Math 231 Homework 1: Solutions

Section 5.5 # 3,14,31,33,43,59,64,67

5.5.3

$$\begin{aligned} I &= \int x^2\sqrt{1+x^3}dx \qquad u = 1+x^3 \qquad du = 3x^2 \\ &= \int \frac{1}{3}\sqrt{u}dx \\ &= \frac{1}{3}\frac{2}{3}u^{\frac{3}{2}} + c = \frac{2}{9}(1+x^3)^{\frac{3}{2}} \end{aligned}$$

5.5.14

$$\begin{aligned} I &= \int e^x \sin(e^x)dx \qquad u = e^x \qquad du = e^x dx \\ &= \int \sin(u)du = -\cos(u) + c \\ &= -\cos(e^x) + c \end{aligned}$$

5.5.31

$$\begin{aligned} I &= \int \frac{\cos(x)}{\sin^2(x)}dx \qquad u = \sin(x) \qquad du = \cos(x)dx \\ &= \int \frac{du}{u^2} = -\frac{1}{u} + c \\ &= -\csc(x) + c \end{aligned}$$

5.5.33

$$\begin{aligned} I &= \int \sqrt{\cot(x)}\csc^2(x)dx \qquad u = \cot(x) \qquad du = -\csc^2(x)dx \\ &= -\int u^{\frac{1}{2}}du = -\frac{2}{3}u^{\frac{3}{2}} + c = -\frac{2}{3}\cot^{\frac{3}{2}}(x) + c \end{aligned}$$

5.5.43 This integral needs to be split into two pieces which are handled differently:

$$\int \frac{1+x}{1+x^2}dx = \int \frac{dx}{1+x^2} + \int \frac{x}{1+x^2}dx$$

In the second we make the substitution $u = x^2$. The first is just $arctan(x)$. Thus we get

$$\int \frac{1+x}{1+x^2}dx = \arctan(x) + \frac{1}{2}\ln|1+x^2| + c$$

5.5.59

$$\begin{aligned} &\int_1^2 \frac{e^{\frac{1}{x}}}{x^2}dx \qquad u = \frac{1}{x} \qquad du = -\frac{1}{x^2}dx \\ &= -\int_1^{\frac{1}{2}} e^u du = -e^{-1} + e^{-\frac{1}{2}} \end{aligned}$$

5.5.64

$$I = \int_0^a x\sqrt{a^2 - x^2}dx \qquad u = a^2 - x^2 \qquad du = -2xdx$$
$$= -\frac{1}{2}\int_{a^2}^0 \sqrt{u}du = -\frac{1}{3}u^{\frac{3}{2}}|_{a^2}^0 = \frac{1}{3}a^3$$

5.5.67

$$I = \int_e^{e^4} \frac{dx}{x\sqrt{\ln|x|}} \qquad u = \ln|x| \qquad du = \frac{dx}{x}$$
$$= \int_1^4 \frac{du}{\sqrt{u}} = 2\sqrt{u}|_1^4 = 4 - 2 = 2$$

Section 7.1 # 1,5,8,15,18,35,43,47,62,63

7.1.1

$$I = \int x^2 \ln|x|dx \qquad u = \ln|x| \quad dv = x^2dx$$
$$du = \frac{dx}{x} \quad v = \frac{x^3}{3}$$
$$= \frac{x^3}{3}\ln|x| - \int \frac{x^2}{3}dx$$
$$= \frac{x^3}{3}\ln|x| - \frac{x^3}{9} + c$$

7.1.5

$$I = \int re^{\frac{r}{2}}dr \qquad x = \frac{r}{2} \qquad dx = \frac{dr}{2}$$
$$= 4\int xe^x dx \qquad u = x \quad dv = e^x dx \qquad du = dx \quad v = e^x$$
$$= 4xe^x - 4\int e^x = 4(x-1)e^x = 4(\frac{r}{2} - 1)e^{\frac{r}{2}} + c$$

7.1.8

$$I = \int x^2\cos(mx)dx \qquad u = x^2 \quad dv = \cos(mx)dx \qquad du = 2xdx \quad v = \frac{\sin(mx)}{m}$$
$$= \frac{x^2}{m}\sin(mx) - \int \frac{2x\sin(mx)}{m}dx \qquad u = \frac{2x}{m} \quad dv = \sin(mx)dx \qquad du = \frac{2}{m}dx \quad v = -\frac{\cos(mx)}{m}$$
$$= \frac{x^2}{m}\sin(mx) + \frac{2\cos(mx)}{m^2} - \frac{2}{m^2}\int \cos(mx)dx$$
$$= \frac{x^2}{m}\sin(mx) + \frac{2\cos(mx)}{m^2} - \frac{2\sin(mx)}{m^3} + c$$

7.1.15

$$\begin{aligned} I &= \int (\ln|x|)^2 dx \qquad u = (\ln(x))^2 \quad dv = dx \qquad du = 2\frac{\ln(x)}{x}dx \quad v = x \\ &= x(\ln(x))^2 - \int 2x\frac{\ln(x)}{x}dx \qquad u = \ln(x) \quad dv = dx \qquad du = \frac{dx}{x} \quad v = x \\ &= x(\ln(x))^2 - 2x\ln(x) + 2\int dx \\ &= x(\ln(x))^2 - 2x\ln(x) + 2x + c \end{aligned}$$

7.1.18

$$\begin{aligned} I &= \int e^{-\theta}\cos(2\theta)d\theta \qquad u = e^{-\theta} \quad dv = \cos(2\theta)d\theta \qquad du = -e^{-\theta}d\theta \quad v = \frac{\sin(2\theta)}{2} \\ &= e^{-\theta}\frac{\sin(2\theta)}{2} + \int \frac{\sin(2\theta)}{2} - e^{-\theta}d\theta \qquad u = e^{-\theta} \quad dv = \frac{\sin(2\theta)}{2}d\theta \qquad du = -e^{-\theta}d\theta \quad v = -\frac{\cos(2\theta)}{4} \\ &= e^{-\theta}\frac{\sin(2\theta)}{2} - e^{-\theta}\frac{\cos(2\theta)}{4} - \int \frac{\cos(2\theta)}{4}e^{-\theta}d\theta \\ \frac{5}{4}I &= e^{-\theta}\frac{\sin(2\theta)}{2} - e^{-\theta}\frac{\cos(2\theta)}{4} \\ I &= \frac{2}{5}e^{-\theta}\sin(2\theta) - \frac{1}{5}e^{-\theta}\cos(2\theta) \end{aligned}$$

7.1.35

$$\begin{aligned} I &= \int \theta^3\sin(\theta^2)d\theta \qquad x = \theta^2 \quad dx = 2\theta d\theta \\ &= \int \frac{1}{2}x\sin(x)dx \qquad u = x \quad dv = \sin(x)dx \qquad du = dx \quad v = -\cos(x) \\ &= \frac{1}{2}\left(-x\cos(x) + \int \cos(x)\right) \\ &= \frac{1}{2}\left(-x\cos(x) + \sin(x)\right) + c \\ &= \frac{1}{2}\left(-\theta^2\cos(\theta^2) + \sin(\theta^2)\right) + c \end{aligned}$$

7.1.43

$$\begin{aligned} \int sin^n(x)dx &= -\frac{1}{n}\cos(x)\sin^{n-1}(x) + \frac{n-1}{n}\int \sin^{n-2}(x)dx \\ \int \sin^2(x)dx &= -\frac{1}{2}\cos(x)\sin(x) + \frac{1}{2}\int dx \\ &= -\frac{1}{2}\cos(x)\sin(x) + \frac{x}{2} + c \end{aligned}$$

7.1.47

$$\begin{aligned} I_n &= \int (\ln|x|)^n dx \qquad u = (\ln|x|)^n \quad dv = dx \qquad du = n\frac{(\ln|x|)^{n-1}}{x} \quad v = x \\ &= x(\ln|x|)^n - \int n\frac{(\ln|x|)^{n-1}}{x} x dx \\ &= x(\ln|x|)^n - \int n(\ln|x|)^{n-1} dx \\ &= x(\ln|x|)^n - I_{n-1} \end{aligned}$$

7.1.62 IMPORTANT: Recall that

$$\log(ab) = \log(a) + \log(b)$$

$$\begin{aligned} \frac{dx}{dt} &= -gt - v_c \ln(1 - \frac{rt}{m}) \\ x(t) &= \int -gt - v_c \ln(1 - \frac{rt}{m}) dt \\ x(t) &= -\frac{gt^2}{2} + \int v_c \int \ln(1 - \frac{rt}{m}) dt \\ y &= 1 - \frac{rt}{m} \quad dy = -\frac{r}{m} dt \\ x(t) &= -\frac{gt^2}{2} + \frac{mv_c}{r} \int \ln|y| dy \end{aligned}$$

Integrating by parts gives

$$\begin{aligned} I &= \int \ln|y| dy \qquad u = \ln|y| \quad dv = dy \qquad du = \frac{dy}{y} \quad v = y \\ &= y \ln|y| - \int \frac{ydy}{y} \\ &= y\ln|y| - y + c \end{aligned}$$

Thus we have

$$x(t) = -\frac{gt^2}{2} + \frac{mv_c}{r}\left((1 - \frac{rt}{m})\ln|1 - \frac{rt}{m}| - (1 - \frac{rt}{m})\right) + c$$

Since the height of the rocket at $t = 0$ is $x(0) = 0$ this gives

$$0 = -\frac{mv_c}{r} + c$$

Thus

$$\begin{aligned} x(t) &= -\frac{gt^2}{2} + \frac{mv_c}{r}\left((1 - \frac{rt}{m})\ln|1 - \frac{rt}{m}| - (1 - \frac{rt}{m})\right) + \frac{mv_c}{r} \\ &= -4.9t^2 + \frac{30,000 \times 3000}{160}(1 - \frac{160t}{30,000})\ln|1 - -\frac{160t}{30,000}| + 3000t \end{aligned}$$

Plugging in $t = 60$ (units are in seconds, problem asks for one minute height) gives

$$x(t) \approx 50,124$$