INVERSE/IMPLICIT FUNCTION THEOREMS
SECTION 2, MATH 25B

BENJAMIN GUNBY

As per discussion in last section, you might look at why the assumptions are necessary in these two theorems. Fortunately, Problem 2.2 on the homework asks that exact thing for the inverse function theorem–that is, why we need the continuously differentiable condition.

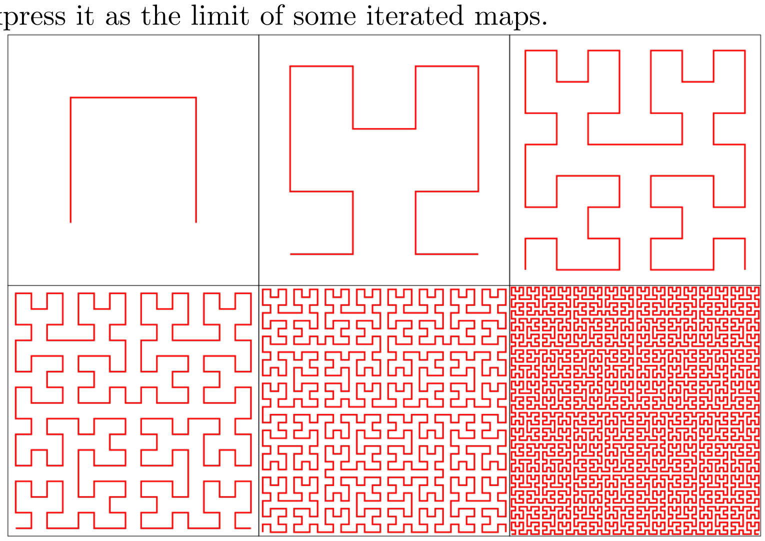
The inverse function theorem is in fact a special case of the implicit function theorem! Consider a function $f : \mathbb{R}^n \to \mathbb{R}^n$ that satisfies the conditions of the inverse function theorem at a point a. Then define $f^\sharp : \mathbb{R}^{2n} \to \mathbb{R}^n$ as follows:

$$\begin{aligned} f^\sharp(y_1, \ldots, y_n, x_1, \ldots, x_n) &= y - f(x) \\ &= \left(-y_1 + f_1(x_1, \ldots, x_n), -y_2 + f_2(x_1, \ldots, x_n), \ldots, -y_n + f_n(x_1, \ldots, x_n)\right). \end{aligned}$$

Now, we want to find an inverse to f; that is, find the x_i's in terms of the y_i's given that $y = f(x)$. Note that $y = f(x)$ is exactly given by the condition $f^\sharp(x, y) = 0$. So we want to find the x_i in terms of the y_i, given the condition $f^\sharp = 0$, around some point $(f(a), a)$. What condition do we need to ensure this exists? Well, if we look at the partials of $f^\sharp$ with respect to the x_i, the y_i parts will vanish, so we have that $\frac{\partial f_i^\sharp}{\partial x_j} = \frac{\partial f_i}{\partial x_j}$. Thus the matrix whose determinant we need to not vanish is exactly the same as the Jacobian of f, which is what we need for the inverse function theorem! This shows that the inverse function theorem is really a case of the implicit function theorem.

Space-filling curves: The homework asks to show that there is no injective C^1 (continuously differentiable) map $\mathbb{R}^2 \to \mathbb{R}$. And this intuitively makes sense; you'd think you shouldn't be able to 'compress' $\mathbb{R}^2$ to $\mathbb{R}$ in a sufficiently continuous manner without sending two elements to the same point. However, sometimes this intuition doesn't always hold.

For example, you might also think there is no surjective C^1 map $\mathbb{R} \to \mathbb{R}^2$, and this is true. However, there *are* C^0 (continuous) surjective maps $(0, 1) \to (0, 1)^2$, which is weird! We can express it as the limit of some iterated maps.



Date: 2/10/2017.

Also, it turns out that wikipedia has good graphics.