Hi, everyone,

I was hoping to have time for an example today. I didn't, so here are three:

1. First, a warm-up. Consider the linear map $g\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$g(x, y) = x - y.$$

Both the left and the right have standard bases: $\mathbb{R}^2$ is spanned by $v_1 = (1, 0)$ and $v_2 = (0, 1)$, and $\mathbb{R}$ has basis $w_1 = (1)$. In order to compute the matrix representation of g, we apply g to v_1 and v_2 and represent the result in terms of w_1:

$$\begin{aligned} g(v_1) &= g(1, 0) = (1) = 1 \cdot w_1 \\ g(v_2) &= g(0, 1) = (-1) = -1 \cdot w_1, \end{aligned}$$

and hence the matrix representation of g has two columns and one row:

$$\begin{pmatrix} 1 & -1 \end{pmatrix}.$$

2. Now consider the linear map $d/dx\colon V_2 \to V_1$, where V_2 is the vector space of real polynomials of degree at most 2 (as considered on the homework) and V_1 is the vector space of real polynomials of degree at most 1. We would like to give a matrix presentation of d/dx.

(a) Pick the basis $(1, x, x^2)$ of V_2 and the basis $(1, x)$ of V_2. Were going to end up computing a map $\mathbb{R}^3 \to \mathbb{R}^2$, which is represented by a matrix M with 3 columns and 2 rows. To figure out the columns, we compute

$$\begin{aligned} M(e_1) &= \frac{d}{dx}(1) = 0 = 0 \cdot 1 + 0 \cdot x, \\ M(e_2) &= \frac{d}{dx}(x) = 1 = 1 \cdot 1 + 0 \cdot x, \\ M(e_3) &= \frac{d}{dx}(x^2) = 2x = 0 \cdot 1 + 2 \cdot x, \end{aligned}$$

and so we get the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

If we had some other polynomial $p(x) = 1 + x + x^2$ in V_2, we could compute the map d/dx using M (and not using direct information about differentiation!) by presenting this vector in terms of the basis.

Start by writing the polynomial as a linear combination of basis vectors:

$$\frac{d}{dx}(1 + x + x^2) = \frac{d}{dx}\left(1 \cdot 1 + 1 \cdot x + 1 \cdot x^2\right)$$

Now remember that $d/dx = \varphi_{V_1} \circ M \circ \varphi_{V_2}^{-1}$:

$$= \varphi_{V_1} \circ M \circ \varphi_{V_2}^{-1}\left(1 \cdot 1 + 1 \cdot x + 1 \cdot x^2\right)$$

To evaluate $\varphi_{V_2}^{-1}$, we strip off the coefficients in the linear combination to form a vector in $\mathbb{R}^3$:

$$= \varphi_{V_1}\left(M \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right)$$

Then apply the matrix M to get a vector in $\mathbb{R}^2$:

$$\begin{aligned} &= \varphi_{V_1}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix}\right) \\ &= \varphi_{V_1}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \end{aligned}$$

Finally, convert that back at a polynomial using the other basis:

$$= 1 \cdot 1 + 2 \cdot x = 1 + 2x.$$

(b) Lets do this again, but with a funny basis. You know from your homework that $(1, 1-x, 2-3x+x^2)$ and $(2, x)$ also form bases for V_2 and V_1 respectively. Again, we figure out the columns:

$$\begin{aligned} M(e_1) &= \frac{d}{dx}(1) = 0 = 0 \cdot 2 + 0 \cdot x, \\ M(e_2) &= \frac{d}{dx}(1-x) = -1 = \frac{-1}{2} \cdot 2 + 0 \cdot x, \\ M(e_3) &= \frac{d}{dx}(2 - 3x + x^2) = -3 + 2x = \frac{-3}{2} \cdot 2 + 2 \cdot x, \end{aligned}$$

so that the matrix representation is

$$M = \begin{pmatrix} 0 & -1/2 & -3/2 \\ 0 & 0 & 2 \end{pmatrix}.$$

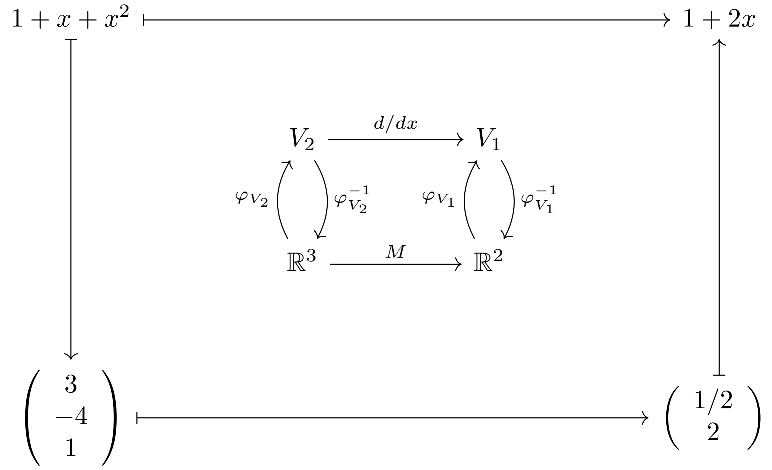
Given the same polynomial $p(x) = 1 + x + x^2 \in V_2$, we calculate $\frac{d}{dx}(1+x+x^2)$ by first calculating the linear combination

$$1 + x + x^2 = 3 \cdot 1 + -4 \cdot (1-x) + 1 \cdot (2 - 3x + x^2),$$

so that

$$\begin{aligned} \frac{d}{dx}(1 + x + x^2) &= \varphi_{V_1}\left(M \cdot \varphi_{V_2}^{-1}(1 + x + x^2)\right) \\ &= \varphi_{V_1}\left(M \cdot \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}\right) \\ &= \varphi_{V_1}\left(3 \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix} + -4 \cdot \begin{pmatrix} -1/2 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} -3/2 \\ 2 \end{pmatrix}\right) \\ &= \varphi_{V_1}\begin{pmatrix} 1/2 \\ 2 \end{pmatrix} \\ &= 1/2 \cdot 2 + 2 \cdot x \\ &= 1 + 2x. \end{aligned}$$

I hope this clarifies whats going on with representing linear maps as matrices. I recommend as you read this to draw those square diagrams and look at how were pushing these various elements around in each step, as in:



$1 + x + x^2$
$1 + 2x$
V_2
d/dx
V_1
φ_{V_2}
$\varphi_{V_2}^{-1}$
φ_{V_1}
$\varphi_{V_1}^{-1}$
$\mathbb{R}^3$
M
$\mathbb{R}^2$
$\begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}$
$\begin{pmatrix} 1/2 \\ 2 \end{pmatrix}$.