sequel memoir, which is an application of the theory presented here.

52.

Let k be an algebraic number field, and let \mathcal{A}_k be a simple k -algebra endowed with an involution ι . For simplicity, we will take the trace function on \mathcal{A}_k to be the reduced trace. Without loss of generality, we may further suppose that k is the central subfield of \mathcal{A}_k formed by those elements which are invariant under ι .

Let G_k be the group of elements $a \in \mathcal{A}_k$ such that $a \cdot a' = 1$, and let G_A be the corresponding adelic group. Let $d_1 a$ be the Haar measure on G_A , normalized so that G_A/G_k has measure 1. Let X_k be a left \mathcal{A}_k -module, and set $X_A = X_k \otimes A_k$. For $\Phi \in \mathcal{S}(X_A)$, we will demonstrate the formula

$$\int_{G_A/G_k} \sum_{\xi \in X_k} \Phi(a\xi) d_1 a = \sum_s \mathbf{r}_k(s) \Phi(0),$$

where the summation on the right-hand side runs over a complete system of representatives of left-cosets of $P(X_k/\mathcal{A}_k) \leq Ps(X_k/\mathcal{A}_k)$. The two expressions are absolutely convergent and the formula holds whenever one has

$$\dim_k(X_k) > 4 \dim_k Q(X_k/\mathcal{A}_k).$$

REFERENCES

- [Bru61] François Bruhat. Distributions sur un groupe localement compact et applications à l'étude des représentations des groupes p -adiques. *Bull. Soc. Math. France*, 89:43–75, 1961.
- [Car64] P. Cartier. über einige Integralformeln in der Theorie der quadratischen Formen. *Math. Z.*, 84:93–100, 1964.
- [Mac65] George W. Mackey. Some remarks on symplectic automorphisms. *Proc. Amer. Math. Soc.*, 16:393–397, 1965.
- [Sch51] Laurent Schwartz. *Théorie des distributions. Tome II*. Actualités Sci. Ind., no. 1122 = Publ. Inst. Math. Univ. Strasbourg 10. Hermann & Cie., Paris, 1951.
- [Seg59] I. E. Segal. Foundations of the theory of dynamical systems of infinitely many degrees of freedom. I. *Mat.-Fys. Medd. Danske Vid. Selsk.*, 31(12):39 pp. (1959), 1959.
- [Seg63] I. E. Segal. Transforms for operators and symplectic automorphisms over a locally compact abelian group. *Math. Scand.*, 13:31–43, 1963.
- [Sha62] David Shale. Linear symmetries of free boson fields. *Trans. Amer. Math. Soc.*, 103:149–167, 1962.