Homework #3

Math 25b

Due: February 22nd, 2017

Guidelines:

- As we calibrate the difficulty of the first few assignments, please write the amount of time it takes you to finish the entire assignment (i.e., the sum of all four parts) at the top of each packet you hand in.
- You must type up your solutions to this assignment in LaTeX. There's a template available on the course website.
- This homework is divided into four parts. You will turn each part in to a separate CA's mailbox on the second floor of the science center. So, be sure to do the parts on *separate* pieces of paper.
- If your submission to any particular CA takes multiple pages, then *staple them together*. If you don't own one, a stapler is available in the Cabot Library in the Science Center.
- Be sure to put your *name* at the top of each part, so that we know who to score!
- If you collaborate with other students, please announce that somewhere (ideally: next to the problems you collaborated on) so that we don't get suspicious of hyper-similar answers.

Failure to meet these guidelines may result in loss of points. (Staple your pages!)[1]

1 For submission to Thayer Anderson

Problem 1.1. Let $g_1, g_2 \colon \mathbb{R}^2 \to \mathbb{R}$ both be continuous functions. Define $f \colon \mathbb{R}^2 \to \mathbb{R}$ by the following integral:

$$f(x, y) = \int_0^x g_1(t, 0)\,\mathrm{d}t + \int_0^y g_2(x, t)\,\mathrm{d}t.$$

(You can think of f as measuring the total charge for moving from $(0, 0)$ to (x, y) along a rectilinear path.)

1. Demonstrate $\frac{\partial f}{\partial y}(x, y) = g_2(x, y)$. (You'll want to invoke the single-variable fundamental theorem of calculus, which we have not yet proven. Go ahead.)
2. How can the definition of f be modified so that $\frac{\partial f}{\partial x}(x, y) = g_1(x, y)$? Can you get both identities to work at once?
3. Find a function f with $\frac{\partial f}{\partial x} = x$ and $\frac{\partial f}{\partial y} = y$. Now find a function f with $\frac{\partial f}{\partial x} = y$ and $\frac{\partial f}{\partial y} = x$.
4. Explain why you don't expect to be able to find a function f with $\frac{\partial f}{\partial x} = y$ and $\frac{\partial f}{\partial y} = -x$.

Problem 1.2. For $f \colon \mathbb{R}^2 \to \mathbb{R}$, show that if $\frac{\partial f}{\partial y} = 0$ then f is independent of the second variable — i.e., $f(x, y_1) = f(x, y_2)$ for any $y_1, y_2 \in \mathbb{R}$. If both $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$, show that f is a constant function.

[1] This version of the homework dates from February 21, 2017.

2 For submission to Davis Lazowski

Problem 2.1. 1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function, and assume $f'(a) \neq 0$ for all $a \in \mathbb{R}$. Show that f is injective.

2. Now consider $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, described by the formula

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^x \cos y \\ e^x \sin y \end{pmatrix}.$$

Show that $\det(D_{(a,b)}f) \neq 0$ for all $(a,b) \in \mathbb{R}^2$, and yet f is *not* injective.[2]

Problem 2.2. The statement of the inverse function theorem for a function f requires that the derivative f' be continuous. Consider the following function:

$$f(x) = \begin{cases} \frac{x}{2} + x^2 \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

1. Check that f is differentiable everywhere and that its derivative fails to be continuous at 0.

2. Check that the conclusion of the inverse function theorem fails at 0.

3 For submission to Handong Park

Problem 3.1. Consider the domain

$$A = \{(x,y) \in \mathbb{R}^2 \mid x < 0\} \cup \{(x,y) \in \mathbb{R}^2 \mid x \geq 0 \text{ and } y \neq 0\} = \mathbb{R}^2 \setminus ([0,\infty) \times \{0\}).$$

1. Let $f\colon A \to \mathbb{R}$ be a differentiable function with $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$. Show f is constant.[3]

2. Find a differentiable function $f\colon A \to \mathbb{R}$ satisfying $\frac{\partial f}{\partial y} = 0$ but which is *not* independent of y.

Problem 3.2. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be differentiable. Given a point $a \in \mathbb{R}^n$ and a direction $v \in T_a\mathbb{R}^n$, the associated *directional derivative* of f is defined by

$$\lim_{t \to 0} \frac{f(a+tv) - f(a)}{t},$$

which we will temporarily denote by $\mathbb{D}_a^v f$.

1. Choosing $v = e_i$ to be a standard basis vector, show $\mathbb{D}_a^{e_i} f = \frac{\partial f}{\partial x_i}$.

2. Choosing a scalar $k \in \mathbb{R}$, show $\mathbb{D}_a^{kv} f = k \cdot \mathbb{D}_a^v f$.

3. Now suppose that f is a differentiable function. Show that $\mathbb{D}_a^v f = (D_a f)(v)$, so that we have not invented a new notion of derivative. Conclude that $\mathbb{D}_a^{v+w} f = \mathbb{D}_a^v f + \mathbb{D}_a^w f$.

[2] The inspiration for this function comes from the complex exponential: $e^{a+bi} = e^a \cdot (\cos b + i \sin b)$.

[3] Note that this is *not* a repeat of Problem 1.2, since the domain is different.

4 For submission to Rohil Prasad

Problem 4.1. Let $A \subseteq \mathbb{R}^n$ be an open set, and let $f\colon A \to \mathbb{R}^n$ be a continuously differentiable injective function with $\det D_a f \neq 0$ for all $a \in A$.

1. Show that the image $f(A) \subseteq \mathbb{R}^n$ is an open set.
2. Show that $f^{-1}\colon f(A) \to A$ is differentiable.
3. Show that for any open $B \subseteq A$, $f(B)$ is also an open set.

Problem 4.2. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function. Show that f *cannot* be injective. Now generalize this to the case of $f\colon \mathbb{R}^n \to \mathbb{R}^m$ with $m < n$.