BARSOTTI–TATE GROUPS AND DIEUDONNÉ CRYSTALS

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Forewords

This is an English translation of Grothendieck's *Groupes de Barsotti–Tate et Cristaux de Dieudonné*, published in 1974 by Les Presses de l'Université de Montréal in their *Séminaire de Mathématiques Supérieures* sequence. While the French original is actually quite legible even to an anglophone, I find it preferable to be able to read this interesting lay-of-the-land document without even the minor distraction of having to mentally pair the cognates "cohomologie" and "cohomology". I hope that this translation encourages a wider audience to become familiar with its contents, rather than (as was the case for me) skimming the main for the biggest ideas and relegating the rest to a rainy day.

I've taken some liberties with the prose, but I've done my best to preserve the letter of the mathematics. Corrections are warmly welcomed at `peterson.eric.c@gmail.com`.

———Eric Peterson

Avertissement

These notes do not include certain material treated by Prof. Grothendieck in his 1970 summer Montreal course. We hope to include in a later edition those materials not published here.

Among the items omitted here, the first is a chapter intended to cover F–crystals, which would have been inserted between the final two. As a compromise, one will instead find a letter from Grothendieck to Barsotti in an appendix at the end of the notes which covers the intended applications of F–crystals. One may also consult the notes of Demazure on p–divisible groups [Dem72], here called Barsotti–Tate groups. The second omission is two other chapters which were to sit at the end of the text and which were to cover the definition of generalized Dieudonné functors and their relationship with the classical functors. We will instead refer the reader to the article, presently in preparation, of Mazur and Messing [MM74].

The existence of these notes is primarily due to the efforts of Monique Hakim and Jean-Pierre Delale, who drafted the majority of the chapters. We sincerely thank them. We must also mention that they are not the ones responsible for our incapacity to reproduce all the materials outlined in the course; we hope nonetheless that these notes, even if incomplete, will give to readers an introduction to the theory of crystals and to Barsotti–Tate groups.

Finally, we thank Ms. Thérèse Fournier for her excellent work in typing these notes.

———J.P. Labute, October 1973

Contents

1. Preliminaries on Witt vectors

To lead off, we describe a variety of results stemming from the study of Witt vectors which will either be of specific use to us later on or which serve as motifs which we will emulate. Throughout this section, p will denote a fixed prime number.

Chapitre I
Preliminaires sur Witt

1.1. **Reminders on Witt vectors.** The scheme $\mathbb{W}$ of Witt vectors and the scheme $\mathbb{W}_n$ of Witt vectors truncated to order n are certain ring-schemes, each affine over $\mathbb{Z}$. To give their definitions, let $n \geq 1$ be an integer and let $\mathbb{E}^n = \operatorname{Spec} \mathbb{Z}[T_1, T_2, \ldots, T_n]$ be affine space of dimension n over $\mathbb{Z}$. The scheme $\mathbb{E}^n$ represents the functor $A \mapsto A^n$ in the category of affine schemes in the category of sets. It is thus endowed with a natural structure of a scheme in rings. We will denote $\mathscr{O}^n$ the scheme $\mathbb{E}^n$ endowed with this structure. One then proves [Ser62, Section II.6] that there exists a unique ring scheme structure on $\mathbb{E}^n$ such that the morphism of schemes

1. Rappels sur les vecteurs de Witt

$$\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n) \colon \mathbb{E}^n \to \mathscr{O}^n,$$
$$\varphi_i \colon (x_1, x_2, \ldots, x_n) \mapsto \sum_{j=1}^{i} p^{j-1} x_j^{p^{i-j}}$$

is a homomorphism of rings.

Definition 1.1.1. $\mathbb{W}_n$ is the scheme $\mathbb{E}_n$ endowed with this ring scheme structure.

This definition of the ring (resp. group) structure comes with a universality property: a map $f \colon X \to \mathbb{W}_n$ is a morphism of ring (resp. group) schemes if and only if for every $1 \leq i \leq n$, the map $\varphi_i \circ f \colon X \to \mathscr{O}$ is a homomorphism of rings (resp. of groups). This entails that the restriction morphisms

$$R_n \colon \mathbb{W}_{n+1} \to \mathbb{W}_n$$
$$(x_1, \ldots, x_n, x_{n+1}) \mapsto (x_1, \ldots, x_n)$$

are morphisms of ring schemes, since $\varphi_i R_n = \varphi_i$.

Definition 1.1.2. The ring scheme of Witt vectors is given by

$$\mathbb{W} = \lim \Big(\cdots \to \mathbb{W}_{n+1} \xrightarrow{R_n} \mathbb{W}_n \to \cdots \Big).$$

The scheme underlying $\mathbb{W}$ is, in particular, isomorphic to $\mathbb{E}^{\mathbb{N}}$.

We define two families of maps:

2. Quelques morphismes remarquables

$$V = V_n \colon \mathbb{W}_n \to \mathbb{W}_n, \qquad T = T_n \colon \mathbb{W}_n \to \mathbb{W}_{n+1},$$
$$(x_1, x_2, \ldots, x_n) \mapsto (0, x_1, \ldots, x_{n-1}), \qquad (x_1, x_2, \ldots, x_n) \mapsto (0, x_1, \ldots, x_n).$$

2.1

The morphism T is additive, since $\varphi_t T = p\varphi_{i-1}$ for $1 \leq i \leq n+1$. It follows then that $V_n = R_n T_n = T_{n-1} R_{n-1}$ is also an additive morphism.

Definition 1.1.3. Passing to the limit, the morphisms V_n define an additive morphism

$$V \colon \mathbb{W} \to \mathbb{W}$$

called *Verschiebung*.[1]

Definition 1.1.4. The *Frobenius* $F\colon \mathbb{W} \to \mathbb{W}$ is defined as the limit of the morphisms 2.2

$$F_n\colon \mathbb{W}_n \to \mathbb{W}_n,$$
$$(x_1, x_2, \ldots, x_n) \mapsto (x_1^p, x_2^p, \ldots, x_n^p).$$

The Frobenius is generally not even additive, but after reducing to characteristic p it defines a ring homomorphism

$$F_{\mathbb{F}_p}\colon \mathbb{W}_{\mathbb{F}_p} \to \mathbb{W}_{\mathbb{F}_p},$$

where $\mathbb{F}_p$ denotes the prime field $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{W}_{\mathbb{F}_p}$ denotes the scheme $\mathbb{W}$ base-changed to $\mathbb{F}_p$.

We may use these morphisms to study the two composition laws on $\mathbb{W}$:

$$(x_1, x_2, \ldots, x_n, \ldots) + (y_1, y_2, \ldots, y_n, \ldots) = (S_1, S_2, \ldots, S_n, \ldots),$$
$$(x_1, x_2, \ldots, x_n, \ldots) \cdot (y_1, y_2, \ldots, y_n, \ldots) = (P_1, P_2, \ldots, P_n, \ldots),$$

where S_i and P_i are integer-coefficient polynomials in the indeterminates $x_1, x_2, \ldots, x_i$, and $y_1, y_2, \ldots, y_i$. Using the fact that $F_{\mathbb{F}_p}$ is a ring morphism, for all $\mathbb{F}_p$–algebras we then have

$$P(x^p, y^p) = [P(x, y)]^p.$$

Proposition 2.3

Proposition 1.1.5 ([Ser62, Section II.6, Corollary of Theorem 7]). *In characteristic p, one has the relations on $\mathbb{W}_{\mathbb{F}_p}$:*

$$F \circ V = V \circ F = p.$$

Proof. Set

$$h = (h_1, h_2, \ldots, h_n, \ldots) = (p - VF)(x_1, \ldots, x_n, \ldots).$$

We know $h_i \in \mathbb{Z}[x_1, \ldots, x_i]$, and it suffices to show that h_i is divisible by p for all i to prove the proposition. We calculate

$$\varphi_i(h) = p\varphi_i(x) - \varphi_i(VF(x)) = p^i x_i,$$

from which it follows first that $h_1 = px_1$, then that

$$\varphi_2(h) = h_1^p + ph_2 = p^2 x^2$$

where $h_2 = p(x_2 - p^{p-2} x_1)$. Continuing in this manner and inducting on i, one may deduce that for all i one has $h_i = pk_i$. □

Remark 1.1.6. From this it follows that $p = (0, 1, 0, \ldots)$ in $\mathbb{W}_{\mathbb{F}_p}$.

3. L'ind-schéma $\underset{\rightarrow}{\mathbb{W}}$ et l'ind-schéma $\underset{\rightarrow}{\mathbb{W}'}$

1.2. **The ind-schemes $\widehat{\mathbb{W}}$ and $\widehat{\mathbb{W}}'$.** In addition to $\mathbb{W}$, we may also make the following dual definition:

Definition 1.2.1. Set $\widehat{\mathbb{W}}$ to be the group ind-scheme

$$\widehat{\mathbb{W}} = \operatorname{colim}\Big(\cdots \to \mathbb{W}_n \xrightarrow{T_n} \mathbb{W}_{n+1} \to \cdots\Big).$$

[1]This is also known as *the décalage morphism*.

Remark 1.2.2. The underlying ind-scheme in sets is isomorphic to $\mathbb{E}^{(\mathbb{N})}$: for all rings A,

$$\widehat{\mathbb{W}}(A) = A^{(\mathbb{N})} = \{(x_1, x_2, \ldots, x_n, \ldots) \in A^{\mathbb{N}} \mid x_i = 0 \text{ except for finitely many indices}\}.$$

However, the additive structure of $\widehat{\mathbb{W}}$ is not the usual structure on $\mathbb{E}^{(\mathbb{N})}$.

There is a second inductive system given by interleaving the above with the Frobenius:

$$\cdots \to \mathbb{W}_n \xrightarrow{T_n F_n} \mathbb{W}_{n+1} \to \cdots.$$

Using the identity $T_n F_n = F_{n+1} T_n$, we find the two systems to be related by

$$\begin{tikzcd} \cdots \arrow[r] & \mathbb{W}_n \arrow[r, "T_n"] \arrow[d, "F_n^{n-1}"] & \mathbb{W}_{n+1} \arrow[r] \arrow[d, "F_{n+1}^{n}"] & \cdots \\ \cdots \arrow[r] & \mathbb{W}_n \arrow[r, "T_n F_n"] & \mathbb{W}_{n+1} \arrow[r] & \cdots. \end{tikzcd}$$

Definition 1.2.3. Set $\widehat{\mathbb{W}}'$ to be the ind-scheme

$$\widehat{\mathbb{W}}' = \operatorname{colim}\left(\cdots \to \mathbb{W}_n \xrightarrow{T_n F_n} \mathbb{W}_{n+1} \to \cdots\right).$$

Remark 1.2.4. In characteristic 0, $\widehat{\mathbb{W}}'$ does not have a natural group structure, but in characteristic p, $\widehat{\mathbb{W}}'$ is an ind-scheme in groups, and there is an additive homomorphism

$$u \colon \widehat{\mathbb{W}}_{\mathbb{F}_p} \to \widehat{\mathbb{W}}'_{\mathbb{F}_p}$$

induced by the above map of systems. Note further that in characteristic p, the transition morphisms TF in this second system are actually multiplication by p, since

$$TFR = FTR = FV = p.$$

Remark 1.2.5. If A is a perfect ring of characteristic p, then u induces an *isomorphism*

$$u(A) \colon \widehat{\mathbb{W}}_{\mathbb{F}_p}(A) \to \widehat{\mathbb{W}}'_{\mathbb{F}_p}(A)$$

because the same is true for each $F_n(A)$.

1.3. **The ring W.**

4. L'anneau W

4.1

In the case of a perfect field k of positive characteristic p, the k-points of $\mathbb{W}$ are of particular interest. In this setting, the ring

$$W = \mathbb{W}(k) = \lim_n W_n = \lim_n \mathbb{W}_n(k)$$

is a complete discrete valuation ring with residue field k and uniformizer $p = (0, 1, 0, 0, \ldots)$, and there is an augmentation homomorphism given by the canonical projection $W \to W_1 = k$ [Ser62, Theorem II.6.7]. One may show that these properties characterize the ring W up to unique isomorphism as the initial such ring [Ser62, Theorem II.5.3].[2] Additionally, its fraction field $K = W[1/p]$ is of characteristic zero.

The k-points of the dual construction are also of interest. Since k is perfect, the homomorphism u defines an isomorphism of groups

$$u(k) \colon \widehat{\mathbb{W}}(k) \to \widehat{\mathbb{W}}'(k).$$

[2]This result is sometimes known as "Cohen's theorem".

The natural projection $\pi_n\colon W \to W_n$ then defines an isomorphism

$$\varepsilon_n\colon W/p^n W \xrightarrow{\sim} W_n,$$

and these participate in a commutative diagram

$$\begin{tikzcd}
W \arrow[rrr, "p=VF"] \arrow[dr] \arrow[dd, "\pi_n"'] & & & W \arrow[dl] \arrow[dd, "\pi_{n+1}"] \\
& W/p^n W \arrow[r, "p"] \arrow[dl, "\sim"', "\varepsilon_n"] & W/p^{n+1} W \arrow[dr, "\sim", "\varepsilon_{n+1}"'] & \\
W_n \arrow[rrr, "TF=FT"] & & & W_{n+1}.
\end{tikzcd}$$

Using the natural isomorphism of W–modules

$$\operatorname{colim}\left(\cdots \to W/p^n W \xrightarrow{p} W/p^{n+1} W \to \cdots\right) = K/W,$$

one deduces that $\widehat{\mathbb{W}}'(k)$ is isomorphic to the dualizing module K/W. Since K/W is a W–module and u is an isomorphism, we may inherit a W–module structure on $\widehat{W}(k)$ by transport of structure.

Proposition 4.2

Proposition 1.3.1. *This extends uniquely to an action of the ring of operators W on the ind-scheme in groups*

$$\widehat{\mathbb{W}}_k = \operatorname{colim}\left(\cdots \to (\mathbb{W}_n)_k \xrightarrow{T} (\mathbb{W}_{n+1})_k \to \cdots\right)$$

by letting $\lambda \in W$ act on $(\mathbb{W}_n)_k$ by multiplication by $F^{1-n}(\lambda)$.

Proof. First, note that the element $F^{1-n}(\lambda) \in W$ is well defined: since k is perfect, $F\colon W \to W$ is an isomorphism. Then, for a k–algebra A, we use the structure map $k \to A$ to induce a map

$$W = \mathbb{W}(k) \xrightarrow{\pi_n} \mathbb{W}_n(k) \to \mathbb{W}_n(A)$$

which allows us to interpret "multiplication by $F^{1-n}(\lambda)$" on $\mathbb{W}_n(A)$, if not yet on $\widehat{\mathbb{W}}_k(A)$.

In order to extend this action to $\widehat{\mathbb{W}}_k$, we must show that the action of $\lambda \in W$ commutes with the morphisms T in the defining system. It suffices to show for $Vx \in \mathbb{W}_n(A)$ and $V\lambda' \in \mathbb{W}_{n+1}(A)$ that

$$\lambda' T(x) = T(FR(\lambda')x)$$

or, using the fact that R is a surjection, for $V\lambda', y \in \mathbb{W}_{n+1}(A)$ that

$$\lambda' V(y) = V(F(\lambda')y).$$

Consider first the case that A is a perfect ring. In this setting, F is an isomorphism, and so this relation follows from

$$F(\lambda' V(y)) = F(\lambda')FV(y) = F(\lambda')py = p[F(\lambda')y] = FV[F(\lambda')y].$$

In the general case, note that $\lambda' V(y)$ and $V(F(\lambda')y)$ may be expressed using polynomials with coefficients in $\mathbb{Z}/p\mathbb{Z}$. If the variables take values in a perfect ring (e.g., the algebraic closure of k), these polynomials evaluate the same, and from this we deduce that they are equal in general.

Finally, we must show that this action on $\widehat{\mathbb{W}}_k$ agrees with the previously defined action on $\widehat{\mathbb{W}}(k)$. One simply traces this through: using the isomorphisms

$$W_n \xrightarrow{F^{n-1}} W_n \xleftarrow{\varepsilon_n} W/p^n W,$$

multiplication by $F^{1-n}(\lambda)$ on $\mathbb{W}_n(k) = W_n$ induces multiplication by λ on $W/p^n W$. This additionally proves the unicity of the structure on $\widehat{\mathbb{W}}_k$. □

Remark 1.3.2. In general, there is not an action of the ring-scheme $\mathbb{W}_k$ on $\widehat{\mathbb{W}}_k$, only of the ring W.

Remark 1.3.3. This construction is functorial in k and is valid for all perfect rings A (but $\mathbb{W}(A)$ is not in general a discrete valuation ring).

1.4. **The Frobenius and Verschiebung of a group scheme.** In this section, we will suppose that all schemes are lie over the prime field $\mathbb{F}_p$. 5. Les morphismes de Frobenius et de Verschiebung d'un schéma en groupes

Definition 1.4.1. For X an $\mathbb{F}_p$–scheme, we define the *absolute Frobenius on X*

$$\mathbf{f}_X : X \to X$$

to be the endomorphism of X which is the identity on the underlying space and which sends a section $s \in \mathcal{O}_X$ to the section s^p. 5.1

Remark 1.4.2. For Y another scheme and $g : X \to Y$ a morphism, the following square commutes:

$$\begin{tikzcd} X \arrow[r, "\mathbf{f}_X"] \arrow[d, "g"] & X \arrow[d, "g"] \\ Y \arrow[r, "\mathbf{f}_Y"] & Y. \end{tikzcd}$$

One may thus think of $\mathbf{f}$ as an endomorphism of the identity functor on $\mathsf{Schemes}_{\mathbb{F}_p}$.

Definition 1.4.3. We may also consider a relative version of this construction by taking S to be a base $\mathbb{F}_p$–scheme and X to be an S–scheme. Denote by $X^{(p/S)}$, or simply by $X^{(p)}$, the S–scheme given by the inverse image of X along the base-change map induced by $\mathbf{f}_S : S \to S$. Using the diagram above, there exists a unique arrow $\mathbf{f}_{X/S}$ such that the following diagram commutes: 5.2

$$\begin{tikzcd} X \arrow[dr, dashed, "\mathbf{f}_{X/S}"] \arrow[drr, "\mathbf{f}_X"] \arrow[ddr] & & \\ & X^{(p)} \arrow[r] \arrow[d] & X \arrow[d] \\ & S \arrow[r, "\mathbf{f}_S"] & S. \end{tikzcd}$$

We name the morphism $\mathbf{f}_{X/S}$ the *relative Frobenius of X over S*.

Remark 1.4.4. The assignment $X \mapsto \mathbf{f}_{X/S}$ determines a natural transformation from the identity functor on $\mathsf{Schemes}_S$ to the functor $X \mapsto X^{(p/S)}$. As this latter functor commutes with finite projective limits, if X is an S–group then so is $X^{(p/S)}$, and $\mathbf{f}_{X/S}$ is additionally a group morphism.

Remark 1.4.5. The construction of $X^{(p/S)}$ commutes with change of base: for all morphisms $T \to S$, we have

$$(X \times_S T)^{(p/T)} \cong X^{(p/S)} \times_S T.$$

If S is the spectrum of the prime field $\mathbb{F}_p$, then we recover the absolute notions: $X^{(p/S)} = X$ and $\mathbf{f}_{X/S} = \mathbf{f}_X$.

5.3 **Definition 1.4.6** ([SGA72, VII$_A$, 4.2–3])**.** Let G be a *flat* commutative group scheme. It is known that, functorially in G, there exists a canonical morphism of groups $\mathbf{v}_G \colon G^{(p/S)} \to G$ called *the Verschiebung homomorphism* of G over S. This construction has the following properties:

$$\mathbf{f}_{G/S} \circ \mathbf{v}_G = p, \qquad \mathbf{v}_G \circ \mathbf{f}_{G/S} = p.$$

Remark 1.4.7. In many cases, the second formula suffices for calculating $\mathbf{v}_G$, as we now describe. If G is smooth, then $\mathbf{f}_{G/S}$ is a epimorphism. Moreover, if H embeds in a smooth group G, $\mathbf{v}_H$ is also determined because it is induced by $\mathbf{v}_G$. The majority of groups which we will consider are finite and locally free over the base, and we will show later that they embed canonically into a smooth group, and hence we may use the explicit presentation of $\mathbf{f}_{G/S}$ to calculate $\mathbf{v}_G$.

Remark 1.4.8. Not only is the morphism $\mathbf{v}_G$ functorial in G, but it also commutes with change of base. It also specializes to a previously discussed concept: for $S = \mathrm{Spec}(\mathbb{F}_p)$ and G is the group $\mathbb{W}$ of Witt vectors, the Frobenius and Verschiebung morphisms coincide with those given in Section 1.1.

5.4 Before moving on, we also consider the effect of iterating these constructions. For every $n \geq 1$, there are schemes

$$X^{(p^n)} = X^{(p^n/S)} = [X^{(p^{n-1}/S)}]^{(p/S)},$$

which come equipped with an n^{th} *iterated relative Frobenius*

$$\mathbf{f}^n_{X/S} \colon X \xrightarrow{\mathbf{f}_{X/S}} X^{(p)} \xrightarrow{\mathbf{f}_{X^{(p)}/S}} X^{(p^2)} \to \cdots \xrightarrow{\mathbf{f}_{X^{(p^{n-1}/S)}}} X^{(p^n)}.$$

Similarly, if G is a flat commutative group scheme over S, we set the n^{th} *iterated Verschiebung* $\mathbf{v}^n_G$ to be the composite

$$\mathbf{v}^n_G \colon G^{(p^n)} \xrightarrow{\mathbf{v}_{G^{(p^{n-1})}}} G^{(p^{n-1})} \to \cdots \xrightarrow{\mathbf{v}_G} G.$$

We then have the following analog of the above relations:

$$\mathbf{v}^n_G \circ \mathbf{f}^n_{G/S} = p^n, \qquad \mathbf{f}^n_{G/S} \circ \mathbf{v}^n_G = p^n.$$

In the non-relative case where the base S is the spectrum of the prime field $\mathbb{F}_p$, these are the iterated composites of $\mathbf{f}_{G/S}$ and $\mathbf{v}_G$ in the usual sense.

2. Locally free finite groups and classical Dieudonné theory

II. Groupes finis localement libres et theorie classique de Dieudonné

In all that follows, all groups are commutative unless otherwise stated.

2.1. **General reminders on locally free finite groups.** Select a base scheme S, and let G be a finite and locally free group scheme over S. Concretely, the scheme G may be presented as $G = \operatorname{Spec} A$, where A is a finite, locally free, quasicoherent $\mathcal{O}_S$–algebra, and

1. Rappels généraux sur les groupes finis localement libres

the group structure of G is described on A by homomorphisms of $\mathcal{O}_S$–algebras

$$\Delta\colon A \to A \otimes_{\mathcal{O}_S} A, \qquad I\colon A \to A, \qquad \varepsilon\colon A \to \mathcal{O}_S$$

where Δ, I, and ε respectively correspond to the composition law, the inversion, and to

1.1 the unit section. These morphisms satisfy various relations expressing assocativity and so on. For example, these relations show that Δ intertwines with the algebra structure of A to form a bialgebra. As a matter of terminology, the kernel J of ε is called the augmentation ideal of the group G.

Definition 2.1.1. For $s \in S$, we will refer to the rank of $A \otimes_{\underline{O}_S} k(s)$ over the residue field $k(s)$ as the *rank* of G over s. Considered as a function of s, the rank of G is locally constant. If the rank is *actually* constant and equal to n, we say that G is a group of rank n over S. 1.2

1.3. Exemples de groupes finis localement libres

Example 2.1.2. Given an (abstract) group G of order n, we define $\underline{G}_n$ to be the constant sheaf on S whose S'–sections for S' an S–scheme is given by

$$\underline{G}_S(S') = G.$$

This sheaf is in fact a scheme, and $\underline{G}_S$ is moreover a free group scheme of rank n over S whose affine algebra is isomorphic to $\mathcal{O}_S^n$. 1.3.1

Example 2.1.3. Let $\mathbb{G}_m|_S$ be the multiplicative group over S, defined for any S–scheme S' by 1.3.2

$$\mathbb{G}_m|_S(S') = \Gamma(S', \mathcal{O}_{S'})^*.$$

For each $n \geq 1$, there is a homomorphism

$$\begin{aligned} n\colon \mathbb{G}_m|_S &\to \mathbb{G}_m|_S, \\ x &\mapsto x^n. \end{aligned}$$

We define $\mu_n|_S$, the group of n^{th} roots of unity over S, to be the kernel of this homomorphism. This is a finite free group of rank n over S whose affine algebra is isomorphic to $\mathcal{O}_S[T]/(T^n - 1)$.

Example 2.1.4. Let $\mathbb{G}_a|_S$ be the additive group over S, defined for an S–scheme S' by 1.3.3

$$\mathbb{G}_a|_S(S') = \Gamma(S', \mathcal{O}_{S'}).$$

If S is a scheme over the prime field $\mathbb{F}_p$, then there is a Frobenius homomorphism

$$\begin{aligned} F\colon \mathbb{G}_a|_S &\to \mathbb{G}_a|_S, \\ x &\mapsto x^p. \end{aligned}$$

We define $\alpha_p|_S$ to be the kernel of F, which is a finite free group of rank p over S, with affine algebra isomorphic to $\mathcal{O}_S[T]/T^p$.

We now describe a variant of Pontryagin duality for group schemes. 1.4. Dualité de Cartier

Definition 2.1.5. For a commutative group S–scheme G, we define its *Cartier dual* to be

$$G^* = \mathsf{GroupSchemes}_{/S}(G, \mathbb{G}_m|_S).$$

This construction defines a contravariant endofunctor of $\mathsf{GroupSchemes}_{/S}$ which commutes with changes of base.

If $G = \mathrm{Spec}(A)$ is finite and locally free over S, this functor is given by linear duality: $G^* = \mathrm{Spec}(A^\vee)$ with $A^\vee = \mathsf{Modules}_{\mathscr{O}_S}(A, \mathscr{O}_S)$, where the bialgebra structure on $A^\vee$ is defined using the transposes of the morphisms Δ and μ of Section 2.1 [SGA72, VII$_A$.3.3.1]. It follows that G^* is also a finite locally free group over S and of the same rank as G. Moreover, in this same setting the isomorphism $A \cong (A^\vee)^\vee$ induces an isomorphism

$$G \xrightarrow{\sim} (G^*)^*,$$

from which it follows that Cartier duality restricts to an antiequivalence of the category of finite locally free groups in S–schemes with itself.

Remark 2.1.6 ([SGA72, VII$_A$.4.3.3]). When S is an $\mathbb{F}_p$–scheme, we defined in Section 1.4 the group scheme $G^{(p/S)}$, the relative Frobenius morphism $\mathbf{f}_{G/S}$, and the Verschiebung $\mathbf{v}_G$. If G is finite locally free over S, then $G^{(p/S)}$ is as well, and one may then show that Cartier duality exchanges the relative Frobenius and Verschiebung morphisms:

$$(\mathbf{v}_G)^* = \mathbf{f}_{G^*/S}, \qquad (\mathbf{f}_{G/S})^* = \mathbf{v}_{G^*}.$$

1.5. Exemples

Example 2.1.7. Let $\underline{G}_S$ be the finite group over S attached to a finite (abstract) group S as in Example 2.1.3. One verifies easily that the affine algebra of $(\underline{G}_S)^*$ is isomorphic to $\mathscr{O}_S[G]$, the group–algebra of G over $\mathscr{O}_S$, with diagonal induced by the diagonal function $G \to G \times G$. Picking $G = \mathbb{Z}/n\mathbb{Z}$, we may explicitly compute

$$(\underline{\mathbb{Z}/n\mathbb{Z}})^*_S = \mu_n|S$$

at the level of affine algebras, though this also follows directly from the definitions. In characteristic p, one may moreover show

$$(\alpha_p|_S)^* = \alpha_p|_S.$$

1.6. Quotients de groupes finis localement libres

Definition 2.1.8 ([SGA72, V.4.1]). Let $\mu\colon G' \to G$ be a monomorphism of finite locally free group schemes over S, and let $G'' = G/G'$ be the quotient sheaf for the fppf topology, so that there is an exact sequence of sheaves

$$0 \to G' \xrightarrow{\mu} G \xrightarrow{\pi} G'' \to 0.$$

One may prove that G'' is itself representable by a finite locally free group scheme over S, giving rise to a notion of *quotient group scheme*. The morphism π can be shown to be faithfully flat, from which it follows that G is finite and locally free over G'', of relative rank equal to the rank of G'. In terms of absolute ranks, this gives the equality

$$\mathrm{rank}(G) = \mathrm{rank}(G') \cdot \mathrm{rank}(G'').$$

Remark 2.1.9. Suppose that we have an exact sequence of sheaves as above and that G' and G'' are representable by finite locally free group schemes over S. It can be shown that G is then also so-representable, i.e., an extension of finite locally free groups is again finite and locally free.

1.7 **Definition 2.1.10.** For ℓ a prime number, the *ℓ–primary component* of a group G over S is the largest subgroup of ℓ–torsion in G:

$$G(\ell) = \operatorname{colim}\left(\cdots \to G \xrightarrow{\ell^n} G \to \cdots\right).$$

Remark 2.1.11 ([SGA72, VIII.7.3]). If G is finite and locally free over S, one may prove that G is locally annihilated by an integer n. It follows that each of its ℓ–primary components is locally annihilated by a power of ℓ, i.e., $G(\ell)$ is an ℓ–group, and that G admits a canonical decomposition (as an fppf sheaf) into its ℓ–primary components:

$$G = \coprod_{\ell} G(\ell).$$

It follows that each $G(\ell)$ is finite and locally free, since it appears as a direct factor in such a group. If S is quasicompact, G is then annihilated by a fixed integer n, and this decomposition is finite.

In the remainder of these lectures, we will primarily concern ourselves with groups for which we have $G = G(p)$, i.e., finite locally free groups over S which are locally annihilated by a power of p. Note that if G is such a p–group, its dual G^* is also a p–group.

2.2. **Particular types of groups.** Let G be a finite locally free group over any base S. 2. Types particuliers de groups

Definition 2.2.1. We set out the following various subclasses of groups.

One says that G is *infinitesimal*[3] (over S) if any of the following conditions is met: 2.1

- The augmentation ideal J of G is locally nilpotent. 2.1.1
- For all $s \in S$, G_s is connected.
- For all $s \in S$, G_s does not contain an isolated point.
- For all $s \in S$, $G_{\bar{s}}$ does not contain an isolated point.[4]

One says that G is *étale* over S if any of the following conditions is met: 2.1.2

- The structure morphism $G \to S$ is étale.
- For all $s \in S$, G_s is étale over the residue field $k(s)$.
- For all $s \in S$, $G_{\bar{s}}$ is reduced.
- G is locally (for either the étale topology or the fppf topology) isomorphic to a constant group.

One says that G is *unipotent* if G^* is infinitesimal.[5] 2.1.3

One says that G is *of multiplicative type* over S if any of the following conditions is met: 2.1.4

- G^* is étale.
- For all $s \in S$, G_s is of multiplicative type.
- Locally in the étale topology, G embeds into $(\mathbb{G}_m|_S)^r$.

One says that G is *bi-infinitesimal* over S if it is both infinitesimal and unipotent (i.e., both G and G^* are infinitesimal). 2.1.5

Proposition 2.2.2. *Suppose that the base scheme S be an $\mathbb{F}_p$–scheme, that G is étale, and that G is locally annihilated by a power of p. The dual group G^* is then infinitesimal.* 2.2

[3] Some authors call this property *radical* instead.

[4] Here $G_{\bar{s}}$ denotes the geometric fiber of G over s.

[5] See [SGA72, XVII] for a general definition of a unipotent group.

Proof. Because G is étale, the fiber $G_{\overline{s}}$ over a point $s \in S$ is isomorphic to the constant group $G(\overline{k})_{\overline{k}}$, where $\overline{k}$ denotes the algebraic closure of the residue field at s. Following Example 2.1.7, the affine algebra of $(G_{\overline{s}})^* = (G^*)_{\overline{s}}$ may be identified with the algebra $\overline{k}[G(\overline{k})]$ of the group $G(\overline{k})$. Using this presentation, the fact that $G(\overline{k})$ is annihilated by a power of p, and the fact that $\overline{k}$ is of characteristic p, one verifies that the ideal I given by

$$I = \left\langle \sum_{g \in G(\overline{k})} \lambda_g g \,\middle|\, \sum_g \lambda_g = 0 \right\rangle \leq \overline{k}[G(\overline{k})]$$

is maximal and nilpotent. This shows $(G^*)_{\overline{s}}$ to be reduced at a point, hence G^* is infinitesimal. □

Corollary 2.2.3. *Over a base of characteristic p, all p–groups of multiplicative type are infinitesimal.* □

2.3 *Remark* 2.2.4. For an exact sequence of finite locally free group schemes:

$$0 \to G' \xrightarrow{\mu} G \xrightarrow{\pi} G'' \to 0,$$

G is infinitesimal (resp. étale, resp. unipotent, resp. of multiplicative type, resp. bi-infinitesimal) if and only if G' and G'' are infinitesimal (resp. étale, resp. unipotent, resp. of multiplicative type, resp. bi-infinitesimal). By reducing to the case where S is the spectrum of an algebraically closed field, one easily checks this assertion in the infinitesimal and étale cases; the others then follow by Cartier duality.

2.4. Exemples de p–groupes en caractéristique p

Example 2.2.5. The group scheme is $(\underline{\mathbb{Z}/p\mathbb{Z}})_S$ is étale over S, and $\mu_p|_S = (\mathbb{Z}/p\mathbb{Z})^*_S$ is thus of multiplicative type. On the other hand, the self-dual group $\alpha_p|_S = (\alpha_p|_S)^*$ is bi-infinitesimal. Of these three groups, $\alpha_p|_S$ and $(\underline{\mathbb{Z}/p\mathbb{Z}})_S$ are the unipotent groups, whereas the infinitesimal groups are $\alpha_p|_S$ and $\mu_p|_S$.

Our interest in these three particular examples stems from the following result:

Proposition 2.2.6. *Let S be the spectrum of an algebraically closed field of characteristic p, and let G be a finite group scheme over S which is annihilated by a power of p. There is then a finite composition series*

$$G \supset G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_{n-1} \supset G_n = 0$$

where each stratum G_i/G_{i+1} is isomorphic to one of the groups α_p, μ_p, or $\underline{\mathbb{Z}/p\mathbb{Z}}$. □

Corollary 2.2.7 ([SGA72, XVII.1.7 and 4.2.1])**.** *A group scheme G in the above setting is infinitesimal (resp. étale, resp. unipotent, resp. of multiplicative type, resp. bi-infinitesimal) if and only if each of the factors G_i/G_{i+1} in the above composition series is isomorphic to α_p or μ_p (resp. $\underline{\mathbb{Z}/p\mathbb{Z}}$, resp. $\underline{\mathbb{Z}/p\mathbb{Z}}$ or α_p, resp. μ_p, resp. α_p).* □

While useful, the composition series is not the only way to determine these types. They may also be read off from the relative Frobenius and Verschiebung as follows:

2.5 **Theorem 2.2.8.** *Let S be an $\mathbb{F}_p$–scheme, and let G be a finite locally free group over S with relative Frobenius and Verschiebung maps $\mathbf{f}_{G/S}$ and $\mathbf{v}_G$ (cf. Section 1.4). Then:*

2.5.1–5

(1) *G is infinitesimal if and only if $\mathbf{f}_{G/S}$ is locally nilpotent.*

(2) *G is unipotent if and only if $\mathbf{v}_G$ is locally nilpotent.*
(3) *G is bi-infinitesimal if and only if both $\mathbf{f}_{G/S}$ and $\mathbf{v}_G$ are locally nilpotent.*
(4) *G is étale if and only if $\mathbf{f}_{G/S}$ is an isomorphism.*
(5) *G is of multiplicative type if and only if $\mathbf{v}_G$ is an isomorphism.*

Proof (sketch). It suffices to prove the interpretations for infinitesimal and étale, since the others may then be deduced by Cartier duality.

Infinitesimal: We begin by considering G as a pointed scheme via its unit section $\varepsilon\colon S \to G$. The "kernel" of $\mathbf{f}^n_{G/S}$ (i.e., the fiber of $\mathbf{f}^n_{G/S}$ over ε) is defined by the ideal J_n generated by the image of the augmentation ideal J through the morphism $F^n\colon x \mapsto x^{p^n}$. Since G is of finite type, we may then suppose that J has fewer than p^k generators for some k, hence

$$J^{p^{n+k}} \subset J_n.$$

The J_n then form a cofinal subsystem of J^{p^n}, and we deduce that $\mathbf{f}_{G/S}$ is locally nilpotent if and only if J is locally nilpotent. For a finite locally free group over S, this is equivalent to being infinitesimal.

Étale: One may show in a general manner [SGA72, XIV] that if G is of locally finite presentation over S, then $\mathbf{f}_{G/S}$ is an isomorphism if and only if G is étale. □

2.3. Decomposition of a finite locally free p–group over a point s of char. p.

3. Décomposition d'un p–groupe fini localement libre sur une base de caractéristique p réduite à un point s

Definition 2.3.1. Functorially in finite locally free groups G over S, there is a "connected-étale" exact sequence:

$$0 \to G^\circ \to G \to G^{\text{ét}} \to 0.$$

The group scheme G° is the connected component of G, and $G^{\text{ét}}$ is defined by $G^{\text{ét}} = G/G^\circ$. Since the subgroup G° is open in G [SGA72, VI_B.3.9], it is thus finite and locally free over S, and it in fact is the largest infinitesimal subgroup of G. The quotient group $G^{\text{ét}}$ is étale [SGA72, VI_A.5.5], and it is finite and locally free over S (cf. Definition 2.1.8). Applying the same construction to G^*, one obtains a subgroup $G^{\text{mult}} = (G^*/(G^*)^\circ)^*$, which is the largest subgroup of multiplicative type of G. 3.1

Remark 2.3.2. If $k(s)$ is of characteristic p and if G is annihilated by a power of p, then Proposition 2.2.2 shows G^{mult} to be infinitesimal, and we therefore have inclusions

$$0 \subset G^{\text{mult}} \subset G^\circ \subset G,$$

where $G/G^\circ = G^{\text{ét}}$ is étale and where G°/G^{mult} is bi-infinitesimal (because it is a quotient of an infinitesimal group and its dual is identified with $((G^\circ)^*)^\circ$). This composition series is functorial in G, and it commutes with finite products.

Remark 2.3.3. If S is the spectrum of a perfect field of characteristic p, the canonical projection $G \to G^{\text{ét}}$ induces an isomorphism $G_{\text{red}} \to G^{\text{ét}}$ and the connected-étale exact sequence therefore splits canonically [SGA72, XVII.1.6], i.e., 3.2

$$G = G^\circ \times G^{\text{ét}}.$$

By the same token, the exact sequence

$$0 \to G^{\text{mult}} \to G^\circ \to G^\circ/G^{\text{mult}} \to 0$$

is split because the dual is. One thus obtains a canonical decomposition, functorial in G:

$$G = G^{\mathrm{mult}} \times G^{\mathrm{bi}} \times G^{\mathrm{\acute{e}t}},$$

where G^{mult}, G^{bi}, and $G^{\mathrm{\acute{e}t}}$ are respectively of multiplicative type, bi-infinitesimal, and étale. The unipotent part of G is thus $G^{\mathrm{bi}} \times G^{\mathrm{\acute{e}t}}$ and the infinitesimal part is $G^{\mathrm{mult}} \times G^{\mathrm{bi}}$.

Corollary 2.3.4. *For G a group scheme over S the spectrum of a perfect field of characteristic p, there are the following equivalences:*

- *G is infinitesimal if and only if $G^{\acute{e}t} = 0$.*
- *G is étale if and only if $G^{\mathrm{mult}} = G^{\mathrm{bi}} = 0$.*
- *G is unipotent if and only if $G^{\mathrm{mult}} = 0$.*
- *G is bi-infinitesimal if and only if $G^{\mathrm{mult}} = G^{\acute{e}t} = 0$.*
- *G is of multiplicative type if and only if $G^{\mathrm{bi}} = G^{\acute{e}t} = 0$.* □

3.3

Remark 2.3.5. Over a perfect field, there is a well-known presentation of the category of finite étale p–group schemes as the category of (abstract) finite p–groups on which the fundamental group $\pi_1(k) = \mathrm{Gal}(\overline{k}/k)$ acts [Dem72]. Both sides of this equivalence participate in dualities:

- By Cartier duality, the category of finite p–group schemes of multiplicative type is equivalent to the opposite category of finite étale p–groups.
- By Pontryagin duality, the category of finite p–groups on which $\pi_1(k)$ acts is equivalent to its opposite.

From these, one deduces a new equivalence between the category of finite p–group schemes of multiplicative type and the category of (abstract) finite p–groups on which $\pi_1(k)$ acts.

4. Théorie de Dieudonné sur un corps parfait

2.4. **Dieudonné theory over a perfect field.** Let k be a perfect field of positive characteristic p and let $\mathsf{GroupSchemes}^{p,\mathrm{fin}}_{/k}$ be the category of finite group k–schemes which are annihilated by a power of p. The goal of Dieudonné theory is to establish an antiequivalence between this category and a certain module category which we will now define.

4.1

Definition 2.4.1. The *Dieudonné ring $\mathscr{D}$* is the noncommutative ring $W[F, V]$ generated over the local ring $W = W(k)$ (cf. Section 1.3) by the indeterminates F and V, subject to the following relations:

$$F\lambda = \lambda^{\sigma} F, \qquad \lambda V = V\lambda^{\sigma}, \qquad FV = VF = p,$$

for all $\lambda \in W$ and where λ^{σ} denotes the image of λ under the Frobenius morphism of W.

Definition 2.4.2. The *category of Dieudonné modules* is the category $\mathsf{Modules}_{\mathscr{D};W\,\mathrm{l.f.}}$ of left $\mathscr{D}$–modules such that the underlying W–module is of finite length. In particular, such a module is annihilated by a power of p.

Remark 2.4.3. An object of $\mathsf{Modules}_{\mathscr{D};W\,\mathrm{l.f.}}$ may be equivalently expressed as a W–module M of finite length equipped with two W–linear morphisms

$$F_M \colon M^{\sigma} \to M, \qquad V_M \colon M \to M^{\sigma},$$

where M^{σ} denotes M with the W–module structure first twisted by the Frobenius σ, and where F_M and V_M additionally satisfy

$$F_M V_M = p, \qquad V_M F_M = p.$$

Théorème 4.2

Theorem 2.4.4 (Dieudonné). *There is an equivalence of categories*

$$D^*\colon (\mathsf{GroupSchemes}^{p,\mathrm{fin}}_{/k})^{\mathrm{op}} \xrightarrow{\simeq} \mathsf{Modules}_{\mathscr{D};W\,\mathrm{l.f.}}.$$

This equivalence interacts with the object-level Frobenius by

$$(D^*(G))^\sigma \cong D^*(G^{(p)})$$

and with the Frobenius and Verschibung morphisms by

$$D^*(\mathbf{f}_{G/k}) = F_{D^*(G)}, \qquad D^*(\mathbf{v}_G) = V_{D^*(G)}.$$

Corollary 2.4.5. *One has the following equivalences:* Corollaire 4.2.1

- *G is infinitesimal if and only if F is a nilpotent operator on $D^*(G)$.*
- *G is unipotent if and only if V is a nilpotent operator on $D^*(G)$.*
- *G is bi-infinitesimal if and only if both F and V are nilpotent operators.*
- *G is étale if and only if F is invertible on $D^*(G)$.*
- *G is of multiplicative type if and only if V is invertible on $D^*(G)$.* □

Construction of the functor from Theorem 2.4.4. For a complete proof of Theorem 2.4.4, we refer to Gabriel and Demazure [DG70]; here we merely construct D^* itself. In view of Remark 2.3.3, we observe that the category $\mathsf{GroupSchemes}^{p,\mathrm{fin}}_{/k}$ is equivalent to the product of the full subcategories spanned by those objects which are unipotent and those which are of multiplicative type. It thus suffices to construct D^* separately on these two subcategories.

Construction of D^* on unipotent groups. For every unipotent group over k, we set 4.3. Cas des groupes unipotents

$$D^*(G) = \mathsf{Groups}_{/k}(G, \widehat{\mathbb{W}}).$$

We know that W acts on $\widehat{\mathbb{W}}$ (cf. Proposition 1.3.1) and that $D^*(G)$ is a W–module of finite length. The action of F and V on $D^*(G)$ are defined for $u \in D^*(G)$ by 4.3.1

$$Fu = F_{\widehat{\mathbb{W}}} \circ u, \qquad Vu = V_{\widehat{\mathbb{W}}} \circ u,$$

where $F_{\widehat{\mathbb{W}}}$ and $V_{\widehat{\mathbb{W}}}$ are the Frobenius and Verschiebung endomorphisms of $\widehat{\mathbb{W}}$, defined by passing to the limit of the corresponding morphisms on $\mathbb{W}_n$. By reviewing the definition of the action of W on $\widehat{\mathbb{W}}$ and using the properties of $F_{\widehat{\mathbb{W}}}$ and $V_{\widehat{\mathbb{W}}}$, one finds that these actions make $D^*(G)$ into a $\mathscr{D}$–module.

We must now check that this definition of D^* satisfies the desired properties: there should be an isomorphism 4.3.2

$$D^*(G^{(p)}) \xrightarrow{\sim} (D^*(G))^\sigma$$

such that

$$D^*(\mathbf{f}_{G/k}) = F_{D^*(G)}, \qquad D^*(\mathbf{v}_G) = V_{D^*(G)}.$$

Since k is perfect, change of base along the isomorphism $\sigma\colon k \to k$ induces a σ–linear isomorphism

$$\mathsf{GroupSchemes}(G, \mathbb{W}) \xrightarrow{\sim} \mathsf{GroupSchemes}(G^{(p)}, \mathbb{W}^{(p)}).$$

However, because $\widehat{\mathbb{W}}$ was defined over the prime field $\mathbb{F}_p$ and because the formation of $\widehat{\mathbb{W}}^{(p)}$ commutes with change of base, we also have $\widehat{\mathbb{W}}^{(p)} = \widehat{\mathbb{W}}$. On the other hand,

$$\begin{aligned} D^*(\mathbf{f}_{G/k})\colon \mathsf{GroupSchemes}(G^{(p)}, \widehat{\mathbb{W}}) &\to \mathsf{GroupSchemes}(G, \widehat{\mathbb{W}}) \\ u &\mapsto u \circ \mathbf{f}_{G/k} \end{aligned}$$

is identified with $F_{D^*(G)}$ because $u \circ \mathbf{f}_{G/k} = f_{\widehat{\mathbb{W}}} \circ u$, and $f_{\widehat{\mathbb{W}}}$ is identified with $F_{\widehat{\mathbb{W}}}$ by Remark 1.4.8. The assertion for $D^*(\mathbf{v}_G)$ is established similarly. □

If G is unipotent, it follows from Theorem 2.2.8 that $\mathbf{v}_G$ is nilpotent, and hence $V_{D^*(G)}$ is nilpotent as well. More generally, one may show that D^* defines an equivalence between the category of affine unipotent p–group schemes over k and the category of $\mathscr{D}$–modules on which V is locally nilpotent, as in the following:

Corollaire 4.3.3

Corollary 2.4.6. *There are equivalences of categories:*

$$\begin{pmatrix} \textit{finite unipotent} \\ p\textit{–groups over } k \end{pmatrix}^{\mathrm{op}} \xrightarrow{D^*} \begin{pmatrix} \textit{subcat. of } \mathsf{Modules}_{\mathscr{D};W\,\mathrm{l.f.}} \\ \textit{where } V \textit{ is nilpotent} \end{pmatrix},$$

$$\begin{pmatrix} \textit{finite étale} \\ p\textit{–groups over } k \end{pmatrix}^{\mathrm{op}} \xrightarrow{D^*} \begin{pmatrix} \textit{subcat. of } \mathsf{Modules}_{\mathscr{D};W\,\mathrm{l.f.}} \\ \textit{where } F \textit{ is an iso.} \end{pmatrix},$$

$$\begin{pmatrix} \textit{finite bi-infinitesimal} \\ p\textit{–groups over } k \end{pmatrix}^{\mathrm{op}} \xrightarrow{D^*} \begin{pmatrix} \textit{subcat. of } \mathsf{Modules}_{\mathscr{D};W\,\mathrm{l.f.}} \\ \textit{where } F \textit{ and } V \textit{ are nilpotent} \end{pmatrix}.$$

Proof. Couple the preceding description of F and of V to Theorem 2.2.8. □

4.4. Cas des groupes de type multiplicatif

Construction of D^* on groups of multiplicative type. We would like to construct an equivalence of categories

$$\begin{pmatrix} \text{finite multiplicative-type} \\ p\text{–groups over } k \end{pmatrix}^{\mathrm{op}} \xrightarrow{D^*} \begin{pmatrix} \text{subcat. of } \mathsf{Modules}_{\mathscr{D};W\,\mathrm{l.f.}} \\ \text{where } V \text{ is an iso.} \end{pmatrix}.$$

The category on the right is equivalent to the category of W–modules of finite length endowed with a W–linear isomorphism $V\colon M \to M^\sigma$.[6] Using the invertibility of V, one may instead present this as the category of W–modules of finite length endowed with an isomorphism $V^{-1}\colon M^\sigma \to M$. This second category is antiequivalent via Corollary 2.4.6 to the category of finite étale p–group schemes over k, which is itself antiequivalent, by Cartier duality, to the category of finite p–group schemes of multiplicative type over k. To obtain the desired functor D^* as the composite of all these equivalences *with the correct variance*, we additionally insert the Pontryagin auto-antiequivalence of the category of finite étale p–group schemes:

$$G \mapsto \mathsf{GroupSchemes}(G, \mathbb{Q}_p/\mathbb{Z}_p).$$

This situation is recorded in the following diagram:

[6] The morphism F is then uniquely determined by the relation $FV = p$.

$$\begin{tikzcd}
\begin{pmatrix} \text{finite } p\text{-groups} \\ \text{of multiplicative type} \end{pmatrix}^{\mathrm{op}} \arrow[r, dotted, "D^*"] \arrow[d, "\text{Cartier duality}"] & \begin{pmatrix} \text{full subcat. of } \mathsf{Modules}_{\mathscr{D};W\,\mathrm{l.f.}} \\ \text{where } V \text{ is invertible} \end{pmatrix} \\
\begin{pmatrix} \text{étale} \\ \text{finite } p\text{-groups} \end{pmatrix} \arrow[d, "\text{Pontryagin duality}"] & \begin{pmatrix} W\text{-modules of finite length} \\ \text{with an isomorphism } V\colon M \to M^{\sigma} \end{pmatrix} \arrow[u] \\
\begin{pmatrix} \text{étale} \\ \text{finite } p\text{-groups} \end{pmatrix}^{\mathrm{op}} \arrow[r, "D^*"] & \begin{pmatrix} \text{full subcat. of } \mathsf{Modules}_{\mathscr{D};W\,\mathrm{l.f.}} \\ \text{where } F \text{ is invertible} \end{pmatrix}. \arrow[u]
\end{tikzcd}$$

□

Conclusion of Proof of Theorem 2.4.4. If G is any finite p–group, Remark 2.3.3 gives 4.5. Cas général

$$G = G^{\mathrm{uni}} \times G^{\mathrm{mult}},$$

where G^{uni} and G^{mult} are respectively unipotent and of multiplicative type. Using this, we set

$$D^*(G) = D^*(G^{\mathrm{uni}}) \times D^*(G^{\mathrm{mult}}).$$

Theorem 2.4.4 will then be proven if this determines an equivalence of categories

$$\mathsf{Modules}_{\mathscr{D};W\,\mathrm{l.f.}} \xrightarrow{D^*} \begin{pmatrix} \text{full subcat. of } \mathsf{Modules}_{\mathscr{D};W\,\mathrm{l.f.}} \\ \text{where } V \text{ is nilpotent} \end{pmatrix} \times \begin{pmatrix} \text{full subcat. of } \mathsf{Modules}_{\mathscr{D};W\,\mathrm{l.f.}} \\ \text{where } V \text{ is invertible} \end{pmatrix}.$$

This results from the following fact: all $\mathscr{D}$–modules M of finite length over W decompose uniquely (and functorially) as a direct sum Lemme

$$M = M' \times M''$$

such that $V|_{M'}$ is nilpotent and $V|_{M''}$ is bijective. To see this, take $N \gg 0$ such that

$$\bigcup_{n\geq 0} \operatorname{Ker} V^n = \operatorname{Ker} V^N, \qquad \bigcap_{n\geq 0} \operatorname{Im} V^n = \operatorname{Im} V^N.$$

The map $V|_{\operatorname{Ker} V^N}$ is then evidently nilpotent, $V|_{\operatorname{Im} V^N}$ is bijective, and

$$M = \operatorname{Ker} V^N \oplus \operatorname{Im} V^N.$$

□

2.5. **Corollaries.**

5. Corollaires et compléments

Corollary 2.5.1. *The functor D^* commutes with changes of base along perfect extensions K of k: for an object $G \in \mathsf{GroupSchemes}^{p,\mathrm{fin}}_{/k}$, there is a functorial isomorphism* Corollaire 5.1

$$D^*(G_K) \xleftarrow{\sim} W(K) \otimes_{W(k)} D^*(G).$$

Proof, after Oda and Oort. There is an evident morphism. To see that it is an isomorphism, it suffices to consider the case where G is unipotent. By considering the composition series formed by $\operatorname{Im} \mathbf{v}_G^n \subset G$, we further reduce to the case where $\mathbf{v}_G = 0$.

For such a group G, we have

$$D^*(G) = \mathsf{GroupSchemes}(G, \widehat{\mathbb{W}}) = \mathsf{GroupSchemes}(G, \mathbb{G}_a).$$

On the other hand, there is an isomorphism

$$\mathsf{GroupSchemes}(G,\mathbb{G}_a)\cong \operatorname{Ker}\Big(H^0(G,\mathcal{O}_G)\xrightarrow{\Delta^*-\pi_1^*-\pi_2^*} H^0(G,\mathcal{O}_G)\otimes_k H^0(G,\mathcal{O}_G)\Big)$$

where $\Delta\colon G\times G\to G$ denotes multiplication on G. By base change along $k\to K$, one has an analogous isomorphism, from which the corollary follows. □

Corollaire 5.2

Corollary 2.5.2. *For G a finite p–group scheme over k,*

$$\operatorname{rank}(G)=p^{\operatorname{length}_W(D^*(G))}.$$

Proof. It again suffices to consider the case where G is unipotent, and by Corollary 2.5.1 we may take k to be algebraically closed. Using this setting to apply Proposition 2.2.6, we see that G admits a composition series without $\mathbb{Z}/p\mathbb{Z}$ and α_p. By establishing

$$D^*(\alpha_p)=k=W/pW \text{ (with } F=V=0),$$
$$D^*(\mathbb{Z}/p\mathbb{Z})=k=W/pW \text{ (with } V=0,\, F(\lambda)=\lambda p),$$

the formula may be verified. □

5.3. Remarques

5.3.1

Remark 2.5.3. Formal groups and Barsotti–Tate groups can be considered as inductive limits of finite p–group schemes, and by applying the Dieudonné functor to such a system we find a projective limit of $\mathscr{D}$–modules of finite length over W. Passing from the system to its limit, we will later show (see Theorem 3.3.9) that there is an equivalence between the category of Barsotti–Tate groups over k and the category of $\mathscr{D}$–modules which are free and of finite type over W.

5.3.2

Remark 2.5.4. For G a finite p–group scheme over k, there is a natural isomorphism of $\mathscr{D}$–modules

$$D^*(G^*)\xrightarrow{\sim}\mathsf{Modules}_W(D^*(G),K/W),$$

where K/W denotes the dualizing module of W (cf. Gabriel and Demazure [DG70]). The $\mathscr{D}$–module structure on the right-hand side is defined for all $\alpha\in\mathsf{Modules}_W(D^*(G),K/W)$ and $x\in D^*(G)$ by

$$F\alpha(x)=[\alpha(V(x))]^\sigma,\qquad V\alpha(x)=[\alpha(F(x))]^{\sigma^{-1}}.$$

6. Annexe: Construction du foncteur quasi-inverse de D^*

2.6. **Construction of a quasi-inverse to D^*.** As an extended aside, we will now give an explicit construction of a quasi-inverse to the equivalence D^*. Reviewing the construction above, it will suffice to focus on Section 2.4 and to construct a quasi-inverse for the restricted functor

$$D^*\colon\begin{pmatrix}\text{unipotent}\\ \text{finite } p\text{–groups}\end{pmatrix}^{\mathrm{op}}\to\begin{pmatrix}\text{full subcat. of } \mathsf{Modules}_{\mathscr{D};W\,\mathrm{l.f.}}\\ \text{where } V \text{ is nilpotent}\end{pmatrix}.$$

In particular, we will construct a functor which we will show to be a left-adjoint to D^*; because D^* is an equivalence, it will then follow that we have constructed a quasi-inverse.

6.1

Let M be a $\mathscr{D}$–module of finite length over W, and suppose that V is nilpotent on M. We define a functor $\mathbb{E}(M)$ from schemes over k to groups by

$$\mathbb{E}(M)(S)=\mathsf{Modules}_{\mathscr{D}}(M,\widehat{\mathbb{W}}(S)).$$

Our goals are to demonstrate that $\mathbb{E}(M)$ is represented by a finite unipotent p–group and to demonstrate that the functor $\mathbb{E}$ itself is left-adjoint to D^*.

6.2. $\mathbb{E}(M)$ est représentable par un schéma affine

We begin by showing that $\mathbb{E}(M)$ is, in fact, an affine scheme.

Definition 2.6.1. Choose $N \gg 0$ such that $V^{N+1}|_M = 0$, and consider the polynomials P_N and S_N which respectively define the last coordinate of multiplication and of addition on $\mathbb{W}_{N+1}$.[7] We then define A_M to be the quotient k–algebra of $k[T_x, x \in M]$ by the following relations:

(1) $T_{F(x)} = T_x^p$.
(2) $T_{x+y} = S_N(T_{V^N(x)}, \ldots, T_x; T_{V^N(y)}, \ldots, T_y)$.
(3) $T_{\lambda x} = P_N(\lambda_1^{p^{-N}}, \lambda_2^{p^{-N}}, \ldots, \lambda_{N+1}^{p^{-N}}; T_{V^N(x)}, \ldots, T_x)$.

Lemma 2.6.2. *The functor* $\mathbb{E}(M)$ *is represented by* $\operatorname{Spec}(A_M)$.

Proof. Since we have chosen $N \gg 0$ such that $V^{N+1}|_M = 0$, it follows that for any choice of $x \in M$ and $u \in \mathsf{Modules}_{\mathscr{D}}(M, \widehat{\mathbb{W}}(S))$, we have $V^{N+1}(u(x)) = 0$. Any particular such element $u(x) \in \widehat{\mathbb{W}}(S)$ comes from the image of $(y_1, y_2, \ldots, y_r) \in \mathbb{W}_r(S)$ in the inductive system defining $\widehat{\mathbb{W}}(S)$, and the identity $V^{N+1}(y_1, \ldots, y_r) = 0$ then forces $y_1 = y_2 = \cdots = y_{r-N-1} = 0$. This implies that $(y_1, y_2, \ldots, y_r)$ comes from a unique element of $\mathbb{W}_{N+1}(S)$ via the transition morphism T^{r-N-1}, from which we deduce in turn that all homomorphisms $u \in \mathsf{Modules}_{\mathscr{D}}(M, \mathbb{W}(S))$ factorize uniquely through $W_{N+1}(S) \to \widehat{\mathbb{W}}(S)$.

This reduces us to constructing the following isomorphism functorially in S:

$$\varphi_S \colon \mathsf{Modules}_{\mathscr{D}}(M, \mathbb{W}_{N+1}(S)) \to \mathsf{Algebras}_k(A_M, \Gamma(\mathscr{O}_S)).$$

Ed.'s Note: This is typeset as homomorphisms of k–modules in the original, not as k–algebras.

For all $u \in \mathsf{Modules}_{\mathscr{D}}(M, \mathbb{W}_{N+1}(S))$, we set

$$\begin{aligned} \varphi_S(u) \colon A_M &\to \Gamma(\mathscr{O}_S), \\ T_x &\mapsto u(x)_{N+1} \end{aligned}$$

to be the morphism which sends T_x to the last coordinate of $u(x) \in \mathbb{W}_{N+1}(S) = \Gamma(\mathscr{O}_S)^{N+1}$. Using the relations on A_M, it is clear that $\varphi_S(u)$ is well-defined and that it is a morphism of algebras. One may define an inverse function by sending $\overline{u} \in \mathsf{Algebras}_k(A_M, \Gamma(\mathscr{O}_S))$ to the $\mathscr{D}$–module homomorphism

$$\begin{aligned} u \colon M &\to W_{N+1}(S), \\ x &\mapsto (\overline{u}(T_{V^N(x)}), \ldots, \overline{u}(T_x)). \end{aligned}$$

□

6.3. Le schéma $\mathbb{E}(M)$ est un p–groupe fini unipotent

Granting this, we see that $\mathbb{E}(M)$ is a(n affine) scheme. We can say more: it is a group scheme, since $\mathscr{D}$–module homomorphisms may be added; and it is a finite scheme, as V is nilpotent, hence M is annihilated by a power of p, hence $\mathbb{E}(M)$ is annihilated by a power of p as well, and so $\mathbb{E}(M)$ is finite. We have tasked ourselves with showing a tiny bit more:

Lemma 2.6.3. *The functor* $\mathbb{E}$ *factors through finite* unipotent *p–group schemes.*

Proof. Let $\{x_1, \ldots, x_r\}$ be a system of generators for the W–module M. The relations used to define the ring A_M entail that we may find a system of monomials in the variables $T_{x_1}, \ldots, T_{x_r}, T_{Vx_1}, \ldots, T_{Vx_r}, \ldots, T_{V^N x_1}, \ldots, T_{V^N x_r}$ of bounded degree such that this family of monomials generates A_M as a k–vector space.

Using Theorem 2.2.8, we may show $\mathbf{v}_{\mathbb{E}(M)} = \mathbf{v}$ is nilpotent in order to show that $\mathbb{E}(M)$ is unipotent. As $\mathbb{E}(M)$ is a subgroup of a smooth group (viz., affine space of dimension

[7] Recall that these are polynomials over $\mathbb{Z}$ in $(2N+2)$ variables.

$r(N+1)$), we may apply Remark 1.4.7 to determine $\mathbf{v}$ from the relation $\mathbf{v}\mathbf{f}_{\mathbb{E}(M)|_k} = p$; we therefore need only exhibit *some* morphism $\mathbf{v}$ satisfying this relation. The following commutative diagram usefully presents $\mathbf{f}_{\mathbb{E}(M)|_k}$:

$$\begin{tikzcd}
\mathsf{Modules}_{\mathscr{D}}(M, \mathbb{W}_{n+1}(S)) \arrow[r, "\alpha \mapsto \alpha \circ F"] \arrow[d, "\varphi_S", "\sim"'] & \mathsf{Modules}_{\mathscr{D}}(M, \mathbb{W}_{N+1}(S, \sigma)) \arrow[d, "\varphi_{S,\sigma}"', "\sim"] \\
\mathsf{Algebras}_k(A_M, \Gamma(\mathscr{O}_S)) \arrow[r, "\mathbf{f}_{\mathbb{E}(M)|_k}(S)"] & \mathsf{Algebras}_k(A_M \otimes_k (k, \sigma), \Gamma(\mathscr{O}_S))
\end{tikzcd}$$

where (S, σ) denotes the k–scheme $S \to \mathrm{Spec}(k) \xrightarrow{\sigma} \mathrm{Spec}(k)$. We use the top horizontal map (and the relation $\mathbf{v}\mathbf{f}_{\mathbb{E}(M)|_k} = p$) to determine a formula for the Verschiebung:

$$\mathbf{v} \colon \mathsf{Modules}_{\mathscr{D}}(M, \mathbb{W}_{N+1}(S, \sigma)) \to \mathsf{Modules}_{\mathscr{D}}(M, \mathbb{W}_{N+1}(S)),$$
$$u \mapsto u \circ V.$$

As V is nilpotent on M, it follows that $\mathbf{v}$ is nilpotent and $\mathbb{E}(M)$ is unipotent. □

6.4. Le foncteur $\mathbb{E}$ is adjoint à gauche de D^*

To complete the argument, we must show that the functor $\mathbb{E}$ is left-adjoint to D^*, i.e., we require the following isomorphism, functorially in G and in M:

$$\mathsf{GroupSchemes}_k(G, \mathbb{E}(M)) \xrightarrow{\sim} \mathsf{Modules}_{\mathscr{D}}(M, D^*(G)).$$

There is a natural equivalence

$$D^*(G) = \mathsf{GroupSchemes}_k(G, \widehat{\mathbb{W}}) = \mathrm{Ker}\Big(\widehat{\mathbb{W}}(G) \xrightarrow{\Delta^* - \pi_1^* - \pi_2^*} \widehat{\mathbb{W}}(G \times G)\Big) =: \mathrm{Prim}(G).$$

On the other hand, the definition of $\mathbb{E}(M)$ gives a description of *all* maps of schemes:

$$\mathsf{Schemes}_k(G, \mathbb{E}(M)) = \mathbb{E}(M)(G) = \mathsf{Modules}_{\mathscr{D}}(M, \widehat{\mathbb{W}}(G)).$$

If we can show that $u \in \mathsf{Modules}_{\mathscr{D}}(M, \widehat{\mathbb{W}}(G))$ defines a group morphism if and only if it factors through $\mathrm{Prim}(G)$, then these will combine to give the desired isomorphism.

The description of $\mathsf{GroupSchemes}_k(G, \mathbb{E}(M))$ may be interpreted at the level of affine algebras as the subset of those $\alpha \in \mathsf{Algebras}_k(A_M, \Gamma(\mathscr{O}_G))$ making the following diagram commute:

$$\begin{tikzcd}
A_M \arrow[r, "\alpha"] \arrow[d, "\Delta"] & \Gamma(\mathscr{O}_G) \arrow[d, "\Delta"] \\
A_M \otimes A_M \arrow[r, "\alpha \otimes \alpha"] & \Gamma(\mathscr{O}_G) \otimes \Gamma(\mathscr{O}_G).
\end{tikzcd}$$

We seek a formula for Δ. Recall the previously discussed isomorphism

$$\varphi_S \colon \mathsf{Modules}_{\mathscr{D}}(M, \mathbb{W}_{N+1}(S)) \xrightarrow{\sim} \mathsf{Algebras}_k(A_M, \Gamma(\mathscr{O}_S)).$$

For $u, u' \in \mathsf{Modules}_{\mathscr{D}}(M, \mathbb{W}_{N+1}(S))$, that φ_S is a homomorphism can be expressed as

$$\varphi_S(u + u')(T_x) = S_N(u(x), u'(x)).$$

Making the abbreviations $\overline{u} = \varphi_S(u)$ and $\overline{u}' = \varphi_S(u')$, the right-hand side may be written

$$S_N(u(x), u'(x)) = S_N[\overline{u}(T_{V^n(x)}), \ldots, \overline{u}(T_x); \overline{u}'(T_{V^N(x)}), \ldots, \overline{u}'(T_x)].$$

It follows that for $x \in M$, the action of $\Delta \colon A_M \to A_M \otimes A_M$ on $T_x \in A_M$ is given by the formula

$$\Delta(T_x) = S_N(T_{V^N(x)} \otimes 1, \ldots, T_x \otimes 1; 1 \otimes T_{V^N(x)}, \ldots, 1 \otimes T_x).$$

In turn, we see that $\overline{u}$ is a group isomorphism if and only if for all $x \in M$ we have

$$\Delta(\overline{u}(T_x)) = S_N[\overline{u}(T_{V^N(x)}) \otimes 1, \ldots, \overline{u}(T_x) \otimes 1; 1 \otimes \overline{u}(T_{V^N(x)}), \ldots, 1 \otimes \overline{u}(T_x)],$$

i.e., $\Delta(\overline{u}(T_x))$ agrees with the final coordinate in $\mathbb{W}_{N+1}(\Gamma(\mathscr{O}_G) \times \Gamma(\mathscr{O}_G))$ of $u(x) \otimes 1 + 1 \otimes u(x)$.

The same reasoning applied in turn to the members of the sequence $x, V(x), \ldots, V^N(x)$ and using $V^{N+1}|_M = 0$ shows that $\overline{u}$ defines a group morphism $G \to \mathbb{E}(M)$ if and only if $u(x)$ belongs to the kernel of

$$\mathbb{W}_{N+1}(\Gamma(\mathscr{O}_G)) \xrightarrow{\Delta^* - \pi_1^* - \pi_2^*} \mathbb{W}_{N+1}(\Gamma(\mathscr{O}_G) \otimes \Gamma(\mathscr{O}_G))$$

which is itself contained in $\mathrm{Prim}(G)$.

3. Barsotti–Tate groups

Chapitre III. Groupes de Barsotti–Tate

1. Notations et définitions préliminaires

3.1. Notation, preliminary definitions, and flatness and representability criteria. Select a base scheme S, and (for preliminary technical reasons[8]) select a topology τ on the category $\mathsf{Schemes}_{/S}$ which refines the fppf[9] topology and is refined by the fpqc[10] topology [SGA72, IV.6.3]. In this section, we will say that G *is a group* when G is a sheaf in commutative groups over $\mathsf{Schemes}_{/S}$ for the topology τ.

Definition 3.1.1. For G a group and $n \geq 1$ an integer, we set

$$\Lambda_n = \mathbb{Z}/p^n\mathbb{Z}, \qquad G(n) = \mathrm{Ker}\left(G \xrightarrow{p^n \mathrm{Id}} G\right), \qquad G(\infty) = \mathrm{colim}_n\, G(n).$$

The subgroup $G(n)$ is a Λ_n–module (i.e., a τ–sheaf in Λ_n–modules), and the subgroup $G(\infty)$ is the largest p–torsion subgroup of G. These nest as $G(n)(n') = G(n')$ whenever $n' \leq n \leq \infty$.

Remark 3.1.2. If we want to emphasize that a group G is annihilated by p^n (i.e., that it is a Λ_n–module), we may thus write $G(n)$ in place of G.

2. Platitude et critère de reprŕesentabilité

Definition 3.1.3. A Λ_n–module G is *flat over* Λ_n if the tensor product with G over the constant sheaf Λ_n preserves monomorphisms.

2.1

Proposition 3.1.4. *Let $G(n)$ be a group annihilated by p^n. The following are equivalent:*

2.2. Proposition

(1) *For all $1 \leq i \leq n$, $G(i)$ is a flat Λ_i–module.*
(2) *$G(n)$ is a flat Λ_n–module.*
(3) *For all $1 \leq i \leq n-1$, $p^{n-i}G(n) = G(i)$.*
(3') *For all $1 \leq i \leq n-1$, the function $p^{n-i} \colon G(n) \to G(i)$ is an epimorphism.*
(4) *For some $1 \leq i \leq n-1$, $p^{n-i}G(n) = G(i)$.*
(5) *$pG(n) = G(n-1)$.*

[8]The introduction of this topology is technical and is not essential for the final announcements (see Remark 3.3.1).

[9]"Fidèlement plate et de preésentation finie", or "faithfully flat and of finite presentation".

[10]"Fidèlement plate et quasi compacte", or "faithfully flat and quasi compact".

(6) *The canonical morphism*

$$\gamma\colon \Lambda_1[T]/T^n \otimes_{\Lambda_1} G(n)/pG(n) \to \operatorname{gr}_p G(n) = \bigoplus_{i=0}^{n-1} p^i G(n)/p^{i+1}G(N),$$

where $\gamma^i\colon G(n)/pG(n) \to p^i G(n)/p^{i+1}G(n)$ *is induced by multiplication by* p^i, *is an isomorphism.*

(7) *The morphism*

$$\beta\colon \operatorname{gr}_p G(n) \to \Lambda_1[T]/T^n \otimes_{\Lambda_1} G(1),$$

where $\beta^i\colon p^i G(n)/p^{i+1}G(n) \to G(1)$ *is induced by multiplication by* p^{n-i-1}, *is an isomorphism.*

Proof. We prove individual equivalences in turn.

(2) $\Leftrightarrow$ (3)**:** For all $1 \le i < n$, there are canonical commutative diagrams of constant Λ_n–modules

$$\begin{tikzcd}
p^i\Lambda_n \arrow[r, "\varepsilon"] & \Lambda_n \\
\Lambda_n/p^{n-i}\Lambda_n. \arrow[u, "p^i", "\sim"'] \arrow[ur, "(p^i)"']
\end{tikzcd}$$

By tensoring with $G(n)$, these give

$$\begin{tikzcd}
p^i\Lambda_n \otimes G(n) \arrow[r, "\varepsilon\otimes 1"] & \Lambda_n \otimes G(n) \\
\Lambda_n/p^{n-i}\Lambda_n \otimes G(n) = G(n)/p^iG(n). \arrow[u, "\sim"'] \arrow[ur, "(p^i)"']
\end{tikzcd}$$

As the only ideals of Λ_n are the $p^i\Lambda_n$, the module $G(n)$ is flat if and only if $\varepsilon \otimes 1$ is injective for all i—that is, if and only if (p^i) is injective, which is equivalent to condition (3).

(3) $\Leftrightarrow$ (1)**:** We use the same tool: the ideals $p^i\Lambda_n$ of Λ_n tensor with $G(n)$, a flat Λ_n–module, to show $p^{n-i}G(n) = G(i)$, and because these are the only ideals of Λ_n, these same equalities can be used to establish flatness of $G(n)$.

(4) $\Rightarrow$ (5)**:** Suppose that for some $1 \le i < n$ we have $G(i) = p^{n-i}G(n)$. Consider the following diagram, whose rows are exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & G(n-1) \arrow[r] \arrow[d, "p^{n-i-1}"] & G(n) \arrow[r, "p^{n-1}"] \arrow[d, "p^{n-i-1}"] & p^{n-1}G = p^{i-1}G(i) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & G(i) \arrow[r] \arrow[d] & G(i+1) \arrow[r, "p^i"] \arrow[d] & p^iG(i+1) = p^{i-1}G(i) \arrow[r] \arrow[d] & 0 \\
& 0 \arrow[r] \arrow[d] & G(i+1)/p^{n-i-1}G(n) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0.
\end{tikzcd}$$

It follows directly that $G(i+1) = p^{n-i-1}G(n)$.

(5) $\Leftrightarrow$ (6)**:** For each $0 \le i < n-1$, we set λ^i to be the epimorphism

$$\lambda^i\colon p^iG(n)/p^{i+1}G(n) \to p^{i+1}G(n)/p^{i+2}G(n)$$

induced by multiplication by p. Then, for each $1 \leq i < n$, we have

$$\gamma^i = \lambda^{i-1} \circ \lambda^{i-2} \circ \cdots \circ \lambda^0.$$

It follows that the epimorphism γ is an isomorphism if and only if γ^{n-1} is a monomorphism, i.e., if (5) holds.

(5) ⇒ (7) ⇒ (3)**:** Setting j to be the canonical injection $p^{n-1}G(n) \to G(1)$ and otherwise retaining the notation from above, we have the identity

$$\beta^i = j \circ \lambda^{n-2} \circ \lambda^{n-3} \circ \cdots \circ \lambda^i.$$

We've just shown that (5) entails that all the λ^i are isomorphisms, and hence β is a monomorphism.

We will show by induction that if β is a monomorphism, then (3) holds. Taking $\beta^0 = j \circ \gamma^{n-1}$ to be a monomorphism, we directly have $pG(n) = G(n-1)$. Next, taking β^i to be a monomorphism, we also have

$$[p^i G(n)](n-i-1) = p^{i+1}G(n).$$

By inductive hypothesis, we have $p^i G(n) = G(n-i)$, from which we conclude $G(n-i-1) = p^{i+1}G(n)$.

Having established (3), it follows immediately that j (and thus β) are isomorphisms. □

Proposition 3.1.5. *Let $G(n)$ be a group annihilated by p^n and flat over Λ_n. Then $G(n)$ is representable by a finite locally free scheme over S if and only if the same is true for one for the $G(i)$, $1 \leq i \leq n$.* 2.4. Proposition

Proof. Let us now suppose that $G(n)$ is a flat Λ_n–module, representable by a finite locally free scheme over S. It follows that $G(i)$ is the kernel of multiplication by p^i and that it is representable, finite, and locally of finite presentation. Moreover, applying the fiber-by-fiber criterion for flatness [DG67, IV.11.3.1] to the epimorphism $p^{n-1}\colon G(n) \to G(i)$, we deduce that $G(i)$ is flat over S, hence locally free. 2.3

Conversely, if $G(1)$ is representable by a finite locally free scheme over S and if $G(n)$ is Λ_n–flat, then $G(n)$ admits a finite composition series $p^i G(n)/p^{i+1}G(n) \cong G(1)$, and by Definition 2.1.8 $G(n)$ is representable by a finite locally free scheme. □

3.2. **Barsotti–Tate groups, in full and truncated at stage n.** We finally introduce the definition of one of the main subjects of these notes, that of Barsotti–Tate groups. These have also been called "p–divisible groups" in the literature (e.g., by Tate [Tat67]), but we reserve this alternative name for the following weaker definition: 3. Groupes de Barsotti–Tate tronqués d'échelon n

Definition 3.2.1. We say that a group G (i.e., a τ–sheaf in groups) is p–divisible if $p\colon G \to G$ is an epimorphism. 4. Groupes de Barsotti–Tate

If G is p–divisible, then it follows from Proposition 3.1.4.(5) that $G(n)$ is a flat Λ_n–module, and the group of p–torsion $G(\infty)$ is also p–divisible. More generally, given a *co-p-adic system*, i.e., an inductive system 4.1. Définition

$$G_1 \to G_2 \to \cdots \to G_n \to G_{n+1} \to \cdots$$

such that G_n is annihilated by p^n and $G_n(n-1) = G_{n-1}$, in which G_n is a flat Λ_n–module, then $\operatorname{colim}_n G_n$ is a p–torsion p–divisible group.

Definition 3.2.2. A *Barsotti–Tate (p-)group* (over S) is a τ–sheaf G valued in p–torsion p–divisible commutative groups such that $G(1)$ is a finite locally free group over S. 4.2. Définition

Remark 3.2.3. It is equivalent to say that any of the $G(n)$ is a finite locally free scheme over S (cf. Proposition 3.1.5).

3.1 It will also be convenient to consider truncated co–p–adic systems in which G_n is a flat Λ_n–module, which one might naively called a truncated Barsotti–Tate group. However, this condition is empty for $n = 1$, and we will find it necessary to substitute this with a different condition in order to make the theory work out. Unfortunately, we have no *a priori* justification for this extra condition, and so we now simply state the definition.

Let G be a commutative group scheme which is flat over S, let S_0 be the closed $\mathbb{F}_p$–scheme of S defined by the ideal $p\mathcal{O}_S$, and let G_0 be the scheme over S_0 obtained by base-change, which comes equipped with morphisms $\mathbf{f}_{G_0/S_0}$ and $\mathbf{v}_{G_0}$. Note that if G_0 is a Λ_1–module, we have $\mathbf{f}_{G_0/S_0}\mathbf{v}_{G_0} = 0$.

3.2. Définition **Definition 3.2.4.** Let $n \geq 1$ be an integer. We define a *Barsotti–Tate p–group truncated at stage n* (over S) to be a commutative group scheme G, finite and locally free over S, annihilated by p^n, and flat over Λ_n. If $n = 1$, we will further suppose that[11]

$$\operatorname{Ker}\mathbf{f}_{G_0/S_0} = \operatorname{Im}\mathbf{v}_{G_0}.$$

3.3 **Lemma 3.2.5.** *If $G(n)$ is a Barsotti–Tate group truncated at stage n, then for each $1 \leq i \leq n$, $G(i)$ is a Barsotti–Tate group truncated at stage i.*

Proof. For $i > 1$, this follows from Proposition 3.1.5. For $i = 1$, we will show that the extra condition is also satisfied. Over $S_0 \subset S$, we have already remarked that the image of $\mathbf{v}_{G(1)_0}\colon G(1)_0^{(p)} \to G(1)_0$ is contained in the kernel K of $\mathbf{f}_{G(1)_0/S_0}\colon G(1)_0 \to G(1)_0^{(p)}$, from which we derive a commutative diagram

$$\begin{tikzcd}
G(1)_0^{(p)} \arrow[r, "\mathbf{v}_{G(1)_0}"] \arrow[dd] & K \arrow[d] \\
 & G(1)_0 \arrow[d] \\
G(2)_0^{(p)} \arrow[r, "\mathbf{v}_{G(2)_0}"] & G(2)_0.
\end{tikzcd}$$

Using $G(2)_0^{(p)}(1) \cong G(1)_0^{(p)}$, it follows immediately that this diagram is Cartesian. On the other hand, because $\mathbf{v}_{G(2)_0}\mathbf{f}_{G(2)_0} = p$ has image $G(1)_0$, it follows that the image of $\mathbf{v}_{G(2)_0}$ contains $G(1)_0$, from which we deduce that $\mathbf{v}\colon G(1)_0^{(p)} \to K$ is surjective. □

Remark 3.2.6. If G is a Barsotti–Tate group, then $G(n)$ is a Barsotti–Tate group truncated at stage n for each $n \geq 1$. Over an algebraically closed field, the converse is also true: if G' is a Barsotti–Tate group truncated at stage n, then there exists a full Barsotti–Tate group G such that $G' = G(n)$.

5. Sorites sur les groupes de Barsotti–Tate

3.3. Facts about Barsotti–Tate groups.

Remark 3.3.1. We have seen that the category of Barsotti–Tate groups is equivalent to the category of co–p–adic systems $\{G_n\}$ where G_n is a finite locally free group scheme over S which is a flat Λ_n–module. It follows that this notion is independent of the choice of topology τ. 5.1

Remark 3.3.2. The category of Barsotti–Tate groups is additive.

Remark 3.3.3. If S' is a scheme over S and G is a Barsotti–Tate group over S, then we can define the inverse image Barsotti–Tate group G' over S': each of $G(n)\times_S S'$ are is a truncated Barsotti–Tate group, and we set $G' = \operatorname{colim}(G(n)\times_S S')$.

Proposition 3.3.4. *Let $0 \to G' \to G \to G'' \to 0$ be an exact sequence of group schemes. Then* 5.2. Proposition

1. *If G' and G'' are Barsotti–Tate groups, then so is G.*
2. *If G' and G are Barsotti–Tate groups, then so is G''.*

Proof. For (1), an extension of p–divisible (resp. p–torsion) groups is p–divisible (resp. p–torsion). Using the snake lemma, the exact sequence induces an exact sequence

$$0 \to G'(1) \to G(1) \to G''(1) \to G'/pG' \to \cdots.$$

However, as G' is p–divisible, it follows that $G(1)$ is an extension of finite locally free groups and hence is one itself.

For (2), a quotient of a p–divisible (resp. p–torsion) group is p–divisible (resp. p–torsion). Then, as before, $G''(1)$ is the quotient of two finite locally free group schemes, which implies that $G''(1)$ is one itself. □

Remark 3.3.5. If G and G'' are Barsotti–Tate groups, then it is in general false that the kernel is a Barsotti–Tate group. For example, there is an exact sequence $0 \to G(1) \to G \xrightarrow{p} G \to 0$, where $G(1)$ is not p–divisible!

Definition 3.3.6. For a Barsotti–Tate group G, its first stage $G(1)$ is a finite locally free group which is annihilated by p. For every point $s \in S$, we thus have 5.3. Rang (ou “hauteur”) d’un groupe de Barsotti–Tate

$$\operatorname{rank}(G(1)_s) = p^{r(s)}$$

and the function r so-defined is locally constant in s. The exact sequences

$$0 \to G(1) \to G(n) \xrightarrow{p} G(n) \to 0$$

show in turn that

$$\operatorname{rank}(G(n)_s) = (\operatorname{rank}(G(1)_s))^n = p^{nr(s)}.$$

The integer $r(s)$ is called the *rank (or height) (at the point $s \in S$).*

Remark 3.3.7. We have described the category of Barsotti–Tate groups over S by using co–p–adic systems, but we actually could have used p–adic systems. To a Barsotti–Tate group G, we associate the p–adic system 5.4

$$T_p(G) = \left(\cdots \to G(n+1) = \frac{G(n+1)}{p^{n+1}G(n+1)} \xrightarrow{p} \frac{G(n+1)}{p^n G(n+1)} = G(n) \to \cdots\right),$$

[11]This condition may be verified fiber-by-fiber.

where $G(1)$ is annihilated by p, representable, and a finite locally free S–scheme.

5.5. Dual d'un groupe de Barsotti–Tate

Definition 3.3.8. For a Barsotti–Tate group G, applying Cartier duality to the p–adic system $T_p(G)$ defines a co–p–adic system of groups $G(n)^*$. These groups are finite, locally free, annihilated by p^n, and flat over Λ_n, and so we use them to define the *dual* Barsotti–Tate group G^*:[12]

$$G^* = \mathrm{colim}_n\, G(n)^*.$$

Finiteness of the group schemes $G(n)$ yields a natural equivalence with the double-dual $G \xrightarrow{\sim} (G^*)^*$, and hence this construction determines a contravariant equivalence, compatible with base change, between the category BT_S of Barsotti–Tate p–groupes over S and its opposite.

5.6. Module de Dieudonné associé à un groupe de BT sur un corps parfait

Theorem 3.3.9 (cf. Remark 2.5.3)**.** *Recall the notation from Section 2.4, and use BT_k to denote the category of Barsotti–Tate p–groups over a perfect field k. There exists an equivalence of categories*

$$D^*\colon \mathsf{BT}_k^{\mathrm{op}} \xrightarrow{\sim} \begin{pmatrix} \mathscr{D}\text{–modules which are free and} \\ \text{of finite type over } W \end{pmatrix}$$

such that

$$[D^*(G)]^\sigma \cong D^*(G^{(p)}), \qquad D^*(\mathbf{f}_{G/k}) = F_{D^*(G)}, \qquad D^*(\mathbf{v}_G) = V_{D^*(G)}.$$

Construction. Let G be such a Barsotti–Tate group. We begin by setting $M(n) = D^*(G(n))$, where D^* is the "classical" functor defined in Section 2.4. As $G(n)$ is annihilated by p^n, $M(n)$ is annihilated by p^n and is thus a module over $W/p^nW = W_n$. For each $1 \le i < n$, the epimorphisms $p^i\colon G(n) \to G(n-i)$ determine monomorphisms

$$M(n)/p^{n-i}M(n) = M(n-i) \xrightarrow{p^i} M(n),$$

which shows $M(n)$ to be flat over W_n. Finally, the co–p–adic system defining G determines a p–adic system built out of the modules $M(n)$: the injections $G(n) \to G(n+1)$ define surjections $M(n+1) \to M(n)$ which identify $M(n)$ with $M(n+1)/p^nM(n+1)$. We make the definition

$$D^*(G) = \lim M(n).$$

The limit $D^*(G)$ is a module over $W = \lim W(n)$, and since the modules in the system are flat, $D^*(G)$ is itself flat over the local ring W, hence is free. As the modules $M(n)$ are finite, $D^*(G)$ is itself of finite type. Finally, $D^*(G)$ is endowed with semilinear morphisms F and V induced by the corresponding morphisms on $M(n)$. □

6. Exemples et types particuliers de groupes de Barsotti–Tate

3.4. Examples and particular types of Barsotti–Tate groups.

6.1

Example 3.4.1. Let A be an abelian scheme over S (i.e., a commutative group scheme, proper and smooth over S, and with connected fibers) of dimension d. For any prime p, the subgroup $A(\infty) = \mathrm{colim}_n\, A(n)$ is a Barsotti–Tate group of rank $2d$.

[12] The dual satisfies the pleasant interchange law $G^*(n) = G(n)^*$.

Construction. Every abelian scheme is p–divisible: this is well-known in the case where S is the spectrum of an algebraically closed field [Mum70] and, if S is arbitrary, the surjectivity of $p\colon A \to A$ results from the algebraically closed case via the fiber-by-fiber flatness criterion [DG67, IV.11.3.10]. Additionally, if d denotes the relative dimension of A over S, then one may show that

$$\operatorname{rank} A(1) = p^{2d}.$$

The group $A(1)$ is thus finite, it is evidently of finite presentation and, being flat, it is locally free. □

Remark 3.4.2. There already exists a notion of duality for abelian varieties: writing A^* for the dual variety of the abelian variety A, one shows easily that its associated Barsotti–Tate group $A^*(\infty)$ may be identified with the dual, in the sense of Barsotti–Tate groups, of $A(\infty)$.

Remark 3.4.3. If all of the residual characteristics of S are different from p, one may show that $A(1)$ is étale over S, and one then says that $A(\infty)$ is ind-étale.

Definition 3.4.4. We say that a Barsotti–Tate group over S is *ind-étale* if $G(1)$ is étale. 6.2

Remark 3.4.5. As the groups $G(n)$ admit a composition series whose factors are isomorphic to $G(1)$, it is equivalent to say that all the groups $G(n)$ are étale.

Remark 3.4.6. The functor $G \mapsto T_p(G)$ defines an equivalence between the category of ind-étale Barsotti–Tate groups and the category of *twisted torsion-free constant p–adic sheaves*, i.e., p–adic systems of finite locally free groups G_n which are étale over S and flat over Λ_n. If S is connected, the category of twisted torsion-free constant p–adic sheaves is well-known. Choosing a geometric point $\bar{s} \in S$, there is an equivalence between this category and that of finite type free $\mathbb{Z}_p$–modules on which the fundamental group $\pi_1(S, \bar{s})$ acts continuously. Namely, one sends a system $(G_n)_{n\geq 1}$ to the $\mathbb{Z}_p$–module $\lim (G_n)_{\bar{s}}$, with $\pi_1(S, \bar{s})$ acting by monodromy on $(G_n)_{\bar{s}}$.

Example 3.4.7. We say that a group scheme T over S is a *torus* if locally isomorphic to 6.3. Exemple
$\mathbb{G}^r_{m,S}$ in the étale topology. A torus T is p–divisible, and $T(1)$ is a finite locally free group of rank p^r for r the relative dimension of T over S,[13] hence $T(\infty)$ is a Barsotti–Tate group of rank r. As a particularly important example, the Barsotti–Tate group μ is defined by

$$\mu = \mathbb{G}_{m,S}(\infty) = \operatorname{colim}_n \mu_{p^n,S}.$$

Example 3.4.8. Let G be a group scheme over S given as an extension of an abelian scheme
A by a torus T. Since G is p–divisible, $G(1)$ is finite and locally free over S and $G(\infty)$ is a 6.4. Exemple
Barsotti–Tate group of rank $2d + r$ (where d and r are respectively the relative dimensions of A and of T over S). Moreover, there is an exact sequence

$$0 \to T(\infty) \to G(\infty) \to A(\infty) \to 0.$$

Definition 3.4.9. Following Example 3.4.7, we will say that a Barsotti–Tate group G is 6.5
toroidal if $G(1)$ is of multiplicative type. It is equivalent to say that all of the $G(n)$ are of multiplicative type, or equivalently that the dual G^* is ind-étale.

[13]These properties are local and can be checked for $\mathbb{G}^r_{m,S}$.

Remark 3.4.10. The functor $G \mapsto T_p(G^*)$ defines an antiequivalence between the category of toroidal Barsotti–Tate groups and that of twisted torsion-free constant p–adic sheaves. Composing this functor with duality in twisted torsion-free constant p–adic sheaves

$$T_p(G^*) \mapsto \mathsf{GroupSchemes}(G^*, \mathbb{Z}_p) \stackrel{\text{def}}{=} (\mathsf{GroupSchemes}(G(n)^*, \Lambda_n))_{n\geq 1},$$

one obtains an *equivalence* rather than an antiequivalence. At the level of group schemes, this composite equivalence behaves as

$$G \mapsto \mathsf{GroupSchemes}(\mu, G) \stackrel{\text{def}}{=} (\mathsf{GroupSchemes}(\mu(n), G(n)))_{n\geq 1}.$$

6.6 **Definition 3.4.11.** One says that a Barsotti–Tate group G over S *has connected fibers* when $G(1)$ is infinitesimal, or, equivalently, when all of the groups $G(n)$ are infinitesimal.

Corollary 3.4.12 (Proposition 2.2.2)**.** *If all the residual characteristics of S are equal to p, the toroidal Barsotti–Tate groups have connected fibers.* □

Remark 3.4.13. We will show later on that if p is locally nilpotent on S, then a Barsotti–Tate group has connected fibers if and only if it is also a formal Lie group.

Ed. note: He does not show this later on.

We will give here a first elementary definition of formal Lie groups and study under what conditions a formal Lie group is a Barsotti–Tate group.

6.7 **Definition 3.4.14.** A *pointed (or augmented) formal Lie variety (over S)* is a τ–sheaf X over S endowed with a section $\varepsilon\colon S \to X$ such that, Zariski–locally on X,

$$X \simeq \operatorname{colim}_n \operatorname{Spec} \mathscr{O}_S[\![T_1,\dots,T_r]\!]/(T_1,\dots,T_r)^n.$$

Remark 3.4.15. One may make the local isomorphism explicit by saying that for all $S' \to S$ with S' quasicompact, we have

$$\begin{aligned} X(S') &= \operatorname{colim}_n \mathsf{Schemes}_S(S', \operatorname{Spec} \mathscr{O}_S[\![T_1,\dots,T_r]\!]/(T_1,\dots,T_r)^n) \\ &= \operatorname{colim}_n \{f_1, f_2, \dots, f_r \in \Gamma(S', \mathscr{O}_S) \mid f_i^n = 0, 1 \leq i \leq r\}. \end{aligned}$$

In particular, X is the inductive limit of finite and radical schemes over S.

Definition 3.4.16. A *formal Lie group (over S)* is a sheaf in groups over S such that its unit section gives it the structure of a formal Lie variety.

Remark 3.4.17. As the product of two formal Lie varieties is again a formal Lie variety, it is equivalent to say that a formal Lie group is a group object in the category of formal Lie varieties.

From here on, we assume all formal Lie groups to be commutative, and we take G to be such a formal Lie group over S. One checks easily that if p is locally nilpotent on S, then G is p–torsion, so that $G = G(\infty) = \operatorname{colim}_n G(n)$. Of course, G is a Barsotti–Tate group if and only if the following two conditions are satisfied:

(1) G is p–divisible.
(2) $G(1)$ is representable by a finite locally free scheme.

Remark 3.4.18. In the case where S is Artinian and local, with unique point $s \in S$, and still assuming G to be a formal Lie group, then these two conditions are equivalent to each other. They're furthermore equivalent to the following pair of conditions, which involve only the geometric fiber of G:

(3) $G_{\bar{s}}$ is p-divisible.
(4) It is impossible to embed the formal additive group $\overline{\mathbb{G}}_a$ (the formal completion of $\mathbb{G}_a$ along its unit section) into $G_{\bar{s}}$.

We suspect that conditions (1) and (2) are equivalent for formal Lie groups in general. This would follow from the following conjecture, which is true in the Artinian local case [SGA72, VII]:

Conjecture 3.4.19. Let $u\colon G \to G'$ be an S-morphism of formal Lie varieties of the same relative dimension over S. The following are then equivalent:

- u is an epimorphism.
- $\mathrm{Ker}(u)$ is representable by a finite scheme.
- $\mathrm{Ker}(u)$ is representable by a finite locally free scheme.

Remark 3.4.20. On the contrary, the fact that the formal Lie group G is a Barsotti–Tate group does not in general imply that the fibers are Barsotti–Tate groups as well. For example, the formal Lie group associated to the modular elliptic curve A over a curve S of characteristic p fails this, because there are points $s \in S$ where the Hasse invariant of A_s is zero.

In turn, this example suggests the following conjecture:

Conjecture 3.4.21. For a formal Lie group G over S to be Barsotti–Tate, it is necessary and sufficient for the fibers G_s to be Barsotti–Tate groups with locally constant rank as s ranges in S.

3.5. **Composition series of a Barsotti–Tate group.**

7. Suite de composition d'un groupe de Barsotti–Tate

Definition 3.5.1. Suppose that S is reduced at a point s. If G is a Barsotti–Tate group over S, we set

$$G^\circ = \mathrm{colim}_n\, G(n)^\circ, \qquad G^{\text{ét}} = \mathrm{colim}_n\, G(n)/G(n)^\circ,$$ 7.1

yielding an exact sequence

$$0 \to G^\circ \to G \to G^{\text{ét}} \to 0.$$

Remark 3.5.2. One checks immediately that G° is a Barsotti–Tate group, hence $G^{\text{ét}}$ is as well, and so G is obtained as an extension of an ind-étale Barsotti–Tate group from a Barsotti–Tate group with connected fibers. This decomposition is canonical and functorial in G.

Definition 3.5.3. Suppose further that the residual characteristic of S is p, and define the toroidal subgroup (which is itself a Barsotti–Tate group) to be 7.2

$$G^{\text{tor}} = \mathrm{colim}_n\, G(n)^{\text{mult}},$$

where $G(n)^{\text{mult}}$ denotes the largest subgroup of $G(n)$ of multiplicative type. We thereby obtain a filtration

$$\{0\} \subset G^{\text{tor}} \subset G^\circ \subset G.$$

We deduce from this that G°/G^{tor} is an ind-unipotent Barsotti–Tate group (i.e., $G^\circ/G^{\text{tor}}(1) = G(1)^\circ/G(1)^{\text{mult}}$ is unipotent) which also has connected fibers. This canonical composition series is functorial in G, and it commutes with change of base and with passing to the dual.

Remark 3.5.4. Using the same construction, one finds canonical isomorphisms

$$(G/G^\circ)^* \cong (G^*)^{\mathrm{tor}}, \qquad (G/G^{\mathrm{tor}})^* \cong (G^*)^\circ.$$

7.3 *Remark* 3.5.5. For an arbitrary base S, one cannot generally write a Barsotti–Tate group G as an extension of an ind-étale Barsotti–Tate group G'' by a Barsotti–Tate group G' with connected fibers. However, if one has such an extension, one may then obtain an exact sequence

$$0 \to G'(1) \to G(1) \to G''(1) \to 0.$$

For each $s \in S$, consider the separable rank of $G(1)_s$:

$$\operatorname{rank}\operatorname{sep}(G(1)_s) = |G(1)(\bar{s})| = \operatorname{rank}\operatorname{sep}(G'(1)_s) \cdot \operatorname{rank}\operatorname{sep}(G''(1)_s).$$

The separable rank of $G'(1)_s$ is 1 because $G'(1)$ is infinitesimal, and the separable rank of $G''(1)_s$ is a locally constant function of $s \in S$ because $G''(1)$ is étale over S. From this, we deduce that the separable rank of $G(1)_s$ is a locally constant function of $s \in S$. However, this is not true in general, as exemplified by the modular elliptic curve A over a curve S in characteristic p, where there are isolated points where the Hasse invariant of A_s is zero.

Avoiding this kind of counterexample, one may deduce the following result:

7.4 **Proposition 3.5.6** (cf. [Mes72, Proposition II.4.9]). *Let G be a Barsotti–Tate group over S. The following properties are equivalent:*

(1) *G is an extension of an ind-étale Barsotti–Tate group by a Barsotti–Tate group with connected fibers.*
(2) *For all $n \geq 1$, $G(n)$ is an extension of a finite étale group by a finite infinitesimal group.*
(2') *$G(1)$ is an extension of a finite étale group by a finite infinitesimal group.*
(3) *For all $n \geq 1$, the function $s \mapsto \operatorname{rank}\operatorname{sep} G(n)_s$ is locally constant.*
(3') *The function $n \mapsto \operatorname{rank}\operatorname{sep} G(1)_s$ is locally constant.*

Proof. Given our work so far, the only implication left to prove is (3) ⇒ (2). For this we rely on the following:

Lemma 3.5.7 ([Mes72, Lemma II.4.8]). *Let $f : X \to S$ be a finite locally free morphism of schemes. For f to factorize as $f' \circ f''$ for a radical and surjective morphism f'' and an étale morphism f', it is necessary and sufficient that the function $s \mapsto \operatorname{rank}\operatorname{sep}(X_s)$ is locally constant. The factorization is then unique and functorial in f.*[14] □

4. Crystals

Chapitre IV. Cristaux

4.1. Reminders on divided powers.

1. Rappels sur les puissances divisées

1.1 **Definition 4.1.1.** Let A be a ring and J an ideal of A. One says that J is endowed with *divided powers* if one is given for $n \geq 1$ a family of functions $\gamma_n : J \to J$ satisfying the following axioms:

(1) $\gamma_1(x) = x$ for all $x \in J$.
(2) $\gamma_n(x+y) = \gamma_n(x) + \sum_{i=1}^{n-1} \gamma_{n-i}\gamma_i(y) + \gamma_n(y)$ for all $x, y \in J$.

[14] Using the functoriality, one may show that if the source is a group scheme, then so is the target in the factorization.

(3) $\gamma_n(xy) = x^n \gamma_n(y)$ for all $x \in A$, $y \in J$.
(4) $\gamma_m \circ \gamma_n(x) = \frac{(mn)!}{(n!)^m m!} \gamma_{mn}(x)$.
(5) $\gamma_m(x)\gamma_n(x) = \frac{(m+n)!}{m!n!} \gamma_{m+n}(x)$.

Remark 4.1.2. Iterating the fifth axiom gives the relation

$$\gamma_{m_1+m_2+\cdots+m_p}(x) \cdot \frac{(m_1+m_2+\cdots+m_p)!}{m_1!m_2!\cdots m_p!} = \prod_{i=1}^{p} \gamma_{m_i}(x).$$

In particular, we have $x^n = (\gamma_1(x))^n = n!\gamma_n(x)$. This last formula is the principle motiviation for the introduction of divided powers.

Example 4.1.3. If the ring A is of characteristic 0 (i.e., if its unit is divisible by all primes), then the formula $\gamma_n(x) = \frac{x^n}{n!}$ shows that any ideal J has one and only one divided power structure.

Definition 4.1.4. To more comfortably write certain series formulas, we make two further notational defintitions:

- The zeroth power: $\gamma_0(x) = 1$.
- A shorthand for the n^{th} divided power: $x^{(n)} = \gamma_n(x)$.

Remark 4.1.5. Divided powers were introduced by H. Cartan for the study of Eilenberg–Mac Lane spaces, and in abstract algebra by N. Roby in his thesis.

4.1.1. *The exponential and logarithm.*

1.2. Exponentielle et logarithme

Definition 4.1.6. Let A be a ring, and let (J, γ_n) be an ideal endowed with divided powers. We may then define exponential and logarithm functions

$$\begin{aligned} &\exp\colon J \to 1+J, && \log\colon 1+J \to J, \\ &\exp(x) = \sum_{n\geq 0} x^{(n)}, && \log(1+x) = \sum_{n\geq 1} (-1)^{n-1}(n-1)!x^{(n)} \end{aligned}$$

where we suppose, for these functions to be defined, that $x^{(n)}$ (resp. $(n-1)!x^{(n)}$) is zero for n sufficiently large.

If this nilpotence-like hypothesis is satisfied, then the usual proof shows that exp and log are inverse isomorphisms between J^+ and $(1+J)^*$. In order to study their convergence and inverse properties more generally, we introduce the following:

Definition 4.1.7. We define a filtration on J:

$$J = J^{(1)} \supset J^{(2)} \supset \cdots \supset J^{(n)} \supset \cdots,$$

where $J^{(n)}$ is the ideal generated by the monomials $x_1^{(a_1)} x_2^{(a_2)} \cdots x_r^{(a_r)}$ with $\sum_i a_i \geq n$.

Definition 4.1.8. One says that (J, γ_n) is *divided-power nilpotent* if there exists an n with $J^{(n)} = 0$. In this case, $x^{(n)} = 0$ for all $x \in J$, and the exponential and logarithm are thus everywhere defined.

Definition 4.1.9. If there exists an n such that $(n-1)!J^{(n)} = 0$, then at least the logarithm is everywhere defined. This condition was introduced by P. Berthelot in his study of crystalline cohomology, and so we will call this *Berthelot's condition*.

4.1.2. *Some examples of divided power structures.*

1.3. Quelques exemples de structure de puissances divisées

Example 4.1.10. If A is of characteristic 0, we have already seen that every ideal possesses exactly one divided power structure: $\gamma_n(x) = \frac{x^n}{n!}$.

Ex. 1)

Ex. 2)

Example 4.1.11. If A is a torsion-free ring, there exists at most one divided power structure on J, as we may apply cancellation to the combined formula

$$n!\gamma_n(x) = x^n = n!\gamma'_n(x).$$

Moreover, such a structure exists if and only if $J \subset J \otimes \mathbb{Q} \subset A \otimes \mathbb{Q}$ is stable under the operations $x \mapsto x^n/n!$.

Ex. 3)

Example 4.1.12. Let $W = W(k)$ be the ring of Witt vectors for a perfect field k and let $J = pW$. For $n \geq 0$, define the p–adic expansion of n to be

$$n = a_0 + a_1 p + \cdots + a_\ell p^\ell,$$

where $0 \leq a_j < p$ and $0 \leq j \leq \ell$, and additionally define the digital sums $s_n = \sum_{j=0}^{\ell} a_j$. The p–adic valuation of $n!$ is given by the formula

$$\nu_p(n!) = \frac{n - s_n}{p-1} \leq n - 1.$$

Hence we have $\gamma_n(p) = \frac{p^n}{n!} \in pW$, and there is thus a unique divided power structure on pW. Moreover, one sees that for $p > 2$, $\frac{p^n}{n!}$ tends to 0 in W. Using completeness of W, we can thus define the exponential by

$$\exp(px) = \sum_{n=0}^{\infty} \frac{p^n x^n}{n!}.$$

For $p = 2$, this is no longer true; for example, we have

$$\frac{2^{2^n}}{2^n!} \equiv 2 \pmod{2^2}.$$

Considering instead the truncated Witt vectors $W_n = W/p^n W$, for $p > 2$ these divided powers are nilpotent, and hence one may define the exponential and logarithm. For $p = 2$, Berthelot's condition is satisfied, hence one may define the logarithm—but not the exponential.

Ex. 4) Extensions des puissances divisées.

Example 4.1.13 (Berthelot's extension examples). Given two rings together with ideals with divided powers, (A, J, γ) and (A', J', γ'), a *divided power homomorphism* $\varphi\colon (A, J, \gamma) \to (A', J', \gamma')$ is then a homomorphism of rings $\varphi\colon A \to A'$ such that $\varphi(J) \subset J'$ and for all $x \in J$ we have $\varphi(\gamma_n(x)) = \gamma'_n(\varphi(x))$. If (A, J, γ) is a ring endowed with an ideal with divided powers and if $\varphi\colon A \to B$ is a morphism of rings, one says that γ *extends to* B if there exists on JB a divided power structure γ' such that $\varphi\colon (A, J, \gamma) \to (B, JB, \gamma')$ is a divided power homomorphism.[15]

Définition

Définition

Proposition

Let (A, J, γ) be a ring endowed with an ideal with divided powers. We claim that if J is principle, then γ can always be extended. To see this, suppose that $J = (j)$ and let

Démonstration

[15] This structure on γ' is then unique.

$\varphi\colon A \to B$ be a homomorphism. If γ extends to γ', we necessarily have

$$\gamma'_n(b\varphi(J)) = b^n \varphi(\gamma_n(j)).$$

We claim that this determines a well-defined function γ'_n: first using the principle property to write $\gamma_n(j) = a_n j$, if $b\varphi(j) = b'\varphi(j)$ then

$$b^n \varphi(\gamma_n(j)) - (b')^n \varphi(\gamma_n(j)) = c(b-b')\varphi(j) = 0.$$

Next, let (A, J, γ) be a ring A with an ideal J endowed with divided powers γ. We claim that any flat A–algebra B admits an extension of γ to JB.[16] To construct $\gamma'_n\colon JB \to JB$, we consider the free module $\mathbb{Z}^{(J\times B)}$ on $J \times B$, and we define a function $g'_n\colon \mathbb{Z}^{(J\times B)} \to JB$ by

Proposition

Démonstration

$$g'_n(a_1(j_1, b_1) + \cdots + a_\ell(j_\ell, b_\ell)) = \sum_{\substack{i_1+i_2+\cdots+i_\ell=n \\ i_k \geq 0}} (a_1 b_1)^{i_1} \gamma_{i_1}(j_1) \cdots (a_\ell b_\ell)^{i_\ell} \gamma_{i_\ell}(j_\ell).$$

Using flatness, we have $J \otimes_A B \simeq JB$, hence to show that γ'_n is well-defined it suffices to define it on $J \otimes_A B$, by passing to the quotient of g'_n. To do this, we must prove that we have $g'_n(\beta + \alpha) = g'_n(\alpha)$ for any $\alpha \in \mathbb{Z}^{(J\times B)}$ and for any element $\beta \in \mathbb{Z}^{(J\times B)}$ which takes one of the following forms:

(1) $(j' + j'', b) - (j', b) - (j'', b)$,
(2) $(j, b + b') - (j, b) - (j, b')$,
(3) $(aj, b) - (j, ab)$.

This verification is left to the reader.

Example 4.1.14. Let V be a discrete valuation ring of mixed characteristic, with maximal ideal $\mathfrak{m}$ and residual characteristic p, and let e be the absolute ramification index, given by $pV = \mathfrak{m}^e$. The ideal $\mathfrak{m}$ is then stable under divided powers if and only if $e < p$, and it carries topologically nilpotent divided powers if and only if $e < p-1$. These are both immediate consequences of the formula recounted earlier above giving the valuation of $n!$.

Ex. 5) Cas d'un anneau de valuation discrète

Example 4.1.15. If A is a $\mathbb{Z}_{(p)}$–algebra, then the operations γ_n are determined by the single operation $\gamma_p = \pi$. Due to the invertibility of $(p-1)!$ in A, the early divided powers γ_0, γ_1, $\ldots$, γ_{p-1} are all fully determined. For a generic n, we take its p–adic expansion

Ex. 6)

$$n = a_0 + a_1 p + \cdots + a_r p^r,$$

with $0 \leq a_i \leq p-1$ and $a_r \neq 0$, from which we then calculate

$$\gamma_n(x) = c_n x^{a_0} \pi(x)^{a_1} (\pi^2(x))^{a_2} \cdots (\pi^r(x))^{a_r},$$

where π^i is the i^{th} iterate of π and where

$$c_n = \frac{1}{n!}(p!)^{\sum_i a_i(1+p+\cdots+p^{i-1})}$$

is an invertible element of $\mathbb{Z}_{(p)}$.

Question 4.1.16. What are some general conditions on π so that π is of the form γ_p? In the specific case where $J^2 = pJ = \{0\}$, a function $\pi\colon J \to J$ to come from a divided power structure if and only if π is additive and p–linear.

Pb.

4.2. The crystalline site of a scheme.

2. Site cristallin d'un schéma

Definition 4.2.1. Let S be a scheme, I a quasi-coherent ideal of $\mathcal{O}_S$, and γ a divided power structure on I for which Berthelot's condition is satisfied: $(n-1)!I^{(n)} = 0$ for n sufficiently large. For an S–scheme X, we now give the definition of the *crystalline site* associated to (X,S,I,γ).

2.1 Objects du site

Objects: The objects consist of triples $(U \subseteq U', \gamma_{U'})_{/S}$, where U is a Zariski open in X, $U \subseteq U'$ is a nilimmersion,[17] and $\gamma_{U'}$ is a divided power structure on the ideal J of $\mathcal{O}_{U'}$ associated to the subscheme $U \subseteq U'$, satisfying an additonal compatibility condition with the divided power structure γ on I, which we express locally in affines $X = \operatorname{Spec} A$, $U = \operatorname{Spec} B$, and $U' = \operatorname{Spec} B'$:

(1) On IB', there exists a divided power structure which extends that of I.[18]
(2) The divided powers on IB' and on J coincide on the intersection $J \cap IB'$.

(Alternatively, there exist divided powers on $J + IB'$ which are compatible with the divided powers of J and of I.)

2.2 Morphismes du site.

Morphisms: A morphism from (U,U') to (V,V') is an inclusion $U \subseteq V$ and a commuting morphism of schemes $U' \to V'$ such that the divided power structures on the associated ideals are compatible.

2.3. Topologie du site.

Topology: Finally, we place the "Zariski topology" on the crystalline site: the least fine topology such that, for all objects $U \subseteq U'$, the covering families are those families of moprhisms $(U_i, U_i') \to (U,U')$ where $U_i' \to U'$ is a covering of U' by Zariski opens and where $U_i = U \times_{U'} U_i'$.

2.4

Definition 4.2.2. The topos associated to the crystalline site is called the *crystalline topos*. We will denote it as $(X|_S, I, \gamma)_{\text{cris}}$ or, when no confusion is possible, as $(X|_S)_{\text{cris}}$ or even as X_{cris}.

2.5 Description d'un objet du topos cristallin.

Definition 4.2.3. Let $F \in X_{\text{cris}}$ be a sheaf of sets over the crystalline site. To every object (U,U') of the crystalline site[19] we associate a sheaf $F_{(U,U')}$ for the Zariski topology on U' by setting

$$F_{(U,U')}(V') = F(U \times_{U'} V', V')$$

for every open $V' \subseteq U'$. To every morphism $(u,u')\colon (V,V') \to (U,U')$, we associate the morphism of sheaves

$$u^{\natural}\colon (u')^* F_{(U,U')} \to F_{(V,V')}.$$

This construction respects composition, and $u^{\natural}$ is an isomorphism if u' is an open immersion. Conversely, giving such a family of sheaves $F_{(U,U')}$ together with transition morphisms $u^{\natural}$ satisfying these properties determines a unique object of X_{cris}.

Remarque.

Remark 4.2.4. There are other variants of the crystalline site, which we will mention only in passing, since we will not make use of them: there is the *infinitesimal topos*, which is the same definition but without divided powers, and there is the *stratified topos*, where one takes as objects the thickenings $U \subseteq U'$ endowed with a retraction.

[16] In fact, it suffices just that $J \otimes B \to JB$ be an isomorphism.
[17] One also says that $U \subseteq U'$ is a "(nilpotent) thickening".
[18] This will hold, for example, if B' is A–flat or if I is principal.
[19] Of course, we are abbreviating $(U \subseteq U', \gamma_{U'})$.

Definition 4.2.5. By considering the sheaf $\mathscr{O}_{U'}$ for each pair (U, U'), Definition 4.2.3 defines a *sheaf of local rings* $\mathscr{O}_{X/S}$ on X_{cris}. As usual, it may also be constructed as the sheaf associated to the presheaf

2.6 Le faisceau d'anneaux $\mathscr{O}_{X/S}$

$$(U, U') \mapsto \Gamma(U', \mathscr{O}_{U'}).$$

This gives rise to a notion of $\mathscr{O}_{X/S}$–modules on the crystalline site: such an $\mathscr{O}_{X/S}$–module F is determined by giving $\mathscr{O}_{U'}$–module structures on the sheaves $F_{(U,U')}$, such that for each $u\colon (V, V') \to (U, U')$ the morphism $u^\natural$ induces a homomorphism of $\mathscr{O}_{V'}$–modules $u^\natural\colon u^*F_{(U,U')} \to F_{(V,V')}$.[20] A sheaf of modules on X_{cris} such that all the maps $u^\natural$ are isomorphisms is called a *special sheaf of modules*.

Definition 4.2.6. A special sheaf of modules F is called a *crystal in modules*. If the sheaves $F_{(U,U')}$ are additionally all quasicoherent (resp. locally free) $\mathscr{O}_{U'}$–modules, then we say that F is a *quasicoherent* (resp. *locally free*) *crystal in modules*.

2.7. Cristaux

Définition

Proposition 4.2.7. *An $\mathscr{O}_{X/S}$–module is quasicoherent (resp. locally free) if and only if it is a quasicoherent (resp. locally free) crystal in modules, i.e., in these cases specialness is redundant.* □

Proposition.

Definition 4.2.8. More generally, let F be a category fibered over $\mathsf{Sch}_{/S}$. An F–crystal assigns to every object (U, U') of the crystalline site an object $F_{(U,U')}$ of $\mathrm{Ob}\,\mathsf{F}(U')$, where $\mathsf{F}(U')$ denotes the fiber category of F over U', and it assigns to every morphism $v\colon (V, V') \to (U, U')$ an *isomorphism*

2.8. F–Cristaux.

$$u\colon u^*F_{(U,U')} \to F_{(V,V')}$$

which altogether satisfy the usual transitivity conditions.

Remark 4.2.9. The notion of crystal is often more interesting than that of a crystalline sheaf.

4.3. **Relation between crystals and Witt vectors.** Let S be a scheme over $\Lambda_N = \mathbb{Z}/p^n\mathbb{Z}$. We will consider three topoi associated to S:

3. Relation entre cristaux et vecteurs de Witt

(1) The Zariski topos S_{Zar}.
(2) The crystalline topos $S_{n-\mathrm{cris}} := (S|_{\Lambda_n}, p\Lambda_n, \gamma)_{\mathrm{cris}}$, i.e., the ringed crystalline topos (of Berthelot) of S relative to Λ_n at the ideal $p\Lambda_n$ with the canonical divided power structure.
(3) The "Witt topos", which we now introduce. Let $\mathbb{W}_n$ be the scheme of Witt vectors of length n, and let $\mathbb{W}_n(\mathscr{O}_S)$ be the sheaf of rings
$$U \mapsto \mathbb{W}_n(\Gamma(U, \mathscr{O}_U))$$
on S_{Zar}. The Witt topos $S_{n-\mathrm{Witt}}$ is then the ringed topos
$$S_{n-\mathrm{Witt}} = (S_{\mathrm{Zar}}, \mathbb{W}_n(\mathscr{O}_S)).$$

Supposing that S is of characteristic p, we will now define morphisms of ringed topoi

$$\psi_n\colon S_{n-\mathrm{Witt}} \to S_{n-\mathrm{cris}}, \qquad \varphi_n\colon S_{n-\mathrm{cris}} \to S_{n-\mathrm{Witt}}$$

[20]Here, the inverse image is being calculated in the sense of modules.

satisfying $\varphi_n \psi_n = F^n_{\mathbb{W}_n}$, by which we mean the identity morphism on the objects of $S_{n-\mathrm{Witt}}$ and the n^{th} power of the Frobenius endomorphism of $(\mathbb{W}_n)_{\mathbb{F}_p}$ on the sections of $\mathbb{W}_n(\mathscr{O}_S)$. We will do this in several steps.

3.1 Lemme

Lemma 4.3.1. *There is a unique divided power structure on the augmentation ideal of $\mathbb{W}_n(\mathscr{O}_{\mathbb{Z}_{(p)}})$ over $\mathscr{O}_{\mathbb{Z}_{(p)}}$.*

Proof. First, note that the Verschiebung induces an isomorphism

$$V\colon \mathbb{W}_n(B) \xrightarrow{\sim} \ker\{\mathbb{W}_n(B) \to B\},$$

so to define γ_N on the augmentation ideal of $(\mathbb{W}_n)_{\mathbb{Z}_p}$ we may instead define the composite $\gamma_N \circ V$ on all of $\mathbb{W}_n$. For an $\mathbb{Z}_{(p)}$–algebra B and a point $x \in \mathbb{W}_n(B)$, we set

$$\gamma_N V(x) = \frac{p^{N-1}}{N!} V(x^N),$$

which is well-defined because $p^{N-1}/N!$ lies in $\mathbb{Z}_{(p)}$ and $\mathbb{W}_n(B)$ is a $\mathbb{Z}_{(p)}$–algebra.

To show that this defines a divided power structure and that it is unique, we consider the homomorphism $\varphi\colon \mathbb{W}_n \to \mathscr{O}^n$ used in Section 1.1 to define the ring structure of $\mathbb{W}_n$. This morphism satisfies

$$\varphi \circ V = p \cdot V' \circ \varphi,$$

where V' is the endomorphism $(x_1, \dots, x_n) \mapsto (0, x_1, \dots, x_{n-1})$ in $\mathscr{O}^n$. Because this morphism is also multiplicative, one may thus write

$$\begin{aligned} \varphi\gamma_N V(x) = \varphi\left(\frac{p^{N-1}}{N!} V(x^N)\right) &= \frac{p^{N-1}}{N!}\varphi(V(x^N)) = \frac{p^{N-1}}{N!} p V'(\varphi(x^N)) \\ &= \frac{p^N}{N!}[V'(\varphi(x))]^N = \frac{1}{N!}[p \cdot V'(\varphi(x))]^N = \frac{1}{N!}[\varphi(V(x))]^N. \end{aligned}$$

Setting $y = V(x)$, this gives

$$\varphi(\gamma_N(y)) = \frac{1}{N!}\varphi(y^N),$$

which shows γ_N to satisfy the axioms of a divided power structure. As $\mathbb{Z}_p$ is integral, this equality uniquely determines

$$\gamma_N(y) = (0, g_1, \dots, g_N) \in \mathbb{W}_n(B),$$

where the g_i are polynomials with coefficients in $\mathbb{Z}_p$ (and not merely in $\mathbb{Q}$), and hence γ_N determines a divided power structure on the augmentation ideal (rather than its rationalization). Since $\mathbb{Z}_{(p)}$ is integral, this divided power structure is furthermore unique. □

Remark 4.3.2. It follows that if X is a scheme of characteristic p, one may consider the scheme $X_{\mathbb{W}_n} = \operatorname{Spec} \mathbb{W}_n(\mathscr{O}_X) = (X, \mathbb{W}_n(\mathscr{O}_X))$ with the same topological space as X but with structure sheaf $\mathbb{W}_n(\mathscr{O}_X)$ as a thickening of X endowed with divided powers compatible with those of Λ_n. For X an open of S, $X_{\mathbb{W}_n}$ is then an object of the crystalline site of S over Λ_n. Thus, every crystal in modules (resp. algebras, resp. …) over S defines a module (resp. algebra, resp. …) over $S_{\mathbb{W}_n}$.

3.2 Définition de $\varphi_n: S_{n-\mathrm{Witt}} \to S_{n-\mathrm{cris}}$

Definition 4.3.3. We define a functor

$$\psi_n^*\colon S_{n-\mathrm{cris}} \to S_{n-\mathrm{Witt}},$$
$$\{F\colon (U,U') \mapsto F(U,U')\} \mapsto \{\psi_n^* F\colon U \mapsto F(U, U_{\mathbb{W}_n})\}$$

where $(U, U_{\mathbb{W}_n})$ is the natural thickening defined above.

Remark 4.3.4. The functor ψ_n^* interacts with $\mathcal{O}_{S_{n-\mathrm{cris}}}$ according to

$$\psi_n^* \mathcal{O}_{S_{n-\mathrm{cris}}}\colon U \mapsto \mathbb{W}_n(\Gamma(U, \mathcal{O}_U)),$$

and the ring homomorphism

$$\psi_n^* \mathcal{O}_{S_{n-\mathrm{cris}}} \to \mathbb{W}_n(\mathcal{O}_S)$$

is the identity.

3.3 Définition de φ_n

Let A be a Λ_n-algebra, and let $A \to A_0$ a surjective ring homomorphism such for each x in the kernel J and each $0 \le i \le n$, we have $p^i x^{p^{n-i}} = 0$.[21] We then consider the ring homomorphism

$$\Phi_{n+1}\colon \mathbb{W}_{n+1}(A) \to A,$$
$$(x_1, x_2, \dots, x_{n+1}) \mapsto x_1^{p^n} + p x_2^{p^{n-1}} + \dots + p^{n-1} x_n^p + p^n x_{n+1}.$$

This homomorphism is zero on $V^n \mathbb{W}_{n+1}(A) = \ker\{R_n\colon \mathbb{W}_{n+1}(A) \to \mathbb{W}_n(A)\}$, and hence it defines the following morphism $\varphi' = \varphi_n F$ on the quotient:

$$\varphi'\colon \mathbb{W}_n(A) \to A,$$
$$(x_1, \dots, x_n) \mapsto x_1^{p^n} + p x_2^{p^{n-1}} + \dots + p^{n-1} x_n^p.$$

Because of our standing assumption on the ideal J, it follows that φ' is zero on $\mathbb{W}_n(J)$ and thus that it factors through a homomorphism $\xi_{n,A}\colon \mathbb{W}_n(A_0) \to A$. The construction of $\xi_{n,A}$ is functorial in A.

Definition 4.3.5. We now use the above to define $\varphi_n\colon S_{n-\mathrm{cris}} \to S_{n-\mathrm{Witt}}$. The underlying morphism of topoi is given by

$$\varphi_n^*\colon S_{n-\mathrm{Witt}} \to S_{n-\mathrm{cris}}$$
$$\{F\colon U \mapsto F(U)\} \mapsto \{\varphi_n^* F\colon (U,U') \mapsto F(U)\}.$$

Using the notation of Definition 4.2.3), this expands to give $\varphi_n^* F(U,U') = i_* F/U$, where i denotes the nil-immersion $U \subseteq U'$. In particular,

$$\varphi_n^* \mathbb{W}_n(\mathcal{O}_S)\colon (U,U') \mapsto \mathbb{W}_n(\Gamma(U, \mathcal{O}_U)),$$

so that we may take for the ring homomorphism $\xi\colon \varphi_n^* \mathbb{W}_n(\mathcal{O}_S) \to \mathcal{O}_{S_{n-\mathrm{cris}}}$ the map

$$(U,U') \mapsto (\xi_{n,\Gamma(U',\mathcal{O}_{U'})}\colon \mathbb{W}_n(\Gamma(U,\mathcal{O}_U)) \to \Gamma(U', \mathcal{O}_{U'})).$$

3.4 Study of the composite $\varphi_n \psi_n\colon S_{n-\mathrm{Witt}} \to S_{n-\mathrm{Witt}}$

Lemma 4.3.6. *These functors satisfy* $\varphi_n \psi_n = F^n_{\mathbb{W}_n}$.

[21]For example, this condition is satisfied if J is an ideal with divided powers. To wit, if $p^n x = 0$ for all $x \in J$, then $p^i x^{p^{n-i}} = p^i p^{n-i}! x^{(p^{n-i})} = p^n y = 0$.

Proof sketch. It is clear that the underlying morphism of topoi $\psi_n^* \varphi_n^*$ is the identity, but we must work to show that the ring homomorphism

$$\psi_n^* F\colon \mathbb{W}_n(\mathcal{O}_S) = \psi_n^* \varphi_n^* \mathbb{W}_n(\mathcal{O}_S) \to \psi_n^* \mathcal{O}_{S_{n-\mathrm{cris}}} \xrightarrow{\mathrm{Id}} \mathbb{W}_n(\mathcal{O}_S)$$

is given by

$$U \mapsto \{F^n\colon \mathbb{W}_n(\Gamma(U, \mathcal{O}_U)) \to \mathbb{W}_n(\Gamma(U, \mathcal{O}_U))\}.$$

To accomplish this, we will use the construction of $\xi_{n,A}$ in the case where A_0 is an $\mathbb{F}_p$-algebra, $A = \mathbb{W}_n(A_0)$, and J is the augmentation ideal. Take ε to be the following multiplicative system of representatives:

$$\varepsilon\colon A_0 \to A = \mathbb{W}_n(A_0),$$
$$\varepsilon\colon x \mapsto (x, 0, \ldots, 0)$$

In general, we lift $(x_1, x_2, \ldots, x_n) \in \mathbb{W}_n(A_0)$ as $(\varepsilon(x_1), \varepsilon(x_2), \ldots, \varepsilon(x_n)) \in \mathbb{W}_n(A)$, which gives

$$\xi_{n,A}(x_1, \ldots, x_n) = \sum_{i=0}^{n-1} p^i \varepsilon(x_{i+1}^{p^{n-1}}) = (x_1^{p^n}, \ldots, x_n^{p^n}),$$

because, following Proposition 1.1.5, one has

$$\sum_{i=0}^{n-1} p^i \varepsilon(y_{i+1}) = (y_1, y_2^p, \ldots, y_n^{p^n}).$$

The remainder of the study of the composite $\psi_n \varphi_n$ on $S_{n-\mathrm{cris}}$ is left as an exercise. □

4. Cas d'un schéma parfait

4.4. **The case of a perfect scheme.** In this section, we take S to be of characteristic p and investigate the crystalline topos $S_{\mathrm{cris}} = (S|_{\mathbb{Z}_{(p)}}, p\mathbb{Z}_{(p)})$ of S over $\mathbb{Z}_{(p)}$. In fact, we may work over either of $\mathbb{Z}_{(p)}$ or $\mathbb{Z}_p$: for any nil-immersion (U, U'), p is locally nilpotent on U', so if U is furthermore quasicompact, then (U, U') is an object of the crystalline site of S over Λ_n for some $n \gg 0$. We may thereby identify the crystalline sites of S over $\mathbb{Z}_{(p)}$ and over $\widehat{\mathbb{Z}}_{(p)} = \mathbb{Z}_p$.

4.1. Notations

Begin by considering a ring A and an ideal I of A. For every integer $n \geq 1$, we define an ideal I_n as

$$I_n = \langle p^{i-1} x^{p^{n-1}} \mid x \in I, 1 \leq i \leq n \rangle.$$

These ideals form a descending sequence, and they are related to the original ideal by $I_1 = I$, $p^{n-1} I \subseteq I_n$. If the original ideal I carries divided powers, then this inclusion becomes the equality $p^{n-1} I = I_n$, as for each $y \in I$ we then have

$$p^{i-1} x^{p^{n-i}} = p^{i-1} p! x^{(p^{n-1})} = p^{n-1} y.$$

If we suppose moreover that p belongs to I, and hence p^n belongs to I_n, we gain the additional inclusions

$$p^n A \subseteq I_n = p^{n-1} I \subseteq p^{n-1} A.$$

4.2. Théorème

Theorem 4.4.1. *Let A be a ring and $I \subseteq A$ an ideal. Suppose that $A_1 = A/I$ is of characteristic $p > 0$ and perfect, and suppose that there is an equivalence $A \xrightarrow{\sim} \lim A/I_n$, i.e., A is* *separated and complete for the topology defined by the ideals I_n.*[22] *There then exists a unique homomorphism $u\colon \mathbb{W}(A_1) \to A$, compatible with the augmentations of A and of $\mathbb{W}(A_1)$, with the following additional properties:*

(1) *For all n, we have*

$$u^{-1}(I_n) \supseteq V^n \mathbb{W}(A_1) = \ker\{\mathbb{W}(A_1) \to \mathbb{W}_n(A_1)\}.$$

(Consequently, u is continuous.)

(2) *For $\varepsilon\colon A_1 \to \mathbb{W}(A_1)$ a multiplicative section, the image $R \subseteq A$ of $u\varepsilon$ is the set of those $x \in A$ such that for all n there is a $y \in A$ with $y^{p^n} = x$. The set R is stable under multiplication, and the function $R \to A_1$ induced by the augmentation $A \to A_1$ is bijective. Hence, it admits an inverse function $\alpha\colon A_1 \to R \subseteq A$ which is multiplicative.*

(3) *For $x = (x_1, x_2, \ldots, x_i, \ldots) \in \mathbb{W}(A_1)$, one has*

$$u(x) = \sum_{i \geq 0} p^i \alpha(x_{i+1}^{p^{-i}})$$

as a convergent series in A.

Proof. We will show first that *every* homomorphism $u\colon \mathbb{W}(A_1) \to A$ compatible with the augmentations automatically possesses the other properties.

(1) Since A_1 is of characteristic p, it follows that $p^n \in I_n$. The map u then sends $p^n \mathbb{W}(A_1) = V^n \mathbb{W}(A_1)$ to I_n.

(2) Because the composition of $u\varepsilon\colon A_1 \to R$ with the augmentation is the identity, it follows that u is injective and hence bijective. Using this, we define $\alpha = u\varepsilon$ and we turn to the characterization of it image. As A_1 is perfect and isomorphic to R by α, for every n we have $R = R^{p^n}$, from which it follows that the image of α is contained in the claimed set. Conversely, we then consider an element $x \in A$ of the desired type: for each n there is a choice of y_n with $y_n^{p^n} = x$. Let $\overline{x}$ and $\overline{y}_n$ be the residue classes of x and of y_n in A_1, so that $\overline{y}_n^{p^n} = \overline{x}$. Appealing to perfection yields $\overline{y}_n = \overline{x}^{p^{-n}}$, hence $v = y_n - \alpha(\overline{x})^{p^{-n}} \in I$. From this we deduce

$$x = y_n^{p^n} = (\alpha(\overline{x})^{p^{-n}} + v)^{p^n} = \alpha(\overline{x}) + v',$$

where $v' = p^n v + \cdots$ belongs to $p^n I \subseteq I_{n+1}$. As A is separated, we at last conclude $x = \alpha(\overline{x})$.

(3) Using $\alpha = u\varepsilon$ and Proposition 1.1.5, one has

$$x = (x_1, x_2, \ldots, x_i, \ldots) = \sum_{i \geq 0} p^i \varepsilon(x_{i_1})^{p^{-i}}$$

and hence

$$u(x) = \sum_{i \geq 0} p^i \alpha(x_i^{p^{-i}}).$$

[22]The second condition is satisfied when I is nilpotent, or when I has divided powers and A is separated and complete for the p-adic topology.

The second property uniquely determines α using the formula in the third property, from which it follows that u is also unique. It remains to actually *construct* such a homomorphism u, and to accomplish this we define a family of morphisms

$$u_n\colon \mathbb{W}_n(A_1) \to A/I_n$$

which are compatible with the transition morphisms and which satisfy $u_1 = \mathrm{Id}(A_1)$. As the kernel I/I_n of $A/I_n \to A/I = A_1$ satisfies the condition in Section 4.3, we produce a homomorphism

$$\xi_{n,A/I_n}\colon \mathbb{W}_n(A_1) \to A/I_n$$

obtained as a quotient of the homomorphism

$$\varphi_n F = \varphi'\colon \mathbb{W}_n(A/I_n) \to A/I_n$$
$$\varphi'\colon (x_1, \ldots, x_n) \mapsto x_1^{p^n} + p x_2^{p^{n-1}} + \cdots + p^{n-1} x_n^p.$$

The morphism $\xi_{n,A/I_n}$ does not commute with the transition map, but one at least has the following commutative diagrams:

$$\begin{tikzcd}
\mathbb{W}_{n+1}(A_1) \arrow[rr, "\xi_{n+1,A/I_{n+1}}"] \arrow[d, "R"] & & A/I_{n+1} \arrow[d] \\
\mathbb{W}_n(A_1) \arrow[r, "F"] & \mathbb{W}_n(A_1) \arrow[r, "\xi_{n,A/I_n}"] & A/I_n.
\end{tikzcd}$$

Again appealing to the perfection of A_1, we may construct homomorphisms

$$u_n\colon \mathbb{W}_n(A_1) \to A/I_n$$

compatible with the transition morphisms by setting $u_n = \xi_{n,A/I_n} \circ F^{-n}$. In particular, for $n = 1$ we have $u_1 = \mathrm{Id}(A_1)$. □

4.3 Proposition

Proposition 4.4.2. *Rather than suppose A_1 to be perfect, instead assume only the second condition of Theorem 4.4.1. Then for all perfect rings B_1 of characteristic $p > 0$ and for all homomorphisms $v_1\colon B_1 \to A_1$, there exists a unique homomorphism $v\colon \mathbb{W}(B_1) \to A$ which is compatible with v_1 on the augmentation ideals. The homomorphism v is continuous and it again satisfies*

$$v^{-1}(I_n) \subseteq V^n \mathbb{W}(B_1) = \ker\{\mathbb{W}(B_1) \to \mathbb{W}_n(B_1)\}.$$

Proof sketch. This proof is a repeat of the preceding calculation after replacing the homomorphism u_n with the homomorphism

$$v_n = \xi_{n,A/I_n} \circ \mathbb{W}_n(v_1) \circ F^{-n}\colon \mathbb{W}_n(B_1) \to A/I_n. \qquad \square$$

4.4 Corollaire

Corollary 4.4.3. *For S a perfect scheme of characteristic p, $(S, S_{\mathbb{W}_n})$ is the terminal object of the crystalline site of S over Λ_n, where $S_{\mathbb{W}_n} = \mathrm{Spec}(\mathbb{W}_n(\mathcal{O}_S))$ denotes the thickening of Lemma 4.3.1.*

Proof. The claim is that for (U, U') any object of the crystalline site of S over Λ_n, there exists a unique homomorphism (in the site)

$$f\colon (U, U') \to (S, S_{\mathbb{W}_n}).$$

One deduces immediately from Theorem 4.4.1 that for every affine open $V' = \operatorname{Spec} A$ of U', one may set $V = V' \times_{U'} U = \operatorname{Spec} A_1$ and thereby determine a unique homomorphism

$$U\colon \mathbb{W}_n(\Gamma(V, \mathscr{O}_V)) \to \Gamma(V', \mathscr{O}_{V'})$$

compatible with the augmentations. These homomorphisms glue to define the desired unique homomorphism

$$u\colon f^*\mathbb{W}_n(\mathscr{O}_S) = \mathbb{W}_n(\mathscr{O}_U) \to \mathscr{O}_{U'},$$

itself also compatible with the augmentations. □

Proposition 4.4.4. *For S be a perfect scheme of characteristic p, the category of crystals in quasicoherent (resp. locally free) modules of the crystalline site of S over Λ_n is equivalent to the category of those sheaves of quasicoherent (resp. locally free) $\mathbb{W}_n(\mathscr{O}_S)$–modules.* 4.5 Proposition

Proof. We claim more generally that if F is a fibered category over $\mathsf{Sch}_{/\Lambda_n}$, then the category of F–crystals on S is equivalent to the fiber $\mathsf{F}(S_{\mathbb{W}_n})$. Under this correspondence, a crystal F is sent to the object $\mathsf{F}_{(S,S_{\mathbb{W}_n})} \in \mathsf{F}(F_{\mathbb{W}_n})$. Conversely, an object G of $\mathsf{F}(S_{\mathbb{W}_n})$ is sent to the crystal defined over every thickening (U, U') by the formula

$$G'_{(U,U')} = f^*G,$$

where $f\colon (U, U') \to (S, S_{\mathbb{W}_n})$ is the canonical morphism. The particular case of quasicoherent (resp. locally free) modules then follows from Proposition 4.2.7. □

Proposition 4.4.5. *Let S be a perfect scheme of characteristic p, and let F be a fibered category over $\mathsf{Sch}_{/\mathbb{Z}_{(p)}}$ which is "gluable for the Zariski topology"—i.e., for every scheme X, the restriction of F to the site of Zariski opens of X is a stack.*[23] *The category of F–crystals over S is equivalent to the category $\lim \mathsf{F}(S_{\mathbb{W}_n})$ of systems of objects $G_n \in \mathsf{F}(S_{\mathbb{W}_n})$ which satisfy $G_n = i^*G_{n+1}$, where i denotes the canonical immersion $S_{\mathbb{W}_n} \to S_{\mathbb{W}_{n+1}}$.* 4.6 Proposition

Proof. To the F–crystal F we associate the system $G_n = F_{(S,S_{\mathbb{W}_n})} \in \mathsf{F}(S_{\mathbb{W}_n})$. Conversely, given a system $(G_n) \in \lim \mathsf{F}(S_{\mathbb{W}_n})$ and a thickening (U, U') in the site, we obtain $F_{(U,U')} \in \mathsf{F}(U')$ by gluing the objects $F_{(V,V')} \in f^*G_n$, where V' is a quasicompact open of V (so that $p^n = 0$ for some n) and where f still denotes the canonical homomorphism

$$f\colon (V, V') \to (S, S_{\mathbb{W}_n}).$$ □

Example 4.4.6. The category crystals in locally free modules of finite type is equivalent to the category of projective $\mathbb{W}(\mathscr{O}_S)$–modules of finite type. Exemple

We now generalize away from the case where S itself is perfect, supposing instead that S is a scheme over a perfect field k of characteristic p. In this setting, we have $W = \mathbb{W}(k)$ and $W_n = \mathbb{W}_n(k)$. 4.7

Proposition 4.4.7. *For (U, U') a thickening of the crystalline site of S over Λ_n, there exists exactly one homomorphism* Proposition

$$g\colon (U, U') \to (\operatorname{Spec} k, \operatorname{Spec} W_n).$$

This homomorphism is additionally compatible with the divided power structures.

[23] Cf. Giraud, *Cohomologie Non Abélienne*.

Proof. The proof is an immediate consequence of Proposition 4.4.2 and is also analogous to Corollary 4.4.3. □

Corollaire

Corollary 4.4.8. *Let S be a scheme over a perfect field k. The crystalline sites of S over $\mathbb{Z}_{(p)}$ (resp. over Λ_n) and over W (resp. over W_n) are isomorphic.* □

5. Cas d'un schéma relatif lisse

Ed. note: NilCris hasn't been introduced yet.

4.5. **Case of a relatively smooth scheme.** Let (S, I, γ) be as in Section 4.2, let X be an S–scheme, and let F be a special module on the site $\mathsf{NilCris}(X/S, I, \gamma)$. We now aim to relate the data of a crystal, which is naively quite extensive and involved, to a more polite presentation in terms of differential geometry.

Let $\Delta^1(X)$ denote the first infinitesimal neighborhood of X in $X \times_S X$. On the ideal determining the closed subscheme X of $\Delta^1 X$, we may define divided powers by declaring γ_1 to be the identity and γ_n to be the zero map for $n \geq 2$, thereby obtaining an object $X \subseteq \Delta^1(X)$ of the nilpotent crystalline site. The commutative diagram

$$\begin{tikzcd} X \arrow[rr, "\mathrm{Id}_X"] \arrow[dr] & & X \\ & \Delta^1(X). \arrow[ur, shift left, "\pi_1"] \arrow[ur, shift right, "\pi_2"'] & \end{tikzcd}$$

interrelates the values of the special module F on X and on $\Delta^1 X$ via the isomorphisms

$$\pi_1^*(F_{X\subseteq X}) \xrightarrow{\sim} F_{X\subseteq \Delta^1 X} \xleftarrow{\sim} \pi_2^*(F_{X\subseteq X}).$$

Définition (5.1)

Definition 4.5.1. A *connection* on an $\mathcal{O}_X$–module M is an infinitesimal descent datum

$$\varphi \colon \pi_1^*(M) \xrightarrow{\sim} \pi_2^*(M)$$

which satisfies the usual cocycle condition for the diagram

$$\Delta^1_2(X) \mathrel{\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}} \Delta^1(X) \rightrightarrows X,$$

where Δ^1_2 is the first infinitesimal neighborhood of X in $X \times_S X \times_S X$.

Remark 4.5.2. Such a datum is equivalent to an S–linear morphism

$$\nabla \colon M \to \Omega^1_{X/S} \otimes_{\mathcal{O}_X} M$$

which, for any local section a of $\mathcal{O}_X$ and any local section m of M, satisfies

$$\nabla(am) = da \otimes m + a\nabla m.$$

Definition 4.5.3. Recall that one says that a connection is *flat* when the $\mathcal{O}_X$–linear morphism

$$\nabla \colon M \to \Omega^1_{X/S} \otimes_{\mathcal{O}_X} M$$

and the morphism

$$\begin{aligned} \Omega^1_{X/S} \otimes M &\to \Omega^2_{X/S} \otimes M \\ \omega \otimes f &\mapsto d\omega \otimes f - \omega \otimes \nabla f \end{aligned}$$

compose to zero.

Définition (5.2)

Definition 4.5.4. Let F and G be two modules with connections φ_F and φ_G respectively. A morphism $u \colon F \to G$ is said to be *horizontal* when the following diagram commutes:

$$\begin{array}{ccc} \pi_1^* F & \xrightarrow{\varphi_F} & \pi_2^* F \\ \downarrow{\scriptstyle \pi_1^*(u)} & & \downarrow{\scriptstyle \pi_2^*(u)} \\ \pi_1^* G & \xrightarrow{\varphi_G} & \pi_2^* G. \end{array}$$

Our discussion of $\Delta^1 X$ defines a functor

$$\left\{ \begin{array}{c} \text{crystals over } X/S \\ \text{in quasicoherent modules} \end{array} \right\} \to \left\{ \begin{array}{c} \text{modules over } X \\ \text{with a connection} \end{array} \right\}.$$

Proposition 4.5.5 (Berthelot)**.** *For X smooth over S, this functor becomes an equivalence of categories between special quasicoherent modules on the nil-crystalline site of X over S and quasicoherent $\mathcal{O}_X$–modules endowed with a flat connection.* □ Proposition (5.4)

We also announce a second theorem of Berthelot, where the nilpotence condition on the crystalline site is relaxed somewhat.

Proposition 4.5.6 (Berthelot)**.** *For X smooth over S and p nilpotent on S, the same functor induces an equivalence of categories between special modules on the Berthelot crystalline site and the category of $\mathcal{O}_X$–modules endowed with a* nilpotent *connection of null curvature.* □ Proposition (5.5)

We won't prove these results, but we will sketch the significance of the nilpotence condition on the connection. Let S_0 be the closed subscheme of S defined by the ideal $p\mathcal{O}_S$, let X_0 be the restriction $X_0 = X \times_S S_0$, and let F be an $\mathcal{O}_X$–module endowed with a flat connection ∇, which in turn induces a flat connection ∇_0 on the inverse image F_0 of F over X_0. One associates to ∇ (and to ∇_0) a morphism

$$\begin{aligned} \overline{\nabla}\colon\ \mathrm{Der}_S(\mathcal{O}_X, \mathcal{O}_X) &\to \mathrm{End}_{\mathcal{O}_S}(F, F), \\ D &\mapsto \left(F \xrightarrow{\nabla} \Omega^1_{X/S} \otimes F \xrightarrow{D \otimes 1} F \right). \end{aligned}$$

This construction enjoys the following properties:

(1) $\overline{\nabla}(D_1 + D_2) = \overline{\nabla}(D_1) + \overline{\nabla}(D_2)$.
(2) $\overline{\nabla}(aD) = a\overline{\nabla}(D)$.
(3) $\overline{\nabla}(D)(af) = D(a)f + a\overline{\nabla}(D)(f)$.

If X is smooth over S, the data of a $\overline{\nabla}$ satisfying these properties is equivalent to that of a connection, and the additional statement that the connection is flat corresponds to the relation $\overline{\nabla}([D_1, D_2]) = [\overline{\nabla}(D_1), \overline{\nabla}(D_2)]$.

Over S_0, we follow Katz and define the *the p–curvature*,

$$\begin{aligned} \Psi\colon\ \mathrm{Der}_{S_0}(\mathcal{O}_{X_0}, \mathcal{O}_{X_0}) &\to \mathrm{End}_{S_0}(F_0, F_0), \\ D &\mapsto \overline{\nabla}_0(D^p) - (\overline{\nabla}_0(D))^p. \end{aligned}$$

This function Ψ possesses the following properties:

(1) For all D, $\Psi(D)$ is $\mathcal{O}_{X_0}$–linear.
(2) Ψ is additive.
(3) $\Psi(aD) = a^p \Psi(D)$.
(4) $\Psi(D)$, $\overline{\nabla}_0(D)$, and $\overline{\nabla}_0(D^p)$ commute with each other.
(5) As D varies, all of $\Psi(D)$ commute with one another.

Définition (5.6)

Definition 4.5.7. One says that the module F is *nilpotent* if the following equivalent conditions are true, each of which is local for the Zariski topology:

(1) The endomorphisms $\overline{\nabla}_0(D_i)$ are nilpotent for a suitable base $\{D_i\}$ of $\mathrm{Der}_S(\mathscr{O}_X, \mathscr{O}_X)$.
(2) For all D, $\Psi(D)$ is nilpotent.
(3) For all nilpotent D, $\overline{\nabla}(D)$ is nilpotent.

6. Cristaux sur un schéma sur un corps parfait de caractéristique p

4.6. **Crystals on a scheme over a perfect field of characteristic p.** Let k be a perfect field of characteristic p, and let X be a k–scheme. Recall that in Corollary 4.4.8 we found the sites $\mathrm{Cris}(X_0/\mathbb{W}(k))$ and $\mathrm{Cris}(X_0/\mathbb{Z}_p)$ to be equivalent. Moreover, for a fibered category F over Sch, recall also that there is an equivalence of categories

$$\mathrm{Cris}_{\mathsf{F}}(X_0/\mathbb{W}(k)) \xrightarrow{\sim} \lim \mathrm{Cris}_{\mathsf{F}}(X_0/\mathbb{W}_n(k)).$$

Let us suppose that X_0 is smooth over k and that for all n, X_0 lifts to a smooth scheme X_n over $\mathbb{W}_n(k)$ such that $X_n \simeq X_{n+1} \otimes \mathbb{W}_n$—i.e., each of the following squares is a pullback:[24]

$$\begin{tikzcd}
\cdots \arrow[r] & X_n \arrow[r] \arrow[d] & X_{n+1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \operatorname{Spec} \mathbb{W}_n(k) \arrow[r] & \operatorname{Spec} \mathbb{W}_{n+1}(k) \arrow[r] & \cdots .
\end{tikzcd}$$

The category of special modules on $\mathrm{Cris}(X_0/\mathbb{W}(k))$ is then equivalent to the category of gluable systems of modules F_n over X_n, each endowed with a compatible flat connection. By passing to the limit, one produces an equivalence with the category of F–modules on the formal $\mathbb{W}(k)$–scheme $\mathscr{X} = \operatorname{colim} X_n$, endowed with a “formal” flat connection.

We now consider the case where X_0 is additionally proper over k and that it lifts to a proper and flat scheme X over $\mathbb{W}(k)$, so that we might apply GAGA. The category of special modules over $\mathrm{Cris}(X_0/\mathbb{W}(k))$ is then equivalent to the category of $\mathscr{O}_X$–modules endowed with a flat connection. Let K be the field of fractions of $\mathbb{W}(k)$, select an embedding $K \subseteq \mathbb{C}$. The data of an $\mathscr{O}_X$–module with a flat connection then defines an object of the same type over X_K—but, by GAGA, this is equivalent to a module with the same structure over the complex variety $X(\mathbb{C})$, which is in turn equivalent to giving a local coefficient system.

Remark 4.6.1. In the case where $\mathscr{X}$ is not algebraizable, one may instead consider Tate's rigid analytic space associated to $\mathscr{X}$. Khiel [Kie67] has shown in this context that the crystalline cohomology of X_0 is essentially the same as the de Rham cohomology of this rigid analytic space.

7. Indications sur la cohomologie cristalline

4.7. **Indications on crystalline cohomology.** Let us return to fully general conditions by suppoing that the ideal I over S satisfies Berthelot's condition. Having set up the crystalline site, we are naturally interested in the associated cohomology objects:

(1) The cohomology groups

$$H^*_{\mathrm{cris}}(X/S) = H^*_{\mathrm{cris}}(X/S, I, \gamma) := H^*((X/S, I, \gamma)_{\mathrm{cris}}, \mathscr{O}_{X/S}).$$

(2) The complex of crystalline cohomology sheaves

$$\underline{H}_{\mathrm{cris}}(X/S) = \underline{H}_{\mathrm{cris}}(X/S, I, \gamma) := \mathbb{R}(f_{\mathrm{cris}})_*(\mathscr{O}_{X/S}),$$

[24]For example, this will be the case when X_0 is affine and smooth over k.

where $f_{\text{cris}}\colon X_{\text{cris}} \to S_{\text{cris}}$ is the morphism of topoi induced by f.

(3) The associated cohomology groups of the sheaves $\underline{H}^*_{\text{cris}}(X/S)$, which are crystals in modules on the ringed topos S_{cris}.[25]

There are a number of indications that these give a "good" cohomology theory, primarily when $X_0 = X \times_S S_0$ is proper and smooth over $S_0 = \mathrm{Var}(I)$. For instance:

Invariance: There are canonical isomorphisms

$$H^*_{\text{cris}}(X/S) \xrightarrow{\sim} H^*_{\text{cris}}(X_0/S),$$
$$\underline{H}_{\text{cris}}(X/S) \xrightarrow{\sim} \underline{H}_{\text{cris}}(X_0/S).$$

Link with de Rham cohomology: If X is smooth over S, one has

$$H^*_{\text{cris}}(X/S) \xrightarrow{\sim} H^*_{\text{dR}}(X/S) := H^*(X, \underline{\Omega}^*_{X/S}),$$
$$\underline{H}^*_{\text{cris}}(X/S)_S \xrightarrow{\sim} \underline{H}^*_{\text{dR}}(X/S)_S := \mathbb{R}^* f_*(\underline{\Omega}^*_{X/S}),$$

where $\underline{\Omega}^*_{X/S}$ is the de Rham complex of X relative to S and

$$\underline{H}^*_{\text{cris}}(X/S)_S = \mathbb{R}^*(f_{\text{cris}})_*(\mathscr{O}_{X/S})_S$$

is the restriction of the complex of crystalline sheaves $\mathbb{R}^*(f_{\text{cris}})_*(\mathscr{O}_{X/S})$ to the underlying Zariski site. The connection on $\mathbb{R}^*(f_{\text{cris}})_*(\mathscr{O}_{X/S})_S$ induces one on $\underline{H}^*_{\text{dR}}(X/S)$, and it turns out to be the Gauss–Manin connection.

These comparison theorems prove in particular that if S is the spectrum of a characteristic 0 field, then the crystalline cohomology of X gives the "good" Betti numbers.[26]

Deligne's results in characteristic 0: Just to pick one of many: if S is the spectrum of the complex field, and if X is locally of finite type over S, then one has a canonical isomorphism

$$H^*_{\text{cris}}(X/S) = H^*(X(\mathbb{C}), \mathbb{C}).$$

Change of base (at the level of the derived category): Given a morphism of bases $u\colon (S', I', \gamma') \to (S, I, \gamma)$, then for X_0 flat and coherent over S_0 there is an induced map

$$u^*(\mathbb{R}(f_{\text{cris}})_*(\mathscr{O}_{X/S})) \xrightarrow{\sim} \mathbb{R}(f'_{\text{cris}})_*(\mathscr{O}_{X'/S'}),$$

where $f'\colon X' \to S'$ is the morphism deduced from $f\colon X \to S$ by base extension from S to S'.

Künneth formula: For two relatively coherent S–schemes X and X' with X_0 and X'_0 smooth,

$$\mathbb{R}((f \times_S f')_{\text{cris}})_*(\mathscr{O}_{X \times_S X'/S}) \xrightarrow{\sim} \mathbb{R}(f_{\text{cris}})_*(\mathscr{O}_{X/S}) \overset{L}{\otimes} \mathbb{R}(f'_{\text{cris}})_*(\mathscr{O}_{X'/S}).$$

Finiteness: If X_0 is proper and smooth over S_0, the complex $\mathbb{R}(f_{\text{cris}})_*(\mathscr{O}_{X/S})$ is perfect—i.e., it is Zariski-locally isomorphic to a complex L^* such that all the L^i are locally free and of finite type, and $L^i = 0$ except for a finite number of i.

[25] One may also take cohomology with more general coefficients, for example with coefficients in a crystal of quasicoherent modules....

[26] In particular, this reduces to the case $k = \mathbb{C}$, and from there one proceeds by the transcendent way, utilizing a theorem of [Gro66].

Chern classes: There exists a theory of crystalline Chern classes [BI70].

The case of a scheme over a field k of characteristic p: Let X_0 be a scheme over k, and let $\Lambda_n = \mathbb{Z}/p^n\mathbb{Z}$. Then, by passing to the limit in the crystalline cohomologies relative to Λ_n, one obtains cohomology groups (and cohomology sheaves)

$$H^*_{\mathrm{cris}}(X_0) := \lim H^*_{\mathrm{cris}}(X_0/\Lambda_n), \qquad \underline{H}_{\mathrm{cris}}(X_0) := \lim \underline{H}_{\mathrm{cris}}(X_0/\Lambda_n).$$

Supposing further that k is perfect and that X_0 is proper and smooth over k, then $\underline{H}_{\mathrm{cris}}(X_0)$ is a perfect complex of $W = \mathbb{W}(k)$–modules (see [SGA72, p. 31-39]) whose cohomology objects are the $H^i_{\mathrm{cris}}(X_0)$. This cohomology is functorial, commutes with base change, and satisfies a Künneth formula. Hence, modulo torsion over W, one has

$$H^*_{\mathrm{cris}}(X_0) \otimes H^*_{\mathrm{cris}}(Y_0) \xrightarrow{\sim} H^*_{\mathrm{cris}}(X_0 \times_k Y_0).$$

If $f_0 \colon X_0 \to k$ lifts to $f \colon X \to W$ with f proper and smooth, then by setting $X_n = X \times_W W_n$ one obtains isomorphisms

$$H^*_{\mathrm{cris}}(X_0) = \lim H^*_{\mathrm{cris}}(X_n/W_n) \xrightarrow{\sim} \lim H^*_{\mathrm{dR}}(X_n/W_n) \xrightarrow{\sim} H^*_{\mathrm{dR}}(X/W),$$

and in the proper case one recovers the invariance of de Rham cohomology for different lifts of X_0.

It may be that one has a lift not to W but to a complete valuation ring V of mixed characteristic and with residue field k. In this case, one has $W \subseteq V$ and, writing L for the field of fractions of V,

$$H^*_{\mathrm{dR}}(X_L) \simeq H^*_{\mathrm{cris}}(X_0) \otimes_W L.$$

The groups $H^*_{\mathrm{dR}}(X_L)$ are endowed with a natural filtration (provided by hypercohomology), and the lift gives a filtration on $H^*_{\mathrm{cris}}(X_0) \otimes_W L$.

Lacunes de la théorie

Remark 4.7.1. There are many lacunae of this theory, certainly generally but also even for schemes of finite type over a field k of characteristic p.

(1) If X_0 is smooth over k but not proper (or vice versa), one finds pathological invariants which do not satisfy any manner of finiteness. To correct this, I imagine that one must apply the constructions of Monsky–Washnitzer and define an "M-W-crystalline" site—but I fear that in this process we will lose torsion phenomena.

(2) Except in characteristic zero (i.e., the case treated by Deligne), I do not know what will constitute a good finiteness condition for coefficients more general than the structure sheaf, especially those which play the role of sheaves of transcendent algebraically constructible complex vector spaces and which will be will be stable under the usual operations. Even over $\mathbb{C}$, the theory of $R^i f_*$ in Deligne's regime has not been developed in a purely algebraic fashion.

(3) Even in the case of a proper and smooth scheme over a perfect field k of positive characteristic, we have not yet produced a duality theorem à la Gysin.

5. Main course

Chapitre V. Programme

We now turn to the main subject of these notes.

Foncteur de Dieudonné

5.1. **The Dieudonné functor.** Let S be a scheme on which p is locally nilpotent, and let S_{cris} be the crystalline topos of S relative to $\mathbb{Z}_{(p)}$. We will sketch a defintion (cf. Remark 5.4.7) of an additive functor

$$\mathbb{D}^*\colon BT(S)^{\mathrm{op}} \to \mathsf{CrisLocFree}(S)$$

compatible with inverse images.[27] Writing S_0 for the reduction of S to characteristic p, there is an equivalence of categories

$$\mathsf{CrisLocFree}(S) \xrightarrow{\sim} \mathsf{CrisLocFree}(S_0)$$

which allows us to reduce to the characteristic p case, i.e., it is equivalent to define

$$\mathbb{D}^*\colon BT(S_0)^{\mathrm{op}} \to \mathsf{CrisLocFree}(S_0).$$

Let us therefore work over S_0. Using $\mathbf{f}_{G_0}$, $\mathbf{v}_{G_0}$, and the compatibility of $\mathbb{D}^*$ with inverse images, we see that $M = \mathbb{D}^*(G_0)$ is endowed with morphisms F_M, V_M

$$M^{(p)} = \mathbf{f}_S^*(M) \underset{V_M}{\overset{F_M}{\rightleftarrows}} M$$

which satisfy $V_M F_M = p$ and $F_M V_M = p$.

Definition 5.1.1. Such a triple (M, F_M, V_M) is called an *F–V–crystal* or a *Dieudonné crystal*. The functor

$$\mathbb{D}^*\colon BT(S_0)^{\mathrm{op}} \to \mathsf{Cris}_{F,V}(S_0)$$

that we are pursuing will then merit the name "*Dieudonné functor*".

We will see that it commutes with change of base and, when S_0 is the spectrum of a perfect field, it coincides with the previous isomorphism with the usual Dieudonné functor. In this latter case, we have noted already that this gives an equivalence of categories, so that $\mathbb{D}^*(G_0)$ can be used to reconstitute G_0. It then follows in the general case that the F–V–crystal $M = \mathbb{D}^*(G_0)$ on S_0 permits the recovery of the fibers of G_0 over the perfect fields over S_0. This gives us the sense that we have grasped the essential parts of the family of fibers $(G_0)_{\bar{s}}$ via the crystal $M = \mathbb{D}^*(G_0)$. This also emboldens us to ask whether we will have accomplished even more:

Problem 5.1.2. Is the Dieudonné functor over a base S_0 of characteristic p fully faithful? Is its essential image formed of those F–V–crystals which are admissible (cf. Definition 5.3.1)?[28] Problème 1

Problem 5.1.3. Develop a Dieudonné theory for finite locally free p–groups over a base S_0 of positive characteristic p. If S_0 is perfect, establish an antiequivalence between the category of finite locally free p–groups over S_0 which are flat over Λ_n and the category Problème 2

[27] "Compatible with inverse images" here means a Cartesian functor on the category fibered in Barsotti–Tate groups over bases where p is locally nilpotent.

[28] To give an answer, it seems that one must formulate Dieudonné theory for schemes in finite locally free p–groups over S_0, or at least for those which are flat over some $\Lambda_n = \mathbb{Z}/p^n\mathbb{Z}$. This should give an equivalence between the category of these groups and a category of F–V–crystals over S_0 which are endowed with supplementary structures (which are likely to be useless unless S_0 is perfect). I have not managed to resolve this, and it very probably involves giving a filtration on a certain object in a derived category…

of locally free $\mathbb{W}_n(\mathscr{O}_S)$–modules M, endowed with F_M and V_M which satisfy the usual conditions.[29]

2. Filtrations associées aux cristaux de Dieudonné

5.2. Filtrations associated to Dieudonné crystals.

For now, we return to the case of a base scheme S on which p is locally nilpotent.

Definition 5.2.1. For a Barsotti–Tate group G over S, we define a filtration of the locally free module $\mathbb{D}^*(G)_S$ over S by a locally direct factor submodule[30] $\mathrm{Fil}^1 = \underline{\omega}_G$, giving rise to an exact sequence

$$0 \to \underline{\omega}_G \to \mathbb{D}^*(G)_S \to \underline{t}_{G^*} \to 0.$$

This sequence is functorial in G and compatible with inverse images along $S' \to S$.[31]

Definition 5.2.2. In general, we define a *filtered F–V–crystal over S* to be an F–V–crystal M in locally free modules over S equipped with a specified filtration of M_S by a locally direct-factor submodule.

Remark 5.2.3. One may package both the Dieudonné module and this filtration into a single functor

$$\mathbb{D}^*\colon BT(S)^{\mathrm{op}} \to \mathsf{CrisFil}_{F,V}(S)$$

which is compatible with inverse images.

Remarque

Remark 5.2.4. The knowledgeable reader should consider the above filtration to be equivalent to the "Hodge filtration" on relative de Rham cohomology in dimension 1. We will work to explain this later on.

Let S' be a thickening of S with divided powers, and let G be a fixed Barsotti–Tate group over S. We propose to find all the (isomorphism classes of) embeddings of G into a Barsotti–Tate group G' over S' such that $G' \times_{S'} S = G$. There is a tight connection between this problem and the filtration discussed above: by sending such a G' to the filtration on $\mathbb{D}^*(G')_{S'} = \mathbb{D}^*(G)_{S'} = \mathbb{D}^*(G_0)_{S'}$, one notes first that this extends the filtration on $\mathbb{D}^*(G)_S = \mathbb{D}^*(G_0)_S$ and then that this assignment is a bijection between the isomorphism classes of embedding and the set of filtration extensions. More precisely:

Théorème de déformation pour les groupes de Barsotti–Tate

Theorem 5.2.5 (Deformation theory for Barsotti–Tate groups)**.** *Let S be a scheme with p locally nilpotent, let S' a thickening of S with divided powers, and consider the functor*

$$BT(S') \to \left\{ (G \in BT(S), M \leq \mathbb{D}^*(G)_S) \;\middle|\; \begin{array}{c} M \textit{ is a direct-factor submodule,} \\ M \textit{ prolongs the submodule } \mathrm{Fil}^1\, \mathbb{D}^*(G)_S = \underline{\omega}_G \end{array} \right\}.$$

This functor is an equivalence of categories if S' has nilpotent divided powers or if one restricts to groups G, G' are which are infinitesimal or ind-unipotent. □

Remarque

Remark 5.2.6. If Problem 5.1.2 were to have an affirmative answer, then we could conclude a description of the category of Barsotti–Tate groups over S in terms of filtered Dieudonné crystals on S: noting that the ideal $p\mathscr{O}_S$ is an ideal with divided powers, we may then apply

[29] An affirmative answer to this problem evidently gives an affirmative answer to the previous problem in the case where S_0 is a perfect scheme.

[30] It is thus locally free of finite type.

[31] In fact, $\underline{\omega}_G$ is the module of invariant differentials on the formal group $\overline{G}$ associated to G (which will be constructed further on), and $\underline{t}_G = \underline{\omega}_{G^*}^{\vee}$ is the Lie algebra associated to the Barsotti–Tate group G^* dual to G.

deformation theory to the case of the pair (S_0, S). Note also that establishing a theory of deformations in the general case (S, S') is equivalent to the particular case of a pair (S_0, S) due to the equivalence $\mathsf{CrisLocFree}(S) \xrightarrow{\sim} \mathsf{CrisLocFree}(S_0)$.

Problem 5.2.7. Find a variant of deformation theory for finite locally free p–groups.

Problème 3

5.3. **Admissible F–V–crystals in characteristic p.** We return to the case of a scheme of characteristic p. Based on the ideas sketched above, the reader might worry that Problem 5.1.2 is definitely negative: since the Dieudonné crystal $\mathbb{D}^*(G_0)$ is endowed with a canonical filtration, perhaps one must instead work with a functor taking values in filtered Dieudonné crystals. In fact, the filtration is uniquely determined by the Dieudonné crystal structure itself, so that Problem 5.1.2 may be left as-stated, which we now explain.

F–V–cristaux admissibles en caractéristique p

Recall the morphisms of ringed topoi considered in Section 4.3:

$$(S_0|_{\mathbb{F}_p}) = (S_0)_{1-\mathrm{cris}} \underset{\psi}{\overset{\varphi}{\rightleftarrows}} (S_0)_{\mathrm{Zar}} = (S_0)_{1-\mathrm{Witt}}.$$

Set $M = \mathbb{D}^*(G_0)$, and let M_0 be its restriction to the crystalline site relative to $\mathbb{F}_p$. Using the identities $FV = p$ and $VF = p$, it follows that the composites in the following sequence are zero:

$$M_0^{(p)} \xrightarrow{F_{M_0}} M_0 \xrightarrow{V_{M_0}} M_0^{(p)} \xrightarrow{F_{M_0}} M_0.$$

In fact, more is true: for every Dieudonné crystal M and for every divided power thickening of S_0 of characteristic P, the preceding sequence is exact, hence one obtains subcrystals in locally free modules:

$$\operatorname{Ker} V_{M_0} = \operatorname{Im} F_{M_0} \subseteq M_0, \qquad \operatorname{Ker} F_{M_0} = \operatorname{Im} V_{M_0} \subseteq M_0^{(p)}.$$

Setting $M_{S_0} = \psi^*(M_0)$, we apply the results of Section 3.2 to deduce

$$\varphi^*(M_{S_0}) = \varphi^*\psi^* M_0 = M_0^{(p)}.$$

From this, we see

$$\varphi^*(\mathrm{Fil}^1(M_{S_0})) = \operatorname{Ker} F_{M_0} (= \operatorname{Im} V_{M_0}) \subseteq M_0^{(p)}.$$

The faithfulness of φ and the preceding relation together completely determine the locally direct-factor submodule Fil^1 of M_{S_0}.

Definition 5.3.1. We will say that an F–V–crystal in locally free modules M_0 on the crystalline site of S_0 over $\mathbb{F}_p$ is *admissible* if there exists a locally direct-factor submodule Fil^1 of $(M_0)_{S_0}$ such that the preceding relation is satisfied. This Fil^1 is unique and determines a filtration which we will call the *canonical filtration*. We also say that a Dieudonné crystal M over S_0 is *admissible* if its restriction M_0 to the crystalline site relative to $\mathbb{F}_p$ is admissible.

Corollary 5.3.2. *For a Barsotti–Tate group G over S_0, $M = \mathbb{D}(G)_{S_0}$ is admissible and the filtration of M_{S_0} envisioned in Section 5.2 is the canonical filtration.* □

5.4. **Deformation theory for abelian schemes.** Many of the considerations made thus far for Barsotti–Tate groups in this Section apply just as well to abelian schemes. Abelian schemes do not privilege any particular prime number, and so we must work with the general nilpotent crystalline site. We seek a contravariant functor from the category of abelian S–schemes to crystals on the nilpotent crystalline site of S, and repackaging the ideas above yields the following:

4. La théorie de déformation pour les schémas abéliens

Definition 5.4.1. There is a functor

$$\mathbb{D}^*\colon \mathsf{AbSch}(S)^{\mathrm{op}} \to \mathsf{CrisLocFree}_{\mathrm{nil}}(S)$$
$$A \mapsto R^1(f_{\mathrm{cris}})_*(\mathscr{O}_{A_{\mathrm{cris}}})$$

where $f_{\mathrm{cris}}\colon A_{\mathrm{nilcris}} \to S_{\mathrm{nilcris}}$ is induced by $f\colon A \to S$.

Remark 5.4.2. A variant of the change-of-base theorem gives

$$\mathbb{D}^*(A)_S = \underline{H}^1_{\mathrm{dR}}(A/S) := R^1 f_*(\Omega^*_{A/S}).$$

In order to calculate $\mathbb{D}^*(A)_{S'}$ for S' a locally nilpotent divided power thickening of S, one selects an abelian variety A' over S' into which A embeds [Mum65] and deduces

$$\mathbb{D}^*(A)_{S'} = \underline{H}^1_{\mathrm{dR}}(A'/S').$$

The fact that $\underline{H}^1_{\mathrm{dR}}(A'/S')$ does not depend (up to isomorphism) on A' is a consequence of the "tapis of crystalline cohomology".

The Hodge filtration on de Rham cohomology induces a filtration on $\mathbb{D}^*(A)_S$:

$$\begin{tikzcd}
0 \arrow[r] & R^0 f_*(\Omega^1_{A/S}) \arrow[r] \arrow[d, equal] & \underline{H}^1_{\mathrm{dR}}(A/S) \arrow[r] \arrow[d, equal] & R^1 f_*(\mathscr{O}_S) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \underline{t}_A^\vee \arrow[r] & \mathbb{D}^*(A)_S \arrow[r] & \underline{t}_{A^*} \arrow[r] & 0,
\end{tikzcd}$$

where $\underline{t}_A^\vee = \underline{\omega}_A$. The functor $\mathbb{D}^*$ and the associated exact sequence are each functorial in A and compatible with inverse images, so that there is a factorization

$$\mathbb{D}^*\colon \mathsf{AbSch}(S)^{\mathrm{op}} \to \mathsf{CrisLocFreeFil}_{\mathrm{nil}}(S)$$

through crystals in "filtered" (by two locally free stages) locally free modules. As before, this construction sends an embedding of A into an abelian scheme A' over S' in the nilpotent crystalline site of S to the extension of the canonical filtration of $\mathbb{D}^*(A)_S$ into that of $\mathbb{D}^*(A)_{S'}$.

Théorème de déformation pour les schémas abéliens

Theorem 5.4.3 (Deformation theory for abelian schemes)**.** *Let S be a scheme, and let S' be a neighborhood with locally nilpotent divided powers of S. The functor*

$$\mathsf{AbSch}(S') \to \left\{ (A \in \mathsf{AbSch}(S), \mathrm{Fil}^1 \leq \mathbb{D}^*(A)_{S'}) \;\middle|\; \begin{array}{c} \mathrm{Fil}^1 \textit{ a locally direct-factor submodule} \\ \textit{prolonging } \mathrm{Fil}^1\mathbb{D}^*(A)_S = \underline{\omega}_A \end{array} \right\}$$

is an equivalence of categories. □

One may give a second interpretation of the functor $\mathbb{D}^*$ on abelian schemes which readily admits a quasi-inverse.

Definition 5.4.4. For an abelian scheme A over S, its *universal vector extension* is an extension

$$0 \to \underline{t}^\vee_{A^*} \to E(A) \to A \to 0$$

which is universal among all extensions

$$0 \to V \to E \to A \to 0$$

with V a vector space.

Lemma 5.4.5. *There is a crystal $\mathbb{E}(A)$ in smooth groups on S such that $E(A) = \mathbb{E}(A)_S$.*

Construction. Whatever the definition of $\mathbb{E}(A)$, it defines a functor

$$\mathsf{AbSch}(S) \to \mathsf{CrisSmoothGps}_{\mathrm{nil}}(S)$$

which is compatible with base change. As the preceding isomorphism is also functorial and compatible with change of base, one would like to define $\mathbb{E}(A)$ by

$$\mathbb{E}(A) = S' \mapsto \mathbb{E}(A)_{S'} = E(A')$$

for some abelian S'–scheme A' into which A embeds, but this hinges on finding a transitive system of canonical isomorphisms between those groups obtained by different choices of A'.

In the case where S is of characteristic 0 (which hints at the involvement of divided powers), this is particularly simple: there exists a unique isomorphism with a reference smooth group scheme E' over the thickening S' into which E embeds.[32][33] In the general case, the procedure is more delicate, but it can be managed by "the method of the exponential", which will be exposed in the context of Barsotti–Tate groups later in the seminar. □

Remark 5.4.6. One may also define $\mathbb{E}(A^*)$ by

$$\mathbb{E}(A^*) = \mathbb{R}^1(f_{\mathrm{cris}})_*((\mathbb{G}_m)_{A_{\mathrm{cris}}}),$$

where A_{cris} denotes the absolute (i.e., relative to $\mathbb{Z}$) nilpotent crystalline topos. This method may be be adapted to Barsotti–Tate groups.

Whichever method is used to define $\mathbb{E}(A)$, one deduces that there is an natural isomorphism

$$\mathbb{D}^*(A) = \underline{\mathrm{Lie}}(\mathbb{E}(A^*))$$

which is compatible with base change, and where the extension structure of $\mathbb{D}^*(A)$ is inherited from that of $\underline{\mathrm{Lie}}$ as induced by $\mathbb{E}(A)_S = E(A)$.

Returning to the conditions of deformation theory, one can see that the data of a prolongation $\mathrm{Fil}^1 \subseteq \mathscr{D}^*(A)_{S'} = \underline{\mathrm{Lie}}(\mathbb{E}(A^*)_{S'})$ which is a locally direct-factor prolongation of $\mathrm{Lie}(\underline{t}_A) = \underline{t}_A \subseteq \mathbb{D}^*(A)_S = \mathrm{Lie}(\mathbb{E}(A^*)_{S'}|_S)$ is equivalent to the data of a smooth vector subgroup of $\mathbb{E}(A^*)_{S'}$ which prolongs the subgroup $\underline{t}_A^\vee$ of $\mathbb{E}(A^*)_S$. For such a subgroup L, it is immediate that $\mathbb{E}(A^*)_S/L$ is an abelian scheme over S', and hence the dual abelian scheme is the desired A'.[34]

Ed. note: The original has a reference into the unpublished part of the manuscript on the "method of the exponential".

Remark 5.4.7. In the context of Barsotti–Tate groups, one finds the same two constructions of $\mathbb{D}^*(G)$ inspired by the above pair of constructions for abelian schemes: one "cohomological" and another via a "universal vector extension" $\mathbb{E}(G)$. The construction of $\mathbb{E}(G)$ itself can also be obtained by two methods, one cohomological and the other by the directly adapting the exponential to the proof of deformation theory.

Remarque

5.5. **Relations between the two Dieudonné theories.**

5. Relations entre les deux théories (pour schémas abéliens et pour groupes de Barsotti–Tate

[32]N.B.: It suffices to do this even over a nil-thickening, as one sees by passing to the limit from the Noetherian case.

[33]This result may be found in an old letter of Tate.

[34]N.B.: It would be more natural to work with $\mathbb{D}_*(A) = \underline{\mathrm{Lie}}(\mathbb{E}(A)) \xrightarrow{\sim} \mathbb{D}^*(A^*)$ than with $\mathbb{D}^*(A)$ for the statements of deformation theory. The equivalence of the two points of view stem from a perfect pairing $\mathbb{D}^*(A) \otimes \mathbb{D}^*(A^*) \to \mathscr{O}_{S_{\mathrm{nilcris}}}$ which is compatible with the filtrations.

Theorem 5.5.1. *Continue to suppose that p is locally nilpotent on S, and return to the Berthelot crystalline site. For A an abelian S-scheme, there is a natural isomorphism*

$$\mathbb{D}^*(A(\infty)) \simeq \mathbb{R}^1(f_{\mathrm{cris}})_*(\mathscr{O}_{A_{\mathrm{cris}}})$$

which is compatible with base change and with the filtrations $\underline{t}_A^\vee$ and $\underline{t}_{A^*}^\vee$.[35] □

Remark 5.5.2. In fact, *postulating* such an isomorphism gives a useful heuristic for the *definition* of $\mathbb{D}^*(G)$ for an arbitrary Barsotti–Tate group.

Corollary 5.5.3 (Serre–Tate)**.** *For a locally nilpotent thickening S' over S with divided powers, there is an identification of the theories of infinitesimal prolongations of A and of $A(\infty)$.*[36] □

6. Infinitesimal properties and deformations of Barsotti–Tate groups

Chapitre VI. Proprietes infinitesimales des groupes de Barsotti–Tate. Deformation de groupes de Barsotti–Tate.

6.1. Infinitesimal neighborhoods and formal Lie groups.

1. Voisinages infinitésimaux. Groupes de Lie formels.

1.1

Definition 6.1.1. Let S be a scheme, and let $i\colon Y \to X$ a monomorphism of sheaves[37] over S. We define the *k^{th} infinitesimal neighborhood of Y in X* to be the subsheaf $\mathrm{Inf}_Y^k(X)$ of X generated by the images of morphisms $X' \to X$ which participate in a commutative diagram

$$\begin{array}{ccc} Y' & \xrightarrow{i'} & X' \\ \downarrow & & \downarrow \\ Y & \xrightarrow{i} & X \end{array}$$

for $i'\colon Y' \to X'$ a nilpotent immersion of order k of S-schemes.

Remark 6.1.2. When X is representable and Y is a closed subscheme defined by an ideal I of $\mathscr{O}_X$, this recovers the notion of an infinitesimal neighborhood defined in EGA [DG67], i.e., $\mathrm{Inf}_Y^k(X) = V(I^{k+1})$.

Remark 6.1.3. As k ranges, the subsheaves Inf^k define a sequence of subfunctors

$$\mathrm{Inf}_Y^k(X) \to \mathrm{Inf}_Y^{k+1}(X) \to \cdots \to \mathrm{Inf}_Y^\infty(X) = \operatorname{colim} \mathrm{Inf}_Y^k(X).$$

The subsheaves Inf^k are also functorial in X, Y, and S.

Définition 1.2

Definition 6.1.4. A particularly important case is where X is pointed[38] over S by a section e_X and Y is given by $Y = e_X(S)$. In this situation, we omit Y and denote this subsheaf as $\mathrm{Inf}^k(X)$, and we also write $\overline{X} = \mathrm{Inf}^\infty(X)$. When $X = \overline{X}$, one says that X is *ind-infinitesimal*.

[35] This indicated isomorphism is compatible with the isomorphism $\underline{t}_{A(\infty)} \simeq \underline{t}_A$ and similarly for A^*.

[36] One may also define a universal vector extension $E(G)$ of a Barsotti–Tate group, and one will find a canonical isomorphism

$$E(A(\infty)) \simeq E(A)(\infty)$$

giving rise, by the usual procedure, to a more general isomorphism of crystals in groups

$$\mathbb{E}(A(\infty)) \simeq \mathbb{E}(A)(\infty).$$

[37] Here "sheaf" is taken relative to any topology between the Zariski topology and the fpqc topology.

[38] For instance, one might take X to be a group scheme and e_X to be the unit section.

Definition 6.1.5. A pointed S–scheme G which is locally of finite presentation is said to be *smooth to order k* if it satisfies the following equivalent conditions:

(1) (Zariski)-locally on S, $\mathrm{Inf}^k(G)$ is isomorphic to a scheme of the form
$$\operatorname{Spec} \mathscr{O}_S[T_1, \dots, T_n]/(T_1, \dots, T_n)^{k+1}.$$

(1') The conormal sheaf along the unit section $\underline{\omega}_{G,e} := e^*(\underline{\Omega}^1_{G/S})$ is locally free of finite type, and $\mathrm{Sym}^i(\underline{\omega}_{G,e}) \to \underline{\mathrm{gr}}^i_e(\mathscr{O}_X)$ is an isomorphism for $i \leq k$.

(2) For every affine S–scheme X_0, every infinitesimal neighborhood X' of order k of X_0, every subscheme X of X' containing X_0, and every S–morphism $f : X \to G$ with $f|_{X_0}$ factoring through S, there exists an S–morphism $f' : X' \to G$ as in the commutative diagram
$$\begin{tikzcd}
X_0 \arrow[rr] \arrow[dr] \arrow[dd] & & X' \arrow[dd, dotted, "f'"] \\
& X \arrow[ur] \arrow[dr, "f"] & \\
S \arrow[rr, "e"] & & G.
\end{tikzcd}$$

(2') For every commutative Cartesian diagram of S–schemes
$$\begin{tikzcd}
& X \arrow[r, "i"] & X' \\
X \cap X'_0 \arrow[r, equal] & X_0 \arrow[u] \arrow[r] & X'_0 \arrow[u]
\end{tikzcd}$$
with i a nilpotent immersin of order k, and for every S–morphism $f : X \to G$ with $f|_{X_0}$ factoring through S, there exists an S–morphism $f' : X' \to G$ which extends f.

(2") Like (2), with X' a neighborhood of order k of X.

Definition 6.1.6. A sheaf X over S pointed by e is called a *pointed formal variety*if it satisfies the following conditions: Définition 1.3

(1) X is ind-infinitesimal: $X = \overline{X} = \mathrm{Inf}^\infty(X)$.
(2) Each $\mathrm{Inf}^k(X)$ is smooth to order k.

Remark 6.1.7. This notion does not depend upon the topology chosen on Sch_S provided it lies between the Zariski and fpqc topologies.

Remark 6.1.8. One may define a formal variety over S (without a pointing) as a sheaf X which (fpqc-)locally is isomorphic to the underlying sheaf of a pointed formal variety, but we will not need this variant. As an aside, the elided section is *not* uniquely determined by this condition.

Definition 6.1.9. A *formal Lie group* over S is a sheaf of groups such that the underlying pointed sheaf is a pointed formal variety over S.

Remark 6.1.10. As the category of formal varieties admits finite products, one may interpret this as a group object in this category. It follows that this notion again does not depend on the topology.

Because of our intense interest in p–power torsion groups, the following result will cause us to limit ourselves to commutative formal Lie groups:

Proposition 1.4

Proposition 6.1.11. *Suppose that p is locally nilpotent on S. If G is a commutative ind-infinitesimal group on S such that the $\mathrm{Inf}^k(G)$ are representable (e.g., if G is a formal Lie group), then G is p–power torsion.* □

6.2. **Results special to characteristic p.** Now assume that the base scheme S is of positive characteristic p.

Definition 6.2.1. For every commutative group S–scheme G and for every integer $i \geq 0$, we set

$$G[i] = \mathrm{Ker}\{f^i_{G/S} \colon G \to G^{(p^i)}\}.$$

Remark 6.2.2. For each n there is an interchange law $G^{(p^n)}[i] = (G[i])^{(p^n)}$. These subschemes relate to the infinitesimal neighborhoods by $\mathrm{Inf}^i\, G \subseteq G[i]$. Additionally, if G is flat over S, these subschemes relate to the p^i–torsion components through $G[i] \subseteq G(i)$, by way of the identity $p^i = \mathbf{v}^i_{G^\circ} \mathbf{f}^i_{G/S}$.

Definition 6.2.3. For each $0 \leq i \leq n$, the Frobenius $\mathbf{f}^i_{G/S}$ induces a morphism

$$\mathbf{f}^i \colon G[n] \to G[n-i]^{(p^i)}.$$

A group scheme G is *of* $\mathbf{f}$*-regular filtration of stage n* when $G = G[n]$ and when $\mathbf{f}^i$ is an epimorphism of τ–sheaves, τ some topology on $\mathsf{Sch}_{/S}$ between fppf and fpqc.

Proposition 2.1

Proposition 6.2.4. *Let G be a finite locally free group over S such that $G = G[n]$. The following statements are equivalent:*

(1) *G is of $\mathbf{f}$–regular filtration of stage n.*
(1') *The morphisms $\mathbf{f}^i$ are flat*[39] *for $0 \leq i \leq n$.*
(2) *For some $1 \leq i \leq n-1$, $\mathbf{f}^i$ is an epimorphism.*
(2') *For some $1 \leq i \leq n-1$, $\mathbf{f}^i$ is flat.*
(3) *Define the τ–sheaf $\mathrm{gr}^{\mathbf{f}}_i(G)$ by $\mathrm{gr}^{\mathbf{f}}_i(G) = G[i]/G[i-1]$. For $0 \leq i < n$, the homomorphisms $\theta_i \colon \mathrm{gr}^{\mathbf{f}}_n(G) \to [\mathrm{gr}^{\mathbf{f}}_{n-i}(G)]^{(p^i)}$ induced by $\mathbf{f}^i$ are all isomorphisms.*
(4) *The groups $G[i]$ are finite locally free and, defining $\mathrm{gr}^{\mathbf{f}}_i(G)$ in the category of flat group schemes over S, the homomorphisms θ_i are again isomorphisms.*
(5) *Zariski locally, the augmented algebra A of G is isomorphic to*

$$\mathscr{O}_S[T_1, \ldots, T_d]/(T_1^{p^n}, T_2^{p^n}, \ldots, T_d^{p^n}).$$

(6) *Zariski locally, A is isomorphic as an augmented algebra to $\mathrm{Sym}_{\mathscr{O}_S}(\underline{\omega})/(\underline{\omega}^{(p^n)})$, where the conormal bundle $\underline{\omega}$ of G along its unit section is a locally free module of finite type over S and $(\underline{\omega}^{(p^n)})$ denotes the ideal generated by the elements of homogeneous degree p^n.*
(6') *The previous condition is satisfied on the geometric fibers of G.*

[39]Hence faithfully flat.

Proof. (1) and (2) respectively imply (1') and (2') using fiber-by-fiber flatness criteria [DG67, 11.3.10]. Conversely, (1') and (2') respectively imply (1) and (2), as $\mathbf{f}^i$ is a morphism of finite presentation.

To show that (1), (2), and (3) are equivalent, we reason along lines similar to those of Proposition 3.1.4. Note first that if $\mathbf{f}^i$ is an epimorphism, then so is $\mathbf{f}^{i+1}$. Since θ_{n-1} factorizes into $(n-1)$ monomorphisms

$$\alpha\colon G[i]/G[i-1]^{(p^{n-i})} \to G[i-1]/G[i-2]^{(p^{n-i+1})},$$

it suffices that $\mathbf{f}^{n-1}$ be an epimorphism in order for all the morphisms α to be isomorphisms, which is equivalent to (3). Finally, if all the α are isomorphisms, it follows by induction on i that the morphisms $\mathbf{f}\colon G[i]^{(p^{n-i})} \to G[i-1]^{(p^{n-i+1})}$ are all epimorphisms.

To show that (3) implies (4), note that the fiber-by-fiber criterion for flatness applies to show that $G[i]$ is flat over S, and hence it is also finite and locally free. This is then also true for $\mathrm{gr}_i^{\mathbf{f}}(G) \approx \mathrm{gr}_1^{\mathbf{f}}(G) = G[1]$, which are thus quotients in the category of flat groups over S. The converse that (4) implies (3) holds trivially.

In showing that (5) through (6') are equivalent, we may suppose that S is affine. It is also trivial that (6) entails (5). To see that (5) entails (6), let $\underline{\omega}$ be the indicated conormal bundle of G along its unit section. Since S is affine and the hypothesis (5) implies that $\underline{\omega}$ is locally free, one may choose a section $\underline{\omega} \to J$ of the canonical morphism $J \to \underline{\omega} = J/J^2$, from which we gain a surjective morphism of augmented algebras $t\colon \mathrm{Sym}(\underline{\omega})/(\underline{\omega}^{(p^n)}) \to \mathscr{A}$. It follows from (5) that this is an isomorphism.

To see that (5) implies (2'), let $J^{[p]}$ denote the ideal of A generated by the p^{th} powers of local sections of J. Noting that $G[n] = G$ is equivalent to $x^{p^n} = 0$ for all local sections of J, if A is as in (5) then the homomorphism $(A/J^{[p]})^{(p^{n-1})} \to A$ induced by $x \mapsto x^{p^{n-1}}$ makes A a flat algebra over $(A/J[p])^{(p^{n-1})}$. This exactly shows that $\mathbf{f}^{n-1}$ is flat.

To finish the proof, we will show that (2') implies (6).[40] Let us suppose first that S is the spectrum of a perfect field k. By the Hopf–Borel–Dieudonné theorem [SGA72, VII$_B$ 5.4], we learn that the affine algebra of the infinitesimal group $G = G[n]$ is of the form

$$A \simeq k[T_1, \ldots, T_d]/(T_1^{p^{n-1}}, \ldots, T_d^{p^{n_d}}).$$

The assumption $J^{[p^n]} = 0$ yields the upper bound $n_i \leq n$. To show $n_i = n$, note that if $n_i < n$ fails to meet the upper bound, then the homomorphism t will not be surjective, contradicting the faithfully flat hypothesis.

The general case will follow immediately from the local case where $S = \mathrm{Spec}\,B$, B a local ring with residue field k. Let ω be the conormal bundle, and let $d = \dim_k \omega \otimes k$. Select a sequence $(T_1, \ldots, T_d)$ of elements of J whose images in $\omega \otimes k$ form a basis, and consider the resulting homomorphism

$$B[T_1, \ldots, T_d]/(T_1^{p^n}, \ldots, T_d^{p^n}) \to A.$$

This is a surjective morphism of free B-modules, and hence to prove that it is an isomorphism it suffices to prove that these two B-modules have the same rank. To see this, we base-change from S to a perfect closure of k, where A has the correct rank p^{nd}. □

Corollaire (2.2)

[40]The equivalence with (6') is also captured by this proof, because (2') is the same condition applied to geometric fibers.

Corollary 6.2.5. *If G is a finite locally free group of* $\mathbf{f}$*–regular filtration of stage n, then G is smooth to order $p^n - 1$ along its unit section.*

Proof. This is clear from an explicit presentation of G. □

Proposition (2.3)

Proposition 6.2.6. *Let $G = G(n)$ be a Barsotti–Tate group over S truncated at stage n. Then*

(1) *$G[n] \subseteq G$ is of* $\mathbf{f}$*–regular filtration of stage n.*
(2) $G[n] = \operatorname{Ker} \mathbf{f}_G^n = \operatorname{Im} \mathbf{v}_G^n$ *and* $\operatorname{Ker} \mathbf{v}_G^n = \operatorname{Im} \mathbf{f}_G^n$.
(3) *$G[n]$ is flat (hence finite and locally free) over S, and so it satisfies the conditions of Proposition 6.2.4.*

Proof. We treat each consequence in turn.

(1) Consider the following diagram:

$$\begin{tikzcd} G \underset{G^{(p^i)}}{\times} G[n-i]^{(p^i)} \arrow[r, equal] & G[n] \arrow[r, "\mathbf{f}^i"] \arrow[d] & G[n-i]^{(p^i)} \arrow[d] \\ & G \underset{G^{(p^i)}}{\times} G(n-i)^{(p^i)} \arrow[r, "h"] \arrow[d] & G(n-i)^{(p^i)} \arrow[d] \\ G^{(p^i)} \arrow[r, "\mathbf{v}_G^i"] & G \arrow[r, "\mathbf{f}_G^i"] & G^{(p^i)}. \end{tikzcd}$$

By assumption, $\mathbf{f}_G^i \circ \mathbf{v}_G^i = p^i$ has image $G(n-i)^{(p^i)}$. It follows that h is an epimorphism, hence $\mathbf{f}^i \colon G[n] \to G[n-i]^{(p^i)}$ is also.

(2) For $n = 1$, this is the definition of a Barsotti–Tate group truncated to stage 1. For $n \geq 2$, we saw in Lemma 3.2.5 that $G(1) \subseteq G(n) = G$ satisfies (2), and hence we may induct to show $G(i) \subseteq G$. Because $\mathbf{f}_G^n \circ \mathbf{v}_G^n = p^n \mathrm{Id}_{G(n)} = 0$, we have $\operatorname{Im} \mathbf{v}_G^n \subseteq G[n]$, giving a commutative diagram

$$\begin{tikzcd} G(n)^{(p^n)} \arrow[r, "\mathbf{v}_G^n"] \arrow[d, "p"] & G[n] \arrow[d, "\mathbf{f}"] \\ G(n-1)^{(p^n)} \arrow[r, "\mathbf{v}_G^{n-1}"] & G[n-1]^{(p)}. \end{tikzcd}$$

If we suppose that $G(n-1)$ satisfies (2), it follows that $\mathbf{v}_G^n$ is an epimorphism modulo $\operatorname{Ker} \mathbf{f} = G[1]$. It thus suffices to show that $G[1]$ is contained in the image of $\mathbf{v}_G^n$, as in

$$G[1] = \mathbf{v}(G(1)^{(p)}) = \mathbf{v}(p^{n-1} G(n)^{(p)}) = \mathbf{v}^n[\mathbf{f}^{n-1} G(n)^{(p)}] \subseteq \mathbf{v}^n[G(n)^{(p^n)}].$$

By the same methods, one proves that $\operatorname{Ker} \mathbf{v}_G^n = \operatorname{Im} \mathbf{f}_G^n$.

(3) Because $G[n] = \operatorname{Im} \mathbf{v}_G^n$, the fiber-by-fiber criterion shows $G[n]$ to be flat. If G is a Barsotti–Tate group, then $G[n] = G(n)[n]$, hence $G(i)[n] = G[n]$ for $i \geq n$. □

Corollaire (2.4)

Corollary 6.2.7. *Let G be a Barsotti–Tate group over S. For all $n \geq 1$, $G(n)$ is smooth to order $p^n - 1$ over S, and* $\operatorname{Inf}^k G \subseteq G[n]$ *for $k \leq p^n - 1$.*

Proof. We will show this last inclusion. One has $\operatorname{Inf}^k(G) = \operatorname{colim}_n \operatorname{Inf}^k(G(n))$ and $\operatorname{Inf}^k(G(n)) \subseteq G(n)[k] = G[k]$ for $n \geq k$. Hence, for $k \leq p^n - 1$ we have $\operatorname{Inf}^k(G) \subseteq$

$G[p^n-1]$ and, by applying Proposition 6.2.4 and referring tp the explicit structure of $G[p^n-1]$, one sees $\operatorname{Inf}^k G \subseteq G[p^n-1][n] = G[n]$. □

6.3. **Formal Lie groups in characteristic p.** Suppose that $G = \operatorname{colim}_k \operatorname{Inf}^k(G)$ is a formal Lie group over S, so that $G[n] = \operatorname{colim}(\operatorname{Inf}^k(G))[n]$. Since $\operatorname{Inf}^k G$ is smooth to order k, it follows for k sufficiently large that $(\operatorname{Inf}^k G)[n]$ is finite and locally free of $\mathbf{f}$–regular filtration of stage n. Zariski-locally on S, for $k \geq d\,p^n$, one has

2.5 Groups de Lie formels en caractéristique p

$$G[n] = (\overset{k}{\operatorname{Inf}} G)[n] = \operatorname{Spec} \mathscr{O}_S[T_1, \dots, T_d]/(T_1^{p^n}, \dots, T_d^{p^n}),$$

where d is given by the rank of the conormal module $\underline{\omega}$ of G.[41] Since $\operatorname{Inf}^k G \subseteq G[k]$, cofinality gives $G = \operatorname{colim}_n G[n]$, hence $\mathbf{f}_{G/S}\colon G \to G^{(p)}$ is an epimorphism.

Definition 6.3.1. From all this, we see that a formal Lie group G gives rise to a system

$$\{G[n] \to G[n+1] \to \cdots\}$$

of finite locally free groups of $\mathbf{f}$–regular filtration of stage n. By analogy with p–coadic systems, such a system will be called $\mathbf{f}$–*coadic*.

Proposition 6.3.2. *The category of (commutative) formal Lie groups G over S of characteristic p is equivalent to the category of* $\mathbf{f}$*–coadic systems of finite locally free group schemes $G[n]$ of* $\mathbf{f}$*–regular filtration of stage n.*

Proposition (2.6)

Proof. One direction is the construction given above. For the converse, an $\mathbf{f}$–coadic system of finite locally free groups $G[n]$ which are annihilated[42] by $\mathbf{f}^n$ gives rise via an inductive limit to a formal Lie group. □

Theorem 6.3.3 (Construction of a formal Lie group associated to a Barsotti–Tate group in characteristic p). *Let G be a Barsotti–Tate group over S. The group*

Théorème (2.7)

$$\overline{G} = \operatorname{Inf}^{\infty} G = \operatorname{colim}_j \operatorname{Inf}^k(G)$$

is formal Lie with $\overline{G} = \operatorname{colim} G[n]$, $G[n] = G(n)[n]$, $\overline{G}[n] = G[n]$, *and*

$$\operatorname{Inf}^k G = \operatorname{Inf}^k \overline{G} = \operatorname{Inf}^k G(n) \subseteq G[n] \subseteq G(n),\ k \leq p^n - 1.$$

Proof. This is the cumulation of the above results. □

6.4. **The Formal Lie group of a Barsotti–Tate group where p is locally nilpotent.** Theorem 6.3.3 generalizes to the case where p is locally nilpotent on S:

3. Groupe de Lie formel assoié à un groupe de B.–T sur une base non nécessairement de caractéristique p

Theorem 6.4.1. *Let G be a Barsotti–Tate group on S with p locally nilpotent on S. Then:*

(1) $\overline{G} = \lim_k \operatorname{Inf}^k(G) \subseteq G$ *is a formal Lie group over S. One says that $\overline{G}$ is the formal Lie group associated to G. Its formation is functorial in G and commutes with change of base.*

(2) *G is formally smooth over S for nil-immersions (i.e., not only for nilpotent immersions). Explicitly, for all nil-immersions $X_0 \to X'$ and all S–morphisms $f\colon X \to G$ such that $X_0 \to X \to X'$ and $f|_{X_0}$ factor through S, there exists an map $f'\colon X' \to G$ of f over S as in the following diagram:*

Théorème (3.1)

[41] In particular, d is a locally constant function of $s \in S$.

[42] Hence of $\mathbf{f}$–regular filtration of stage n.

$$\begin{tikzcd} X_0 \arrow[r] \arrow[dr] & X \arrow[r] \arrow[dr, "f"] & X' \arrow[d, dotted, "f'"] \\ & S \arrow[r, bend right, "e"'] & G. \arrow[l] \end{tikzcd}$$

One proves this theorem by dévissage to the characteristic p case, a maneuver which requires the relative cotangent complex formalism. Since this formalism is also used in the theory of deformations, we will announce the principle result before giving indications on the underlying theory.

4. Déformations infinitésimales des groupes de Barsotti–Tate (énoncé)

6.5. Infinitesimal deformations of Barsotti–Tate groups (announcements).

Théorème (4.1)

Theorem 6.5.1. *Let $i\colon S_0 \to S$ be a nil-immersion with S affine, and let G_0 be a Barsotti–Tate*[43] *group over S_0.*

(1) *There exists a Barsotti–Tate group G over S which deforms G_0.*
(2) *If one denotes $E(G_0, S)$ (resp. $E(G_0(n), S)$) the set of such deformations up to isomorphism, the natural function $E(G_0, S) \to E(G_0(n), S)$ is surjective.*
(3) *If the immersion i is nilpotent of order k and if $p^N = 0$ on S_0, then the above function is bijective for all $n \geq kN$.*
(4) *If S is an infinitesimal neighborhood of first order of S_0 (i.e., if $k = 1$) and if p is nilpotent on S_0, then $E(G_0, S)$ is a torsor under $\underline{t}_{G_0^*} \otimes \underline{t}_{G_0}$, where $\underline{t}_{G_0}$ denotes the Lie algebra of the formal Lie group $\overline{G}_0$ associated to G_0.*

5. Complexe cotangent relatif

6.6. The relative cotangent complex.

We now give a brief description of the relative cotangent complex and explain how its theory applies to the study of certain deformation problems. In particular, we will use it to study the deformation problems for group schemes and for Barsotti–Tate groups. The local definition of the relative cotangent complex is due to M. André [And67] and to D. Quillen [Qui67] and the global definition is due to L. Illusie [Ill72], who applied it to all sorts of deformation problems. The study of the deformation of flat group schemes over S is due to P. Deligne and to L. Illusie.

Let X be an S–scheme. The relative cotangent complex $L_*^{X/S}$ of X over S is an object of the derived category $D^-(\mathcal{O}_X)$, presentable by a complex of $\mathcal{O}_X$–modules as in

$$L_*^{X/S} = \{\cdots \to L_2 \to L_1 \to L_0 \to 0\}.$$

It is functorial in the following sense: for all Cartesian diagrams

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

there is a natural $D^-(\mathcal{O}_{X'})$–morphism $\mathbb{L}f^*(L_*^{X/S}) \to L_*^{X'/S'}$.

Complex cotangent tronqué

Definition 6.6.1. The truncation of complex to order 1,

$$\tau_{\leq 1}(L_*^{X/S}) = \{0 \to L_1' \to L_0 \to 0\}$$

[43] In the case where G_0 is a formal Lie group, this deformation theory is due to Lazard, who has determined the moduli space of formal Lie group laws.

with L_1 replaced by $L_1' = \operatorname{Im} L_2$, is also of interest. It has the same cohomology objects as $L_*^{X/S}$ at orders 0 and 1, and these have previously been studied by A. Grothendieck [Gro68]. One may give a direct construction of this truncated complex whenever there is an embedding of X into an S–scheme X' which is formally smooth over S (which is always locally true): writing I be the ideal of $\mathcal{O}_X$ defining the immersion $X \subseteq X'$, the defining formula is then

$$\tau_{\leq 1}(L_*^{X/S}) \approx \{0 \to I/I^2 \to \Omega^1_{X'/S} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_X \to 0\}.$$

In this case, $L_0 = \Omega^1_{X'/S} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_X$ is a locally free $\mathcal{O}_X$–module of finite type.

Remark 6.6.2. It follows that the sheaves $\mathscr{H}^i(L_*^{X/S})$ are quasicoherent for $i = 0, 1$. In fact, this property is true for all i.

Remark 6.6.3. If S is Noetherian and if X is locally of finite type, then $\mathscr{H}^i(L_*^{X/S})$ is even coherent for $i = 0$ and $i = 1$. This property is *also* true for all i.

Example 6.6.4. If X is a relative complete intersection,[44] then $\mathscr{H}^i(L_*^{X/S}) = 0$ for $i \geq 2$, so that the complex $L_*^{X/S}$ is isomorphic in the derived category to the truncated version. Moreover, as the immersion $X \subseteq X'$ defining the complete intersection is regular, the quotient I/I^2 is a locally free $\mathcal{O}_X$–module, and hence cotangent complex is a perfect complex of perfect amplitude in $[-1, 0]$.

Remark 6.6.5. Even when these conditions are satisfied, the large complex $L_*^{X/S}$ remains relevant, as it enjoys far superior functoriality and transitivity properties. Letting $f : X \to Y$ and $g : Y \to S$ be two morphisms of schemes, there is an exact triangle

$$\begin{tikzcd} & L_*^{X/Y} \arrow[dl, "\text{degree} + 1"] & \\ \mathbb{L}f^*(L_*^{Y/S}) \arrow[rr] & & L_*^{X/S} \arrow[ul] \end{tikzcd}$$

which generalizes the exact sequence of differentials

$$f^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0$$

via $\mathscr{H}^0(L_*^{X/S}) \simeq \Omega^1_{X/S}$.

Example 6.6.6. One may check this isomorphism in the case of a complete intersection using the exact sequence

$$I/I^2 \to \Omega^1_{X'/S} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_X \to \Omega^1_{X/S} \to 0.$$

Making the further definition

$$\mathscr{N}_{X/S} = \operatorname{Ker}(I/I^2 \to \Omega^1_{X'/S} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_X),$$

the above exact triangle gives rise to a long exact squence of cohomology

$$f^*\mathscr{N}_{Y/S} \to \mathscr{N}_{X/S} \to \mathscr{N}_{X/Y} \to f^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0.$$

In the case where X is a complete intersection relative to Y, one has an exact sequence with six terms.

[44] That is, if X is of finite presentation over S and if there exists a regular immersion of X into an X' which is formally smooth over S

As for change of base, there is the following result:

Proposition

Proposition 6.6.7. *Consider a Cartesian diagram of schemes:*

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S. \end{tikzcd}$$

If X and S' are "Tor-independent",[45] *then the morphism $\mathbb{L}f^*(L_*^{X/S}) \to L_*^{X'/S'}$ is an isomorphism. This hypothesis holds when X is S-flat or when S' is S-flat.* □

Problèmes de déformation typiques

6.7. Example deformation problems.

We now describe the application of the relative cotangent complex to some common deformation problems, as exposited more fully in [Gro68, Ill69a, Ill69b, Ill72].

1°) Classification des voisinages infinitésimaux du premier ordre

Example 6.7.1. For X an S-scheme and J a quasicoherent $\mathscr{O}_X$-module, we seek a classification of the infinitesimal neighborhoods $X \subseteq X'$ in S-schemes, such that X is defined by the square-zero ideal J of $\mathscr{O}_{X'}$, i.e., there is an exact sequence

$$0 \to J \to \mathscr{O}_{X'} \to \mathscr{O}_X \to 0.$$

Such neighborhoods are thus classified by $\mathrm{Ext}^1_{\mathscr{O}_S}(\mathscr{O}_X, J)$, and hence there is an isomorphism

$$\mathrm{Ext}^1_{\mathscr{O}_S}(\mathscr{O}_X, J) \simeq \mathrm{Ext}^1_{\mathscr{O}_X}(L_*^{X/S}, J) \simeq \mathrm{Hom}_{D(\mathscr{O}_X)}(L_*^{X/S}, J[1]),$$

and the set of solutions is a torsor under

$$\mathrm{Ext}^0_{\mathscr{O}_X}(L_*^{X/S}, J) \simeq \mathrm{Hom}(\Omega^1_{X/S}, J).$$

2°) Déformation de morphismes de schémas

Example 6.7.2. Let X and Y be two S-schemes, and let Y_0 be a subscheme of Y defined by a square-zero ideal J of $\mathscr{O}_Y$. Given an S-morphism $f_0\colon Y_0 \to X$ be an S-morphism, we seek a classification of the morphisms $f\colon Y \to X$ which prolong f_0:

$$\begin{tikzcd} X \arrow[d] & Y_0 \arrow[l, "f_0"'] \arrow[d] \arrow[r, equal] & V(J) \\ S & Y. \arrow[l] \arrow[ul, "f"] & \end{tikzcd}$$

The obstruction to such a prolongation is given by

$$\partial f_0 \in \mathrm{Ext}^1_{\mathscr{O}_Y}(\mathbb{L}f_0^*(L_*^{X/S}), J),$$

and the set of solutions to $\partial f_0 = 0$ is a torsor under

$$\mathrm{Ext}^0_{\mathscr{O}_Y}(\mathbb{L}f_0^* L_*^{X/S}, J) \simeq \mathrm{Hom}(f_0^* \Omega^1_{X/S}, J).$$

3°) Déformation des schémas plats

Example 6.7.3. Let S be a scheme, S_0 a subscheme of S defined by a square-zero ideal J, and X_0 a flat S_0-scheme. We seek a classification of the flat S-schemes X which prolong X_0, i.e., which participate in a Cartesian diagram

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X \arrow[d] \\ S_0 \arrow[r] & S. \end{tikzcd}$$

[45] That is, if for all $i > 0$ one has $\underline{\mathrm{Tor}}_i^{\mathscr{O}_S}(\mathscr{O}_X, \mathscr{O}_{S'}) = 0$.

The obstruction to the existence of such a scheme is

$$\partial(X_0, S) \in \mathrm{Ext}^2_{\mathscr{O}_{X_0}}(L_*^{X_0/S_0}, J \otimes_{\mathscr{O}_{S_0}} \mathscr{O}_{X_0}).$$

If this class is null, the set of solutions in thus a torsor under

$$\mathrm{Ext}^1_{\mathscr{O}_{X_0}}(L_*^{X_0/S_0}, J \otimes_{\mathscr{O}_{S_0}} \mathscr{O}_{X_0}),$$

and the group of automorphisms of any solution is isomorphic to

$$\mathrm{Ext}^0_{\mathscr{O}_{X_0}}(L_*^{X_0/S_0}, J \otimes_{\mathscr{O}_{S_0}} \mathscr{O}_{X_0}) \simeq \mathrm{Hom}(\Omega^1_{X_0/S_0}, J).$$

Appendix A. A letter from M. A. Grothendieck to Barsotti

Dear Barsotti,

I would like to tell you about a result on specialization of Barsotti–Tate groups in characteristic p, although you have perhaps known it for a long time, as well as a corresponding conjecture (or, rather, question), whose answer you may again already know well.

First, some terminology. Let k a perfect field of characteristic $p > 0$, W the ring of Witt vectors over k, and K its field of fractions. For us, an *F-crystal* over k will mean a free module M of finite type over W, together with a σ-linear endomorphism $F_M\colon M \to M$ (where $\sigma\colon W \to W$ is the Frobenius automorphism) such that F_M is injective.[46] I am interested in considering F-crystals up to isogeny, an equivalence class of which I will call an F-*iso*-crystal. Such an F-iso-crystal can be interpreted as a finite dimensional K-vector space E and a σ-linear automorphism $F_E\colon E \to E$; an F-isocrystal we will additionally call *effective* when there exists a lattice $M \subset E$ mapped into itself by F_E. The category of F-isocrystals is obtained from that of effective F-isocrystals and its natural internal tensor product by formally inverting formally the *Tate crystal*,

$$K(-1) = (K, F_{K(-1)} = p\sigma).$$

This is to say that the isocrystals (E, F_E) such that $(E, p^n F_E)$ is effective (i.e., those for which the set of iterates of $(p^n F_E)$ is bounded in the natural norm structure) are precisely those of the form $E_0(n) = E_0 \otimes K(-1)^{\otimes(-n)}$, with E_0 an effective F-isocrystal.

Let us now assume k to be algebraically closed. As presented in Manin's report, Dieudonné's classification theorem states that the category of F-isocrystals over k is semi-simple and that the isomorphism classes of simple objects can be indexed by $\mathbb{Q}$—equivalently, by pairs of relatively prime integers $r, s \in \mathbb{Z}$, $r \geq 1$, $(s, r) = 1$. Over $\mathbb{F}_p$, such a pair is sent to the simple object $\mathbb{E}_{s/r} = \mathbb{E}_{r,s}$ of rank r given by the formula as

$$E_{s/r} = \begin{cases} \mathbb{Q}_p[F_{s/r}]/(F_{s/r}^r - p^s) & s > 0, \\ E_{-\lambda} = (E_\lambda)^\vee & s \leq 0, \end{cases}$$

where $(-)^\vee$ denotes the linear-algebraic dual endowed with the contragredient F automorphism. In Manin's report, only effective F-crystals are considered—and then only those such that F_E is topologically nilpotent—but the observation about the Tate twist implies the result as I state it now.

[46] That is, $F(M)$ contains $p^n M$ for some $n \geq 0$.

Indexing by $\mathbb{Q}$ rather than by pairs (s,r) has the advantage that we have the simple formula

$$E_\lambda \otimes E_{\lambda'} = E_{\lambda+\lambda'}.$$

More generally, if we decompose each crystal in its isotypic component corresponding to the various "slopes"[47] $\lambda \in \mathbb{Q}$, this gives a natural grading on it over the group $\mathbb{Q}$, and this grading is compatible with the tensor product structure in the following sense:

$$E(\lambda) \otimes E'(\lambda') \subset (E \otimes E')(\lambda + \lambda').$$

Let's define the sequence of slopes of a crystal (E, F_E) by its isotypic decomposition, where each λ appears rank $E(\lambda)$ many times (bearing in mind that if $\lambda = s/r$ with $(s,r) = 1$, then rank $E(\lambda)$ is a multiple of r). It is also convenient to give an increasing order to this sequence. This definition is still appropriate even if k is not algebraically closed: by passing over to the algebraic closure of k, we can produce this sequence of numbers, but in fact the isotypic decomposition over $\overline{k}$ *descends* to k, so we even get a canonical "iso-slope"[48] decomposition over k:[49]

$$E = \bigoplus_{\lambda \in \mathbb{Q}} E(\lambda).$$

If we further specialize to $k = \mathbb{F}_q = \mathbb{F}_{p^a}$, and if (E, F_E) is a crystal over k, then F_E^a is a linear endomorphism of E over K, and the slopes of the crystal are the valuations of the proper values of F_E^a, using the valuation of $\overline{\mathbb{Q}}_p$ normalized so that $v(q) = 1$ (i.e., $v(p) = 1/a$).[50] Thus, the sequence of slopes of the crystal defined above is just the sequence of slopes of the *Newton polygon* of the characteristic polynomial of the arithmetic Frobenius endomorphism F_E^a, and that data is equivalent to the data of the p–adic valuations of the proper values of the Frobenius!

Let us return to a generic perfect field k. The effective crystals are those whose slopes are positive, and those which are Dieudonné modules[51] are those whose slopes are in the closed interval $[0,1]$. Those of slope zero corresponds to ind–étale groups, and those of slope one correspond to multiplicative groups. Moreover, an arbitrary crystal decomposes canonically into a direct sum

$$E = \bigoplus_{i \in \mathbb{Z}} E_i(-i),$$

where $(-i)$ are Tate twists[52] and the E_i have slopes $0 \le \lambda < 1$ (or, if we prefer $0 < \lambda \le 1$), hence correspond to isogeny classes of Barsotti–Tate groups over k without multiplicative component (resp. which are connected). This remark is interesting because if X is a proper and smooth scheme over k, then the crystalline cohomology groups $H^i(X)$ can be viewed

[47]The terminology "slope" here, as well as the sequence of slopes occuring in any crystal, is I believe due to you, as you presented for formal groups in Pisa about three years ago. I did not appreciate then the full appropriateness of the notation and of the terminology.

[48]In French: *isopentique*.

[49]N.B. This is true only because we assumed k perfect. There is a reasonable notion of F–crystal when k is not perfect, but then we should get only a *filtration* of a crystal by increasing slopes...

[50]This is essentially the "technical lemma" in Manin's report, without his unnecessary restrictive conditions.

[51]That is: those which correspond to (not necessarily connected) Barsotti–Tate groups over k.

[52]This corresponds to multiplying the F endomorphism by p^i.

as F–crystals, where H^i has slopes between 0 and i,[53] and hence this defines a whole avalanche of (isogeny classes of) Barsotti–Tate groups over k. Moreover, these are quite remarkable invariants whose knowledge should be thought as essentially equivalent with the knowledge of the characteristic polynomials of the "arithmetic" Frobenius, acting on (any reasonable) cohomology of X.[54]

Now the result about specialization of Barsotti–Tate groups. Select a Barsotti–Tate group G and a second group G' which is a specialization of G. Let $\lambda_1, \ldots, \lambda_h$ ($h =$ "height") be the slopes of G, and $\lambda'_1, \ldots, \lambda'_h$ the ones for G'. Then we have the equality

$$\sum \lambda'_i = \sum \lambda_i (= \dim G = \dim G')$$

as well as the inequalities

$$\sum_{i=1}^{j} \lambda_i \leq \sum_{i=1}^{j} \lambda'_i.$$

In other words, the "Newton polygon" of G (i.e., of the polynomial $\prod_i (1 + (p^{\lambda_i})T)$) lies below the one of G', and they have the same endpoints: $(0,0)$ and (h,N).

I arrived at this result through a generalization of Dieudonné theory for Barsotti–Tate groups over an arbitrary base S of characteristic p, which allows me to manufacture an F–crystal over S, heuristically thought of as an S–*family* of F–crystals in the sense outlined above. Using this, the result just stated is but a particular case of the analogous statement about specialization of arbitrary crystals. Now this latter statement is not hard to prove at all: passing to $\bigwedge^h E$ and $\bigwedge^h E'$, the equality is reduced to the case of a family of rank one crystals, and even further to the statement that such a family is just a twist of some fixed power of the (constant) Tate crystal. The general inequality (2) is reduced, passing to $\bigwedge^j E$ and $\bigwedge^j E'$, to just the first inequality $\lambda_1 \leq \lambda'_1$. Raising both E and E' to an r^{th} tensor power such that $r\lambda_1$ is an integer, we may assume that λ_1 is an integer, and a Tate twist allows us to assume that $\lambda_1 = 0$, so the statement boils down to the following: if the general member of the family is an *effective* crystal, so are all others. This is readily checked in terms of the explicit definition of a crystal over S.

The conjecture I have in mind is as follows: the equality and inequality family above are necessary conditions for G' to be a specialization of G, and I would like them to also be sufficient. More explicitly, start with a Barsotti–Tate group $G_0 = G'$, and take its formal modular deformation in characteristic p (over a modular formal variety S of dimension dd^*, $d = \dim G_0$, $d^* = \dim G_0^*$). For the Barsotti–Tate group G over S so-obtained, we want to know if every sequence of rational numbers λ_i, lying between 0 and 1 and satisfying the equality and inequality family, occurs as the sequence of slopes of a fiber of G at some point of S. This does not seem too unreasonable, as the set of all (λ_i) satisfying these conditions is finite, as is the set of slope-types of all possible fibers of G over S.

I should mention that the inequality family was suggested to me by the following beautiful conjecture of Katz: if X is smooth and proper over a finite field k, with Hodge numbers in dimension i given by $h^0 = h^{0,i}$, $h^1 = h^{1,i-1}$, …, $h^i = h^{i,0}$, and if we consider the characteristic polynomial of the arithmetic Frobenius F^a operating on some reasonable

[53] This is not proved now in complete generality, but is proved in X lifts formally to characteristic zero, and I certainly believe it to be true generally.

[54] However, the arithmetic Frobenius is not really defined, unless k is finite!

cohomology group of X (say, ℓ-adic for $\ell \neq p$, or crystalline), then the Newton polygon of this polynomial should be *above* the one of the polynomial $\prod_i (1 + p^i T)^h i$. In a very heuristic and also very suggestive way, this could now be interpreted (without needing to assume k finite) as stating that $H^i_{\text{cris}}(X)$ is a specialization of a crystal whose sequence of slopes is 0 h^0 times, 1 h^1 times, ..., i h^i times. If X lifts formally to characteristic 0, then we can introduce also the Hodge numbers of the lifted variety, which satisfy

$$(h')^0 \leq h^0, \ldots, (h')^i \leq h^i,$$

and one should expect a strengthening of Katz's conjecture to hold, with the $(h')^j$ replaced by the h^j. Thus the transcendental analog of an F-crystal in characteristic p seems to be something like a Hodge structure of a Hodge filtration, and the sequence of slopes of such a structure should be defined as the sequence in which j enters with multiplicity $(h')^j = \operatorname{rank} \operatorname{gr}^j$.[55] I have some idea how Katz's conjecture with the h^is (not the $(h')^i$s, at least for the time being) may be attacked by the machinery of crystalline cohomology, at least at the level of the first inequality among the family. At the same time, the formal argument involving exterior powers, outlined afterwards, gives the feeling that it is really the first inequality $\lambda_1 \leq \lambda_1'$ that is essential, and the others should follow once we have a good general framework.

I would very much appreciate your comments on this general nonsense—again, I imagine that most of it is quite familiar to you, under a different terminology.

Very sincerely yours,

A. Grothendieck
Bures May 11, 1970

[55]N.B.: Katz made his conjecture only for global complete intersections. However, I would not be as cautious as he!

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