Practice Math 25b Final #2

This is a practice midterm for Math 25b. To give you the experience of a real exam, we include the following bits of information and warnings.

- *Do not* open the test booklet until told to do so.
- There are *10* questions on this midterm. Make sure you have all of them.
- There are two pages for scratch work included at the end.
- No outside materials are allowed for reference: no friends, no phones, no books, no notes, no pages from other exams, no wandering eyes, ... The only thing you may use for the duration of this exam is your pencil.
- The exam is to last 50 minutes.
- You are allowed to cite results from Axler, from Spivak, or from the classroom. (The one exception is if a question were to ask you to re-prove such a result. Stating "we did this in class" is not a sufficient answer in that case.)
- Agree to the following by signing on the blank line:

> I, ______________________________, am bound by the Harvard Honor Code, which I recently signed when registering for classes. Accordingly, I understand the serious consequences that would befall me if I were to cheat on this midterm. I hereby affirm that I have not cheated.

Unsigned exams will be left ungraded and the examinee marked as absent.

Problem 1. Show that if a sequence $(a_n) \in \mathbb{R}^n$ converges to some limit a, show that any subsequence $(a_{n_k}) \in \mathbb{R}^n$ also converges, with the same limit.

Problem 2. A function f is *uniformly continuous* if for every $\varepsilon > 0$ there is $\delta > 0$ such that for every x, $\|x - y\| < \delta \implies \|f(x) - f(y)\| < \varepsilon$. (That is, whereas δ is normally allowed to depend upon ε and x, a uniformly continuous function's δ is independent of x.) Prove that if $f\colon X \to \mathbb{R}^n$ is continuous and X is a compact subset of $\mathbb{R}^m$, then f is automatically uniformly continuous.

Problem 3. If $f : A \to \mathbb{R}$ is non-negative and $\int_A f = 0$ show that $\{x \mid f(x) \neq 0\}$ has measure 0. (Hint: Prove that $\{x \mid f(x) > 1/n\}$ has content 0.

Problem 4. Define a vector field F on $\mathbb{R}^3$ by the formula

$$F(x, y, z) = (0, 0, cz)$$

for some constant $c \in \mathbb{R}$. Think of F as the downward pressure of a fluid of density c in $\{(x, y, z) \mid z \leq 0\}$. Since a fluid exerts equal pressures in all directions, we define the buoyant force on a 3–manifold M as

$$-\int_{\partial M} \langle F, n \rangle \, \mathrm{d}A,$$

where n is a normal vector to the surface ∂M. Prove that the buoyant force on M is equal to the weight of the fluid displaced by M.

Problem 5. In class, we gave the following second characterization of manifolds: a subset $M \subseteq \mathbb{R}^n$ is a k–manifold if and only if for each $x \in M$ there is an open neighborhood $U \ni x$, an open set $W \subseteq \mathbb{R}^k$, and a bijective differentiable function $f\colon W \to \mathbb{R}^n$ such that

1. $f(W) = M \cap U$.
2. For each $y \in W$, $D_y f$ has rank k.
3. $f^{-1}\colon f(W) \to W$ is continuous.

Find a *counterexample* to this claim if condition 3 is omitted.

Problem 6. Consider the spherical coordinates map $\mathbb{R}^3 \to \mathbb{R}^3$:

$$\begin{pmatrix} \rho \\ \varphi \\ \theta \end{pmatrix} \mapsto \begin{pmatrix} \rho\cos\varphi\cos\theta \\ \rho\cos\varphi\sin\theta \\ \rho\sin\varphi \end{pmatrix}.$$

In what regions of the domain $\mathbb{R}^3$ is this function orientation-preserving? In what regions is this function orientation-reversing?

Problem 7. Let X be a manifold. A point $x \in X$ is a *Lefschetz fixed point* of $f : X \to X$ if $f(x) = x$ and 1 is not an eigenvalue of $D_x f$. The function f itself is a *Lefschetz map* if all its fixed points are Lefschetz. Prove that if X is compact and f is Lefschetz, then f has only finitely many fixed points.

Problem 8. What is the integral of the form

$$\omega = x\,\mathrm{d}y \wedge \mathrm{d}z + y\,\mathrm{d}z \wedge \mathrm{d}x + z\,\mathrm{d}x \wedge \mathrm{d}y$$

over the part S of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

where $x, y, z \geq 0$ and with orientation given by the outward-pointing normal? (Feel free to use either of Stokes's theorem or to compute the integral directly through parametrization.)

Problem 9. Suppose that M_1 is an n–dimensional manifold with boundary and that $M_2 \subseteq M_1$ is another n–dimensional manifold with boundary such that $\partial M_1 \cap \partial M_2 = \varnothing$. Demonstrate the equality

$$\int_{\partial M_1} \omega = \int_{\partial M_2} \omega$$

for any $(n-1)$–form ω on $\mathbb{R}^n$.

Problem 10. Suppose we have V, a finite dimensional vector space, along with $\alpha \in \Omega^p(V)$ and v, a vector in V. Consider the contraction of α by v, denoted $i_v\alpha$, which is the $p-1$ form defined by

$$i_v\alpha(w_1, \ldots, w_{p-1}) = \alpha(v, w_1, \ldots, w_{p-1}).$$

Show that if V is any p-dimensional vector space, v is a nonzero vector in V, and α is a nonzero element of $\Omega^p(V)$, then $i_v\alpha$ defines a nonzero element of $\Omega^{p-1}(W)$, where W is the orthogonal complement of v.