Homework #7 Solutions

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1 For submission to Thayer Anderson

Problem 1.1. Suppose that $f, g\colon K^3 \to K^3$ are two linear functions that each have eigenvalues 2, 6, and 7. Show that there exists a linear function $h\colon K^3 \to K^3$ satisfying $f = h \circ g \circ h^{-1}$.

Solution. Because each has enough eigenvalues to exhaust the 3–dimensional space K^3, f and g are both diagonalizable — i.e., there exist bases x_1, x_2, x_3 and y_1, y_2, y_3 such that f expressed in the first basis and g expressed in the second basis both give the same matrix presentation

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 7 \end{pmatrix}.$$

Write d for this diagonal matrix, b_x for the change-of-basis operator associated to the x–basis, and b_y for the change-of-basis operator associated to the y–basis. Algebraically, we thus have the two equations

$$b_x^{-1} d b_x = f, \quad b_y^{-1} d b_y = g.$$

By sharing d across the two equations, we get

$$f = (b_y^{-1} b_x)^{-1} g (b_y^{-1} b_x). \tag{ECP}$$

Problem 1.2. A *norm* on V is a function $\| - \|\colon V \to \mathbb{R}_{\geq 0}$ satisfying

- $\|u\| = 0$ if and only if $u = 0$.
- $\|k \cdot u\| = |k| \cdot \|u\|$ for any scalar k.
- $\|u + v\| \leq \|u\| + \|v\|$.

In this problem, we will show that when a norm arises from an inner product by $\|v\| = \sqrt{\langle v, v\rangle}$, we can recover the inner product from the norm.

1. Suppose that V is a real inner product space. Show that

$$\langle u, v\rangle = \frac{\|u+v\|^2 - \|u-v\|^2}{4}.$$

2. Suppose that V is a complex inner product space. Show that

$$\langle u, v\rangle = \frac{\|u+v\|^2 - \|u-v\|^2 + \|u+iv\|^2 i - \|u-iv\|^2 i}{4}.$$

Solution. 1. This is a matter of expanding out the fractions.

$$\begin{aligned} \frac{\|u+v\|^2 - \|u-v\|^2}{4} &= \frac{1}{4}\left(\langle u+v, u+v\rangle - \langle u-v, u-v\rangle\right) \\ &= \frac{1}{4}\left(\langle u, u\rangle + \langle v, v\rangle + 2\langle u, v\rangle - \langle u, u\rangle - \langle v, v\rangle + 2\langle u, v\rangle\right) \\ &= \langle u, v\rangle. \end{aligned}$$

2. This is also a matter of expanding out the fractions, but this time we have to be careful to track the conjugate-linearity of the right-hand argument of the inner product.

$$\text{———//———} = \frac{1}{4}(\langle u+v, u+v\rangle - \langle u-v, u-v\rangle + \langle u+iv, u+iv\rangle i - \langle u-iv, u-iv\rangle i)$$

Start by dealing just with the factors without any is:

$$\begin{aligned} &= \frac{1}{4}(\langle u,u\rangle + \langle v,v\rangle + \langle u,v\rangle + \overline{\langle u,v\rangle} - \langle u,u\rangle - \langle v,v\rangle + \langle u,v\rangle + \overline{\langle u,v\rangle} \\ &\qquad + \langle u+iv, u+iv\rangle i - \langle u-iv, u-iv\rangle i) \\ &= \frac{1}{4}(4\operatorname{Re}\langle u,v\rangle + \langle u+iv, u+iv\rangle i - \langle u-iv, u-iv\rangle i) \end{aligned}$$

Now look at the terms with is:

$$\begin{aligned} &= \frac{1}{4}(4\operatorname{Re}\langle u,v\rangle + i\langle u,u\rangle + i\langle v,v\rangle + \langle u,v\rangle - \overline{\langle u,v\rangle} - (i\langle u,u\rangle + i\langle v,v\rangle - \langle u,v\rangle + \overline{\langle u,v\rangle})) \\ &= \frac{1}{4}(4\operatorname{Re}\langle u,v\rangle + 4i\operatorname{Im}\langle u,v\rangle) = \langle u,v\rangle. \end{aligned} \qquad \text{(ECP)}$$

2 For submission to Davis Lazowski

Problem 2.1. Suppose that $S\colon V \to V$ is a linear operator on an inner product space V. Define a new pairing by

$$\langle u,v\rangle_S = \langle Su, Sv\rangle.$$

1. Suppose that S is injective. Show that this new pairing is also an inner product on V.

2. Suppose that S *fails* to be injective. Show that this same pairing is *not* an inner product on V.

Solution. **Part 1**

- *positive definite*: $\langle u,u\rangle_S \geq 0$ because $\langle Su, Su\rangle \geq 0$ due to the definiteness of the original inner product.
- *nondegenerate*: If $\langle Su, Su\rangle = 0$, then $Su = 0$, by nondegeneracy of the original inner product. By injectivity of S, then $u = 0$, as required.
- *Linear in the first argument*: $\langle S(u+\lambda w), Sm\rangle = \langle Su + \lambda Sw, Sm\rangle$ by the linearity of S. By the linearity of the original inner product, therefore $\langle Su+\lambda Sw, Sm\rangle = \langle Su, Sm\rangle + \lambda \langle Sw, Sm\rangle$ as required.
- *Conjugacy*: $\langle Su, Sw\rangle = \overline{\langle Su, Sw\rangle}$, because this is true for the original inner product.

Part 2
In this case, there exists $u \neq 0 : S = 0$. Then $\langle Su, Su\rangle = \langle 0, 0\rangle = 0$, so this pairing is degenerate. (DL)

Problem 2.2. Suppose V is a finite–dimensional real vector space, and suppose $\langle -,-\rangle_1$ and $\langle -,-\rangle_2$ are two inner products on V.

1. Show that there exists a number $c > 0$ with $\|v\|_1 \leq c\|v\|_2$.

2. Suppose further that $\langle v,w\rangle_1 = 0$ if and only if $\langle v,w\rangle_2 = 0$. Show that there is a number $c > 0$ such that $\langle -,-\rangle_1 = c \cdot \langle -,-\rangle_2$.

Solution. **Part 1**

If $||v||_1^2 \leq c||v||_2^2$, then by positivity $||v||_1 \leq \sqrt{c}||v||_2$. Therefore, we'll work with norms squared.

Let $v_1....v_n$ an orthonormal basis of V with respect to $\langle\rangle_1$. Then

$$||v||_1^2 = ||\sum_{j=1}^{n} \alpha_j v_j||_1^2 = \left\langle\sum_{j=1}^{n} \alpha_j v_j, \sum_{j=1}^{n} \alpha_j v_j\right\rangle_1 = \sum_{j=1}^{n} |\alpha_j|^2$$

Let $m = \max\{||v_1||_2^2, ||v_2||_2^2 \ldots ||v_n||_2^2\}$. Then

$$||v||_2^2 = ||\sum_{j=1}^{n} \alpha_j v_j||_2^2$$
$$= \left\langle\sum_{j=1}^{n} \alpha j v_j, \sum_{i=1}^{n} \alpha_i v_j\right\rangle_2$$
$$= \sum_{j=1}^{n}\sum_{i=1}^{n} \alpha_j \alpha_i^* \langle v_j, v_i\rangle_2$$

By Cauchy-Schwarz, $\langle v_j, v_i\rangle \leq ||v_j||||v_i|| \leq \max\{||v_j||_2^2, ||v_i||_2^2\} \leq m$. Furthermore, $|\alpha_j \alpha_i^*| \leq \max\{|\alpha_j|^2, |\alpha_i|^2\} \leq |\alpha_j|^2 + |\alpha_i|^2$.

Therefore

$$\sum_{j=1}^{n}\sum_{i=1}^{n} \alpha_j \alpha_i^* \langle v_j, v_i\rangle_2$$
$$\leq \sum_{j=1}^{n}\sum_{i=1}^{n} \alpha_j \alpha_i^* m$$
$$\leq m \sum_{j=1}^{n}\sum_{i=1}^{n} |\alpha_j|^2 + |\alpha_i|^2$$
$$\leq m[||v||_1^2 + ||v||_1^2]$$
$$\implies ||v||_2^2 \leq 2m||v||_1^2$$

As required.

Part 2

In this case, an orthonormal basis $e_1....e_n$ of $\langle,\rangle_1$ is also an orthogonal basis of $\langle,\rangle_2$, because $\langle e_i, e_j\rangle_1 = 0 \implies \langle e_i, e_j\rangle_2 = -$. It's enough to show that $\langle e_j, e_j\rangle_2 = c$ for every j, because then we can expand any vector in terms of this orthonormal basis to achieve the desired result.

Suppose that $\langle e_i, e_i\rangle_2 = c_1$ and $\langle e_j, e_j\rangle_2 = c_2$. Then

$$\left\langle e_i - \sqrt{\frac{c_1}{c_2}} e_j, e_i + \sqrt{\frac{c_1}{c_2}} e_j\right\rangle_2$$
$$= \langle e_i, e_i\rangle_2 - \frac{c_1}{c_2}\langle e_j, e_j\rangle_2$$
$$= c_1 - c_1 = 0$$

Therefore, by our assumptions,

$$\left\langle e_i - \sqrt{\frac{c_1}{c_2}} e_j, e_i + \sqrt{\frac{c_1}{c_2}} e_j\right\rangle_1 = 0$$

Expanding this out, we recover

$$\langle e_i, e_i \rangle - \frac{c_1}{c_2} \langle e_j, e_j \rangle = 1 - \frac{c_1}{c_2}$$

Therefore, for this to be zero, we must have $c_1 = c_2$. Therefore done. (DL)

Solution. Here's a slightly shorter version of Part 1 (that is very much the same in spirit). For a basis $e_1, \ldots, e_n$ which is orthonormal for the second inner product, we have

$$\|v\|_2 = \|a_1 e_1 + \cdots + a_n e_n\|_2 = \sqrt{a_1^2 + \cdots + a_n^2}.$$

For the first inner product, we consider the same sum:

$$\|v\|_1 = \|a_1 e_1 + \cdots + a_n e_n\|_1.$$

However, this decomposition is *not* necessarily orthonormal for the first inner product, so we have to use the triangle inequality.

$$\begin{aligned} &\leq \|a_1 e_1\|_1 + \cdots + \|a_n e_n\|_1 \\ &= |a_1| \|e_1\|_1 + \cdots + |a_n| \|e_n\|_1. \end{aligned}$$

Taking $m = \max\{\|e_1\|_1, \ldots, \|e_n\|_1\}$, we get a bound

$$\begin{aligned} &\leq |a_1| m + \cdots + |a_n| m \\ &= (|a_1| + \cdots + |a_n|) m. \end{aligned}$$

Finally, each of the terms $|a_j|$ is individually bounded above by our explicit formula for $\|v\|_2$, hence we have yet another bound

$$\begin{aligned} &\leq (\|v\|_2 + \cdots + \|v\|_2) m \\ &= \|v\|_2 \cdot n \cdot m. \end{aligned}$$

(ECP)

3 For submission to Handong Park

Problem 3.1. What happens if Gram–Schmidt is applied to a list of vectors that is not linearly independent?

Solution. It breaks: for a list of vectors $v_1, \ldots, v_n$ with intermediate subspaces

$$U_j = \operatorname{span}\{v_1, \ldots, v_j\},$$

Gram–Schmidt operates by forming the vectors $w_j = v_j - P_{U_{j-1}} v_j$ and normalizing them. If $v_j \in U_{j-1}$ witnesses a linear combination of the preceding vectors, then the resulting vector w_j is zero — but the zero vector cannot be normalized. (ECP)

Problem 3.2. Suppose V is a finite–dimensional complex vector space, and suppose $f \colon V \to V$ is a linear function whose eigenvalues are all of absolute value less than 1. For any $\varepsilon > 0$, show there exists a positive integer m with $\|T^m v\| < \varepsilon \|v\|$ for every $v \in V$. (Hint: you could begin with an upper-triangular presentation of f.)

Solution. We will, in fact, begin with an orthonormal upper-triangular presentation M of f. In class, we gave a formula for the entries of a product matrix:

$$(AB)_{ik} = \sum_{j=1}^{n} A_{ij} \cdot B_{jk}.$$

This generalizes to powers as follows:

$$(M^m)_{ik} = \sum_{j_1,\ldots,j_{n-1}} M_{ij_1} M_{j_1 j_2} \cdots M_{j_{m-2} j_{m-1}} M_{j_{m-1} k}.$$

The upper-triangular property of M means that $M_{yx} = 0$ whenever $y > x$. This makes our sum much smaller:

$$(M^m)_{ik} = \sum_{j_1 \leq \cdots \leq j_{m-1}} M_{ij_1} M_{j_1 j_2} \cdots M_{j_{m-2} j_{m-1}} M_{j_{m-1} k},$$

where the sum is now taken over weakly increasing sequences of integers between i and k. The main observation is that for $m \gg 0$, these sequence must mostly consist of repeated elements: at most $k - i$ different elements can appear, so at least $m - (k - i)$ entries of the form A_{jj} — i.e., diagonal entries — must appear. Since the diagonal entries all satisfy $|A_{jj}| < 1$, we have $\lim_{m\to\infty} |A_{jj}|^m = 0$ for any j. Taking m large enough so that $|A_{jj}|^m < \varepsilon/(n \cdot \prod_{i<k} A_{ik})$ for any choice of j ensures that the entries of the linear combination coefficients expressing any vector $v \in V$ get scaled down by at least ε. (ECP)

4 For submission to Rohil Prasad

Problem 4.1. 1. On $P_2(\mathbb{R})$, consider the inner product given by

$$\langle p, q\rangle = \int_0^1 p(x)q(x)dx.$$

Apply the Gram–Schmidt procedure to the basis $(1, x, x^2)$ to produce an orthonormal basis of $P_2(\mathbb{R})$.

2. Find a polynomial $q \in P_2(\mathbb{R})$ such that for every $p \in P_2(\mathbb{R})$,

$$p\left(\frac{1}{2}\right) = \langle p, q\rangle$$

under the same inner product.

Solution. 1. This problem is largely computational.

1: We need only check that 1 is a normal vector:

$$\sqrt{\int_0^1 1 \cdot 1dx} = \sqrt{1} = 1.$$

x: First, we remove the projection of x onto the subspace spanned by 1:

$$x - 1 \cdot \int_0^1 x \cdot 1dx = x - \frac{1}{2}.$$

Then, we calculate the norm

$$\left\| x - \frac{1}{2} \right\| = \sqrt{\int_0^1 \left(x - \frac{1}{2}\right)^2 dx} = \sqrt{\frac{1}{3}\left(x - \frac{1}{2}\right)^3 \Bigg|_{x=0}^{1}} = \frac{1}{2\sqrt{3}},$$

so that we can normalize the result:

$$\frac{x - 1/2}{1/2\sqrt{3}} = 2\sqrt{3}x - \sqrt{3}.$$

x^2: Again, remove the projection of x^2 onto the subspace spanned by 1 and x:

$$x^2 - 1 \cdot \int_0^1 x^2 \cdot 1 dx - (2\sqrt{3}x - \sqrt{3}) \cdot \int_0^1 x^2 \cdot (2\sqrt{3}x - \sqrt{3}) dx = x^2 - x + \frac{1}{6}.$$

Then, we calculate the norm

$$\left\| x^2 - x + \frac{1}{6} \right\| = \sqrt{\int_0^1 \left(x^4 - 2x^3 + \frac{4}{3}x^2 - \frac{1}{3}x + \frac{1}{36} \right) dx} = \frac{1}{6\sqrt{5}},$$

so that we can normalize the result:

$$\frac{x^2 - x + \frac{1}{6}}{\frac{1}{6\sqrt{5}}} = 6\sqrt{5}x^2 - 6\sqrt{5}x + \sqrt{5}.$$

2. Write $\varphi(p)$ for the functional $\varphi(p) = p(1/2)$. Our by-hand proof of Riesz's theorem gives an explicit formula for the polynomial q satisfying $\langle p, q \rangle = \varphi(p)$:

$$q = \overline{\varphi(e_1)} e_1 + \overline{\varphi(e_2)} e_2 + \overline{\varphi(e_3)} e_3,$$

for any orthonormal basis (e_1, e_2, e_3) of our space of polynomials. In the previous part, we calculated such a basis. Hence:

$$\begin{aligned} q &= \varphi(1) \cdot 1 + \varphi(2\sqrt{3}x - \sqrt{3}) \cdot (2\sqrt{3}x - \sqrt{3}) + \varphi\left(6\sqrt{5}x^2 - 6\sqrt{5}x + \sqrt{5}\right) \cdot \left(6\sqrt{5}x^2 - 6\sqrt{5}x + \sqrt{5}\right) \\ &= 1 \cdot 1 + 0 \cdot (2\sqrt{3}x - \sqrt{3}) + \frac{-\sqrt{5}}{2} \cdot \left(6\sqrt{5}x^2 - 6\sqrt{5}x + \sqrt{5}\right) \\ &= -15x^2 + 15x - \frac{3}{2}. \end{aligned}$$

(ECP)

Problem 4.2. The Fibonacci sequence $F_1, F_2, \ldots$ is defined by

$$F_1 = 1, \qquad F_2 = 1, \qquad F_n = F_{n-2} + F_{n-1}.$$

We also define a linear function $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x, y) = (y, x + y)$.

1. Show that $T^n(0, 1) = (F_n, F_{n+1})$.[1]
2. Find the eigenvalues of T.
3. Find a basis of $\mathbb{R}^2$ consisting of eigenvectors of T.
4. Use the solution to the previous part to compute $T^n(0, 1)$ in closed form.
5. Conclude more lazily that the n^{th} Fibonacci number F_n is the nearest integer to

$$\frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n.$$

[1] This used to read (F_n, F_{n-1}), which was a typo. Sorry!

Solution.

1. We give an inductive proof. The claim holds for $n = 1$:

$$T^1(0,1) = (1, 0+1) = (1,1) = (F_1, F_2).$$

The inductive step follows from

$$T^{n+1}(0,1) = TT^n(0,1) = T(F_n, F_{n+1}) = (F_{n+1}, F_n + F_{n+1}) = (F_{n+1}, F_{n+2}).$$

2. The eigenvalue equation for T is

$$\begin{pmatrix} y \\ x+y \end{pmatrix} = T \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix}.$$

Since $\lambda = 0$ cannot satisfy this relation, we are free to divide by λ and combine the two equations to get $0 = \lambda^2 - \lambda - 1$, or

$$\lambda = \frac{1 \pm \sqrt{5}}{2}.$$

3. Write λ_+ and λ_- for the positive and negative eigenvalues respectively. By picking $x = 1$ and using the first row of the eigenvector equation, an eigenvector for λ_+ is $\begin{pmatrix} 1 \\ \lambda_+ \end{pmatrix}$ and an eigenvector for λ_- is $\begin{pmatrix} 1 \\ \lambda_- \end{pmatrix}$. These are linearly independent and of the right length, hence they form a basis.

4. We seek a_+ and a_- solving

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = a_+ \begin{pmatrix} 1 \\ \lambda_+ \end{pmatrix} + a_- \begin{pmatrix} 1 \\ \lambda_- \end{pmatrix}.$$

The first row forces $a_+ = -a_-$, and the second row gives $1 = a_+\sqrt{5}$. Hence, we have

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ \lambda_+ \end{pmatrix} + \frac{-1}{\sqrt{5}} \begin{pmatrix} 1 \\ \lambda_- \end{pmatrix}.$$

Applying T^n to this equation gives

$$T^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{\lambda_+^n}{\sqrt{5}} \begin{pmatrix} 1 \\ \lambda_+ \end{pmatrix} + \frac{-\lambda_-^n}{\sqrt{5}} \begin{pmatrix} 1 \\ \lambda_- \end{pmatrix}.$$

The first entry reads $F_n = \frac{1}{\sqrt{5}}(\lambda_+^n - \lambda_-^n)$.

5. We know that F_n always gives an integer value, but both λ_+^n and λ_-^n are always irrational values. The observation here is that because $|\lambda_-| < 1$, $\frac{1}{\sqrt{5}}\lambda_-^n < \frac{1}{2}$, so that this term never disturbs the sum by very much. In particular, F_n is always the nearest integer to the first term alone:

$$F_n \approx \frac{\lambda_+^n}{\sqrt{5}}. \tag{ECP}$$